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计算机程序设计艺术

第4卷 第3册（双语版）

生成所有组合和分划

The Art of Computer Programming, Volume 4
Generating All Combinations and Permutations

Fascicle 3

苏运霖 译

（美）Donald E. Knuth 著

机械工业出版社
China Machine Press

关于算法分析的这多卷论著已经长期被公认为经典计算机科学的定义性描述。这一册以及刚刚出版的第4卷第2册揭开了人们急切等待的《计算机程序设计艺术 第4卷 组合算法》的序幕。作为关于组合查找的冗长一章的一部分，这一册开始关于生成所有组合和分划的讨论。在Knuth讨论这两个主题的过程中，读者不仅会看到很多新内容，并且会发现本册与卷1至卷3及计算机科学和数学的其他方面的丰富联系。一如既往，书中包括了大量的习题和富有挑战性的难题。

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图书在版编目（CIP）数据

计算机程序设计艺术 第4卷 第3册，生成所有组合和分划（双语版）/（美）克努特（Knuth, D. E.）著．苏运霖译．－北京：机械工业出版社，2006.8

书名原文：The Art of Computer Programming, Volume 4, Fascicle 3, Generating All Combinations and Partitions

ISBN 7-111-17774-6

Ⅰ.计… Ⅱ.①克… ②苏… Ⅲ.程序设计 Ⅳ.TP311.1

中国版本图书馆CIP数据核字（2006）第126014号

机械工业出版社（北京市西城区百万庄大街22号 邮政编码 100037）
责任编辑：朱起飞
北京瑞德印刷有限公司印刷·新华书店北京发行所发行
2006年8月第1版第1次印刷
170mm × 242mm · 20.5印张
定价：45.00元

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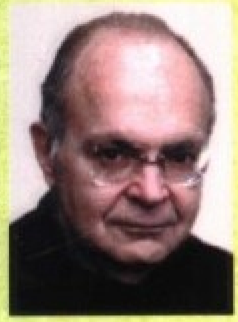

Donald E. Knuth

（唐纳德·E.克努特，中文名高德纳）

是算法和程序设计技术的先驱者，并发明了计算机排版系统TEX和METAFONT，他因这些成就和大量创造性的、影响深远的论著而誉满全球。作为斯坦福大学计算机程序设计艺术的荣誉退休教授，Knuth现正投入全部的时间来完成其关于计算机科学的史诗性的七卷集。Knuth教授获得了许多奖项和荣誉，包括美国计算机协会图灵奖（ACM Turing Award），美国前总统卡特授予的科学金奖（Medal of Science），美国数学学会斯蒂尔奖（AMS Steele Prize），以及极受尊重的京都奖（Kyoto Prize）。

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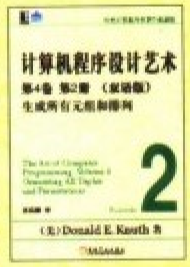

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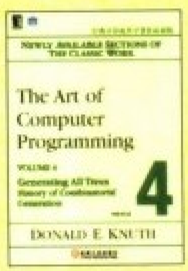

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PREFACE

In my preface to the first edition,
I begged the reader not to draw attention to errors.
I now wish I had not done so
and am grateful to the few readers who ignored my request.
— STUART SUTHERLAND, *The International Dictionary of Psychology* (1996)

THIS BOOKLET is Fascicle 3 of *The Art of Computer Programming*, Volume 4: *Combinatorial Algorithms.* As explained in the preface to Fascicle 1 of Volume 1, I'm circulating the material in this preliminary form because I know that the task of completing Volume 4 will take many years; I can't wait for people to begin reading what I've written so far and to provide valuable feedback.

To put the material in context, this fascicle contains Sections 7.2.1.3, 7.2.1.4, and 7.2.1.5 of a long, long chapter on combinatorial searching. Chapter 7 will eventually fill three volumes (namely Volumes 4A, 4B, and 4C), assuming that I'm able to remain healthy. It will begin with a short review of graph theory, with emphasis on some highlights of significant graphs in The Stanford GraphBase, from which I will be drawing many examples. Then comes Section 7.1, which deals with bitwise manipulation and with algorithms relating to Boolean functions. Section 7.2 is about generating all possibilities, and it begins with Section 7.2.1: Generating Basic Combinatorial Patterns. Details about various useful ways to generate n-tuples appear in Section 7.2.1.1, and the generation of permutations is discussed in Section 7.2.1.2. That sets the stage for the main contents of the present booklet, namely Section 7.2.1.3 (which extends the ideas to combinations of n things taken t at a time); Section 7.2.1.4 (about partitions of an integer); and Section 7.2.1.5 (about partitions of a set). Then will come Section 7.2.1.6 (about trees) and Section 7.2.1.7 (about the history of combinatorial generation), in Fascicle 4. Section 7.2.2 will deal with backtracking in general. And so it will go on, if all goes well; an outline of the entire Chapter 7 as currently envisaged appears on the `taocp` webpage that is cited on page ii.

I had great pleasure writing this material, akin to the thrill of excitement that I felt when writing Volume 2 many years ago. As in Volume 2, where I found to my delight that the basic principles of elementary probability theory and number theory arose naturally in the study of algorithms for random number generation and arithmetic, I learned while preparing Section 7.2.1 that the basic principles of elementary combinatorics arise naturally and in a highly motivated way when we study algorithms for combinatorial generation. Thus, I found once again that a beautiful story was "out there" waiting to be told.

For example, in the present booklet we find many of the beautiful patterns formed by combinations, with and without repetition, and how they relate to famous theorems of extremal combinatorics. Then comes my chance to tell the extraordinary story of partitions; indeed, the theory of partitions is one of the nicest chapters in all of mathematics. And in Section 7.2.1.5, a little-known triangle of numbers, discovered by C. S. Peirce, turns out to simplify and unify the study of set partitions, another vital topic. Along the way I've included expositions of two mathematical techniques of great importance in the analysis of algorithms: Poisson's summation formula, and the powerful saddle point method. There are games and puzzles too, as in the previous fascicles.

My original intention was to devote far less space to these subjects. But when I saw how fundamental the ideas were for combinatorial studies in general, I knew that I could never be happy unless I covered the basics quite thoroughly. Therefore I've done my best to build a solid foundation of theoretical and practical ideas that will support many kinds of reliable superstructures.

I thank Frank Ruskey for bravely foisting an early draft of this material on college students and for telling me about his classroom experiences. Many other readers have also helped me to check the first drafts; I wish to thank especially George Clements and Svante Janson for their penetrating comments.

I shall happily pay a finder's fee of $2.56 for each error in this fascicle when it is first reported to me, whether that error be typographical, technical, or historical. The same reward holds for items that I forgot to put in the index. And valuable suggestions for improvements to the text are worth 32¢ each. (Furthermore, if you find a better solution to an exercise, I'll actually reward you with immortal glory instead of mere money, by publishing your name in the eventual book:–)

Notations that are used here and not otherwise explained can be found in the Index to Notations at the end of Volumes 1, 2, or 3. Those indexes point to the places where further information is available. Of course Volume 4 will some day contain its own Index to Notations.

Machine-language examples in all future editions of *The Art of Computer Programming* will be based on the `MMIX` computer, which is described in Volume 1, Fascicle 1.

Cross references to yet-unwritten material sometimes appear as '00' in the following pages; this impossible value is a placeholder for the actual numbers to be supplied later.

Happy reading!

Stanford, California D. E. K.
June 2005

CONTENTS

目 录

假若一个词不能在一行的末尾被分开，那它必定要在一个音节的末尾被分划。

——亚历山大·何姆(Alexander Hume)，Orthographie…of the Britan Tongue

(约1620年)

7.2.1.3. Generating all combinations. Combinatorial mathematics is often described as "the study of permutations, combinations, etc.," so we turn our attention now to combinations. A *combination of n things, taken t at a time*, often called simply a t-combination of n things, is a way to select a subset of size t from a given set of size n. We know from Eq. 1.2.6–(2) that there are exactly $\binom{n}{t}$ ways to do this; and we learned in Section 3.4.2 how to choose t-combinations at random.

Selecting t of n objects is equivalent to choosing the $n-t$ elements not selected. We will emphasize this symmetry by letting

$$n = s + t \qquad (1)$$

throughout our discussion, and we will often refer to a t-combination of n things as an "(s,t)-combination." Thus, an (s,t)-combination is a way to subdivide $s+t$ objects into two collections of sizes s and t.

If I ask how many combinations of 21 can be taken out of 25,
I do in effect ask how many combinations of 4 may be taken.
For there are just as many ways of taking 21 as there are of leaving 4.
— AUGUSTUS DE MORGAN, *An Essay on Probabilities* (1838)

There are two main ways to represent (s,t)-combinations: We can list the elements $c_t \ldots c_2c_1$ that have been selected, or we can work with binary strings $a_{n-1} \ldots a_1a_0$ for which

$$a_{n-1} + \cdots + a_1 + a_0 = t. \qquad (2)$$

The latter representation has s 0s and t 1s, corresponding to elements that are unselected or selected. The list representation $c_t \ldots c_2c_1$ tends to work out best if we let the elements be members of the set $\{0,1,\ldots,n-1\}$ and if we list them in *decreasing* order:

$$n > c_t > \cdots > c_2 > c_1 \geq 0. \qquad (3)$$

Binary notation connects these two representations nicely, because the item list $c_t \ldots c_2c_1$ corresponds to the sum

$$2^{c_t} + \cdots + 2^{c_2} + 2^{c_1} = \sum_{k=0}^{n-1} a_k 2^k = (a_{n-1} \ldots a_1a_0)_2. \qquad (4)$$

Of course we could also list the positions $b_s \ldots b_2 b_1$ of the 0s in $a_{n-1} \ldots a_1 a_0$, where

$$n > b_s > \cdots > b_2 > b_1 \geq 0. \tag{5}$$

Combinations are important not only because subsets are omnipresent in mathematics but also because they are equivalent to many other configurations. For example, every (s,t)-combination corresponds to a combination of $s+1$ things taken t at a time *with repetitions permitted*, also called a *multicombination*, namely a sequence of integers $d_t \ldots d_2 d_1$ with

$$s \geq d_t \geq \cdots \geq d_2 \geq d_1 \geq 0. \tag{6}$$

One reason is that $d_t \ldots d_2 d_1$ solves (6) if and only if $c_t \ldots c_2 c_1$ solves (3), where

$$c_t = d_t + t - 1, \quad \ldots, \quad c_2 = d_2 + 1, \quad c_1 = d_1 \tag{7}$$

(see exercise 1.2.6–60). And there is another useful way to relate combinations with repetition to ordinary combinations, suggested by Solomon Golomb [*AMM* **75** (1968), 530–531], namely to define

$$e_j = \begin{cases} c_j, & \text{if } c_j \leq s; \\ e_{c_j - s}, & \text{if } c_j > s. \end{cases} \tag{8}$$

In this form the numbers $e_t \ldots e_1$ don't necessarily appear in descending order, but the multiset $\{e_1, e_2, \ldots, e_t\}$ is equal to $\{c_1, c_2, \ldots, c_t\}$ if and only if $\{e_1, e_2, \ldots, e_t\}$ is a set. (See Table 1 and exercise 1.)

An (s,t)-combination is also equivalent to a *composition* of $n+1$ into $t+1$ parts, namely an ordered sum

$$n + 1 = p_t + \cdots + p_1 + p_0, \qquad \text{where } p_t, \ldots, p_1, p_0 \geq 1. \tag{9}$$

The connection with (3) is now

$$p_t = n - c_t, \quad p_{t-1} = c_t - c_{t-1}, \quad \ldots, \quad p_1 = c_2 - c_1, \quad p_0 = c_1 + 1. \tag{10}$$

Equivalently, if $q_j = p_j - 1$, we have

$$s = q_t + \cdots + q_1 + q_0, \qquad \text{where } q_t, \ldots, q_1, q_0 \geq 0, \tag{11}$$

a composition of s into $t+1$ *nonnegative* parts, related to (6) by setting

$$q_t = s - d_t, \quad q_{t-1} = d_t - d_{t-1}, \quad \ldots, \quad q_1 = d_2 - d_1, \quad q_0 = d_1. \tag{12}$$

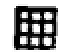
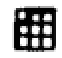
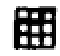
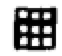
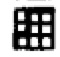
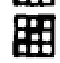
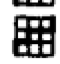
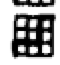
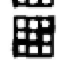
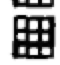
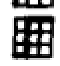
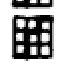
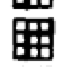
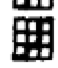
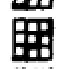
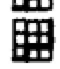
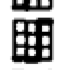
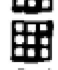
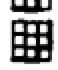
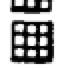
Furthermore it is easy to see that an (s,t)-combination is equivalent to a path of length $s+t$ from corner to corner of an $s \times t$ grid, because such a path contains s vertical steps and t horizontal steps. Thus, combinations can be studied in at least eight different guises. Table 1 illustrates all $\binom{6}{3} = 20$ possibilities in the case $s = t = 3$.

These cousins of combinations might seem rather bewildering at first glance, but most of them can be understood directly from the binary representation $a_{n-1} \ldots a_1 a_0$. Consider, for example, the "random" bit string

$$a_{23} \ldots a_1 a_0 = 011001001000011111101101, \tag{13}$$

Table 1

THE (3,3)-COMBINATIONS AND THEIR EQUIVALENTS

$a_5a_4a_3a_2a_1a_0$	$b_3b_2b_1$	$c_3c_2c_1$	$d_3d_2d_1$	$e_3e_2e_1$	$p_3p_2p_1p_0$	$q_3q_2q_1q_0$	path
000111	543	210	000	210	4111	3000	
001011	542	310	100	310	3211	2100	
001101	541	320	110	320	3121	2010	
001110	540	321	111	321	3112	2001	
010011	532	410	200	010	2311	1200	
010101	531	420	210	020	2221	1110	
010110	530	421	211	121	2212	1101	
011001	521	430	220	030	2131	1020	
011010	520	431	221	131	2122	1011	
011100	510	432	222	232	2113	1002	
100011	432	510	300	110	1411	0300	
100101	431	520	310	220	1321	0210	
100110	430	521	311	221	1312	0201	
101001	421	530	320	330	1231	0120	
101010	420	531	321	331	1222	0111	
101100	410	532	322	332	1213	0102	
110001	321	540	330	000	1141	0030	
110010	320	541	331	111	1132	0021	
110100	310	542	332	222	1123	0012	
111000	210	543	333	333	1114	0003	

which has $s = 11$ zeros and $t = 13$ ones, hence $n = 24$. The dual combination $b_s \ldots b_1$ lists the positions of the zeros, namely

$$23\ 20\ 19\ 17\ 16\ 14\ 13\ 12\ 11\ 4\ 1,$$

because the leftmost position is $n - 1$ and the rightmost is 0. The primal combination $c_t \ldots c_1$ lists the positions of the ones, namely

$$22\ 21\ 18\ 15\ 10\ 9\ 8\ 7\ 6\ 5\ 3\ 2\ 0.$$

The corresponding multicombination $d_t \ldots d_1$ lists the number of 0s to the right of each 1:

$$10\ 10\ 8\ 6\ 2\ 2\ 2\ 2\ 2\ 2\ 1\ 1\ 0.$$

The composition $p_t \ldots p_0$ lists the distances between consecutive 1s, if we imagine additional 1s at the left and the right:

$$2\ 1\ 3\ 3\ 5\ 1\ 1\ 1\ 1\ 1\ 2\ 1\ 2\ 1.$$

And the nonnegative composition $q_t \ldots q_0$ counts how many 0s appear between "fenceposts" represented by 1s:

$$1\ 0\ 2\ 2\ 4\ 0\ 0\ 0\ 0\ 0\ 1\ 0\ 1\ 0;$$

thus we have

$$a_{n-1} \ldots a_1 a_0 = 0^{q_t} 1 0^{q_{t-1}} 1 \ldots 1 0^{q_1} 1 0^{q_0}. \qquad (14)$$

The paths in Table 1 also have a simple interpretation (see exercise 2).

Lexicographic generation. Table 1 shows combinations $a_{n-1}\ldots a_1a_0$ and $c_t\ldots c_1$ in lexicographic order, which is also the lexicographic order of $d_t\ldots d_1$. Notice that the dual combinations $b_s\ldots b_1$ and the corresponding compositions $p_t\ldots p_0$, $q_t\ldots q_0$ then appear in *reverse* lexicographic order.

Lexicographic order usually suggests the most convenient way to generate combinatorial configurations. Indeed, Algorithm 7.2.1.2L already solves the problem for combinations in the form $a_{n-1}\ldots a_1a_0$, since (s,t)-combinations in bitstring form are the same as permutations of the multiset $\{s\cdot 0, t\cdot 1\}$. That general-purpose algorithm can be streamlined in obvious ways when it is applied to this special case. (See also exercise 7.1–00, which presents a remarkable sequence of seven bitwise operations that will convert any given binary number $(a_{n-1}\ldots a_1a_0)_2$ to the lexicographically next t-combination, assuming that n does not exceed the computer's word length.)

Let's focus, however, on generating combinations in the other principal form $c_t\ldots c_2c_1$, which is more directly relevant to the ways in which combinations are often needed, and which is more compact than the bit strings when t is small compared to n. In the first place we should keep in mind that a simple sequence of nested loops will do the job nicely when t is very small. For example, when $t=3$ the following instructions suffice:

For $c_3 = 2, 3, \ldots, n-1$ (in this order) do the following:
 For $c_2 = 1, 2, \ldots, c_3-1$ (in this order) do the following:
  For $c_1 = 0, 1, \ldots, c_2-1$ (in this order) do the following:
   Visit the combination $c_3c_2c_1$. (15)

(See the analogous situation in 7.2.1.1–(3).)

On the other hand when t is variable or not so small, we can generate combinations lexicographically by following the general recipe discussed after Algorithm 7.2.1.2L, namely to find the rightmost element c_j that can be increased and then to set the subsequent elements $c_{j-1}\ldots c_1$ to their smallest possible values:

Algorithm L (*Lexicographic combinations*). This algorithm generates all t-combinations $c_t\ldots c_2c_1$ of the n numbers $\{0,1,\ldots,n-1\}$, given $n\ge t\ge 0$. Additional variables c_{t+1} and c_{t+2} are used as sentinels.

L1. [Initialize.] Set $c_j \leftarrow j-1$ for $1\le j\le t$; also set $c_{t+1}\leftarrow n$ and $c_{t+2}\leftarrow 0$.

L2. [Visit.] Visit the combination $c_t\ldots c_2c_1$.

L3. [Find j.] Set $j\leftarrow 1$. Then, while $c_j+1=c_{j+1}$, set $c_j\leftarrow j-1$ and $j\leftarrow j+1$; eventually the condition $c_j+1\ne c_{j+1}$ will occur

L4. [Done?] Terminate the algorithm if $j>t$.

L5. [Increase c_j.] Set $c_j\leftarrow c_j+1$ and return to L2. ▌

The running time of this algorithm is not difficult to analyze. Step L3 sets $c_j\leftarrow j-1$ just after visiting a combination for which $c_{j+1}=c_1+j$, and the number of such combinations is the number of solutions to the inequalities

$$n > c_t > \cdots > c_{j+1} \ge j; \tag{16}$$

but this formula is equivalent to a $(t-j)$-combination of the $n-j$ objects $\{n-1,\ldots,j\}$, so the assignment $c_j \leftarrow j-1$ occurs exactly $\binom{n-j}{t-j}$ times. Summing for $1 \le j \le t$ tells us that the loop in step L3 is performed

$$\binom{n-1}{t-1}+\binom{n-2}{t-2}+\cdots+\binom{n-t}{0} = \binom{n-1}{s}+\binom{n-2}{s}+\cdots+\binom{s}{s} = \binom{n}{s+1} \quad (17)$$

times altogether, or an average of

$$\binom{n}{s+1}\Big/\binom{n}{t} = \frac{n!}{(s+1)!\,(t-1)!}\Big/\frac{n!}{s!\,t!} = \frac{t}{s+1} \quad (18)$$

times per visit. This ratio is less than 1 when $t \le s$, so Algorithm L is quite efficient in such cases.

But the quantity $t/(s+1)$ can be embarrassingly large when t is near n and s is small. Indeed, Algorithm L occasionally sets $c_j \leftarrow j-1$ needlessly, at times when c_j already equals $j-1$. Further scrutiny reveals that we need not always search for the index j that is needed in steps L4 and L5, since the correct value of j can often be predicted from the actions just taken. For example, after we have increased c_4 and reset $c_3c_2c_1$ to their starting values 210, the next combination will inevitably increase c_3. These observations lead to a tuned-up version of the algorithm:

Algorithm T (*Lexicographic combinations*). This algorithm is like Algorithm L, but faster. It also assumes, for convenience, that $t < n$.

T1. [Initialize.] Set $c_j \leftarrow j-1$ for $1 \le j \le t$; then set $c_{t+1} \leftarrow n$, $c_{t+2} \leftarrow 0$, and $j \leftarrow t$.

T2. [Visit.] (At this point j is the smallest index such that $c_{j+1} > j$.) Visit the combination $c_t \ldots c_2c_1$. Then, if $j > 0$, set $x \leftarrow j$ and go to step T6.

T3. [Easy case?] If $c_1 + 1 < c_2$, set $c_1 \leftarrow c_1 + 1$ and return to T2. Otherwise set $j \leftarrow 2$.

T4. [Find j.] Set $c_{j-1} \leftarrow j-2$ and $x \leftarrow c_j + 1$. If $x = c_{j+1}$, set $j \leftarrow j+1$ and repeat this step until $x \ne c_{j+1}$.

T5. [Done?] Terminate the algorithm if $j > t$.

T6. [Increase c_j.] Set $c_j \leftarrow x$, $j \leftarrow j-1$, and return to T2. ▌

Now $j = 0$ in step T2 if and only if $c_1 > 0$, so the assignments in step T4 are never redundant. Exercise 6 carries out a complete analysis of Algorithm T.

Notice that the parameter n appears only in the initialization steps L1 and T1, not in the principal parts of Algorithms L and T. Thus we can think of the process as generating the first $\binom{n}{t}$ combinations of an *infinite* list, which depends only on t. This simplification arises because the list of t-combinations for $n+1$ things begins with the list for n things, under our conventions; we have been using lexicographic order on the decreasing sequences $c_t \ldots c_1$ for this very reason, instead of working with the increasing sequences $c_1 \ldots c_t$.

Derrick Lehmer noticed another pleasant property of Algorithms L and T [*Applied Combinatorial Mathematics*, edited by E. F. Beckenbach (1964), 27–30]:

Theorem L. *The combination $c_t \ldots c_2c_1$ is visited after exactly*

$$\binom{c_t}{t} + \cdots + \binom{c_2}{2} + \binom{c_1}{1} \tag{19}$$

other combinations have been visited.

Proof. There are $\binom{c_k}{k}$ combinations $c'_t \ldots c'_2c'_1$ with $c'_j = c_j$ for $t \geq j > k$ and $c'_k < c_k$, namely $c_t \ldots c_{k+1}$ followed by the k-combinations of $\{0, \ldots, c_k - 1\}$. ▌

When $t = 3$, for example, the numbers

$$\binom{2}{3}+\binom{1}{2}+\binom{0}{1},\ \binom{3}{3}+\binom{1}{2}+\binom{0}{1},\ \binom{3}{3}+\binom{2}{2}+\binom{0}{1},\ \ldots,\ \binom{5}{3}+\binom{4}{2}+\binom{3}{1}$$

that correspond to the combinations $c_3c_2c_1$ in Table 1 simply run through the sequence 0, 1, 2, ..., 19. Theorem L gives us a nice way to understand the *combinatorial number system* of degree t, which represents every nonnegative integer N uniquely in the form

$$N = \binom{n_t}{t} + \cdots + \binom{n_2}{2} + \binom{n_1}{1}, \qquad n_t > \cdots > n_2 > n_1 \geq 0. \tag{20}$$

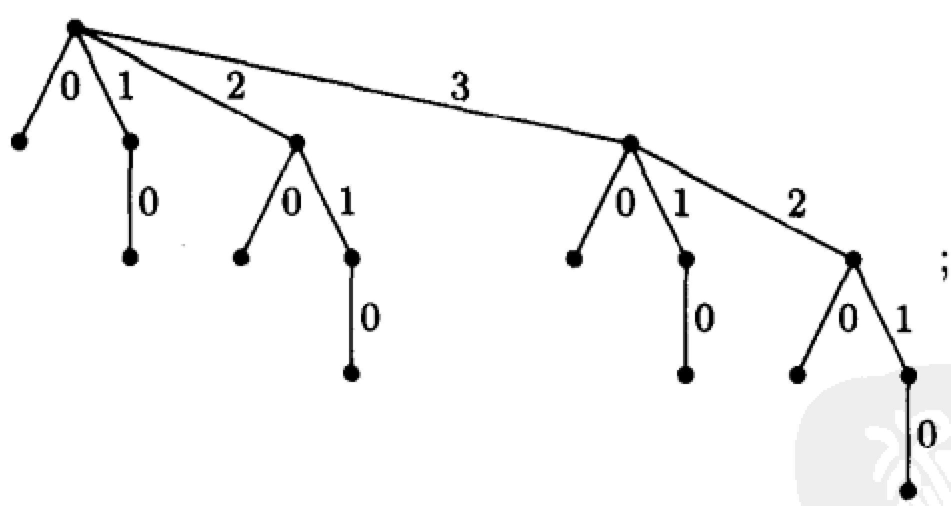
[See Ernesto Pascal, *Giornale di Matematiche* **25** (1887), 45–49.]

Binomial trees. The family of trees T_n defined by

$T_0 = \bullet$, $T_n =$

for $n > 0$

arises in several important contexts and sheds further light on combination generation. For example, T_4 is



; (22)

and T_5, rendered more artistically, appears as the frontispiece to Volume 1 of this series of books.

Notice that T_n is like T_{n-1}, except for an additional copy of T_{n-1}; therefore T_n has 2^n nodes altogether. Furthermore, the number of nodes on level t is the binomial coefficient $\binom{n}{t}$; this fact accounts for the name "binomial tree." Indeed, the sequence of labels encountered on the path from the root to each node on level t defines a combination $c_t \ldots c_1$, and all combinations occur in lexicographic order from left to right. Thus, Algorithms L and T can be regarded as procedures to traverse the nodes on level t of the binomial tree T_n.

The infinite binomial tree T_∞ is obtained by letting $n \to \infty$ in (21). The root of this tree has infinitely many branches, but every node except for the overall root at level 0 is the root of a finite binomial subtree. All possible t-combinations appear in lexicographic order on level t of T_∞.

Let's get more familiar with binomial trees by considering all possible ways to pack a rucksack. More precisely, suppose we have n items that take up respectively w_{n-1}, ..., w_1, w_0 units of capacity, where

$$w_{n-1} \geq \cdots \geq w_1 \geq w_0; \tag{23}$$

we want to generate all binary vectors $a_{n-1} \ldots a_1 a_0$ such that

$$a \cdot w = a_{n-1}w_{n-1} + \cdots + a_1 w_1 + a_0 w_0 \leq N, \tag{24}$$

where N is the total capacity of a rucksack. Equivalently, we want to find all subsets C of $\{0, 1, \ldots, n-1\}$ such that $w(C) = \sum_{c \in C} w_c \leq N$; such subsets will be called *feasible*. We will write a feasible subset as $c_1 \ldots c_t$, where $c_1 > \cdots > c_t \geq 0$, numbering the subscripts differently from the convention of (3) above because t is variable in this problem.

Every feasible subset corresponds to a node of T_n, and our goal is to visit each feasible node. Clearly the parent of every feasible node is feasible, and so is the left sibling, if any; therefore a simple tree exploration procedure works well:

Algorithm F (*Filling a rucksack*). This algorithm generates all feasible ways $c_1 \ldots c_t$ to fill a rucksack, given w_{n-1}, ..., w_1, w_0, and N. We let $\delta_j = w_j - w_{j-1}$ for $1 \leq j < n$.

F1. [Initialize.] Set $t \leftarrow 0$, $c_0 \leftarrow n$, and $r \leftarrow N$.

F2. [Visit.] Visit the combination $c_1 \ldots c_t$, which uses $N - r$ units of capacity.

F3. [Try to add w_0.] If $c_t > 0$ and $r \geq w_0$, set $t \leftarrow t+1$, $c_t \leftarrow 0$, $r \leftarrow r - w_0$, and return to F2.

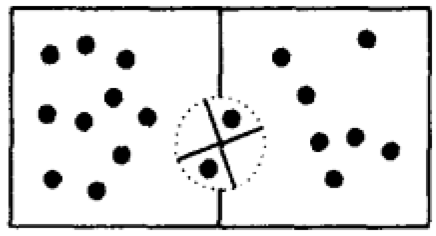
F4. [Try to increase c_t.] Terminate if $t = 0$. Otherwise, if $c_{t-1} > c_t + 1$ and $r \geq \delta_{c_t+1}$, set $c_t \leftarrow c_t + 1$, $r \leftarrow r - \delta_{c_t}$, and return to F2.

F5. [Remove c_t.] Set $r \leftarrow r + w_{c_t}$, $t \leftarrow t - 1$, and return to F4. ▮

Notice that the algorithm implicitly visits nodes of T_n in preorder, skipping over unfeasible subtrees. An element $c > 0$ is placed in the rucksack, if it fits, just after the procedure has explored all possibilities using element $c-1$ in its place. The running time is proportional to the number of feasible combinations visited (see exercise 20).

Incidentally, the classical "knapsack problem" of operations research is different: It asks for a feasible subset C such that $v(C) = \sum_{c \in C} v(c)$ is maximum, where each item c has been assigned a value $v(c)$. Algorithm F is not a particularly good way to solve that problem, because it often considers cases that could be ruled out. For example, if C and C' are subsets of $\{1, \ldots, n-1\}$ with $w(C) \leq w(C') \leq N - w_0$ and $v(C) \geq v(C')$, Algorithm F will examine both $C \cup \{0\}$ and $C' \cup \{0\}$, but the latter subset will never improve the maximum. We will consider methods for the classical knapsack problem later; Algorithm F is intended only for situations when *all* of the feasible possibilities are potentially relevant.

Gray codes for combinations. Instead of merely generating all combinations, we often prefer to visit them in such a way that each one is obtained by making only a small change to its predecessor.



For example, we can ask for what Nijenhuis and Wilf have called a "revolving door algorithm": Imagine two rooms that contain respectively s and t people, with a revolving door between them. Whenever a person goes into the opposite room, somebody else comes out. Can we devise a sequence of moves so that each (s,t)-combination occurs exactly once?

The answer is yes, and in fact a huge number of such patterns exist. For example, it turns out that if we examine all n-bit strings $a_{n-1}\ldots a_1a_0$ in the well-known order of Gray binary code (Section 7.2.1.1), but select only those that have exactly s 0s and t 1s, the resulting strings form a revolving-door code.

Here's the proof: Gray binary code is defined by the recurrence $\Gamma_n = 0\Gamma_{n-1}, 1\Gamma^R_{n-1}$ of 7.2.1.1–(5), so its (s,t) subsequence satisfies the recurrence

$$\Gamma_{st} \;=\; 0\Gamma_{(s-1)t},\; 1\Gamma^R_{s(t-1)} \tag{25}$$

when $st > 0$. We also have $\Gamma_{s0} = 0^s$ and $\Gamma_{0t} = 1^t$. Therefore it is clear by induction that Γ_{st} begins with 0^s1^t and ends with 10^s1^{t-1} when $st > 0$. The transition at the comma in (25) is from the last element of $0\Gamma_{(s-1)t}$ to the last element of $1\Gamma_{s(t-1)}$, namely from $010^{s-1}1^{t-1} = 010^{s-1}11^{t-2}$ to $110^s1^{t-2} = 110^{s-1}01^{t-2}$ when $t \ge 2$, and this satisfies the revolving-door constraint. The case $t = 1$ also checks out. For example, Γ_{33} is given by the columns of

000111	011010	110001	101010	
001101	011100	110010	101100	
001110	010101	110100	100101	(26)
001011	010110	111000	100110	
011001	010011	101001	100011	

and Γ_{23} can be found in the first two columns of this array. One more turn of the door takes the last element into the first. [These properties of Γ_{st} were discovered by D. T. Tang and C. N. Liu, *IEEE Trans.* **C-22** (1973), 176–180; a loopless implementation was presented by J. R. Bitner, G. Ehrlich, and E. M. Reingold, *CACM* **19** (1976), 517–521.]

When we convert the bit strings $a_5a_4a_3a_2a_1a_0$ in (26) to the corresponding index-list forms $c_3c_2c_1$, a striking pattern becomes evident:

210	431	540	531	
320	432	541	532	
321	420	542	520	(27)
310	421	543	521	
430	410	530	510	

The first components c_3 occur in increasing order; but for each fixed value of c_3, the values of c_2 occur in *decreasing* order. And for fixed c_3c_2, the values of c_1 are again increasing. The same is true in general: *All combinations* $c_t\ldots c_2c_1$

appear in lexicographic order of

$$(c_t,\ -c_{t-1},\ c_{t-2},\ \ldots,\ (-1)^{t-1}c_1) \tag{28}$$

in the revolving-door Gray code Γ_{st}. This property follows by induction, because (25) becomes

$$\Gamma_{st} \;=\; \Gamma_{(s-1)t},\ (s{+}t{-}1)\Gamma^R_{s(t-1)} \tag{29}$$

for $st > 0$ when we use index-list notation instead of bitstring notation. Consequently the sequence can be generated efficiently by the following algorithm due to W. H. Payne [see *ACM Trans. Math. Software* **5** (1979), 163–172]:

Algorithm R (*Revolving-door combinations*). This algorithm generates all t-combinations $c_t \ldots c_2c_1$ of $\{0, 1, \ldots, n-1\}$ in lexicographic order of the alternating sequence (28), assuming that $n > t > 1$. Step R3 has two variants, depending on whether t is even or odd.

R1. [Initialize.] Set $c_j \leftarrow j-1$ for $t \ge j \ge 1$, and $c_{t+1} \leftarrow n$.

R2. [Visit.] Visit the combination $c_t \ldots c_2c_1$.

R3. [Easy case?] If t is odd: If $c_1 + 1 < c_2$, increase c_1 by 1 and return to R2, otherwise set $j \leftarrow 2$ and go to R4. If t is even: If $c_1 > 0$, decrease c_1 by 1 and return to R2, otherwise set $j \leftarrow 2$ and go to R5.

R4. [Try to decrease c_j.] (At this point $c_j = c_{j-1} + 1$.) If $c_j \ge j$, set $c_j \leftarrow c_{j-1}$, $c_{j-1} \leftarrow j-2$, and return to R2. Otherwise increase j by 1.

R5. [Try to increase c_j.] (At this point $c_{j-1} = j-2$.) If $c_j + 1 < c_{j+1}$, set $c_{j-1} \leftarrow c_j$, $c_j \leftarrow c_j + 1$, and return to R2. Otherwise increase j by 1, and go to R4 if $j \le t$. ▌

Exercises 21–25 explore further properties of this interesting sequence. One of them is a nice companion to Theorem L: *The combination* $c_tc_{t-1}\ldots c_2c_1$ *is visited by Algorithm R after exactly*

$$N = \binom{c_t{+}1}{t} - \binom{c_{t-1}{+}1}{t-1} + \cdots + (-1)^t\binom{c_2{+}1}{2} - (-1)^t\binom{c_1{+}1}{1} - [t \text{ odd}] \tag{30}$$

other combinations have been visited. We may call this the representation of N in the "alternating combinatorial number system" of degree t; one consequence, for example, is that every positive integer has a unique representation of the form $N = \binom{a}{3} - \binom{b}{2} + \binom{c}{1}$ with $a > b > c > 0$. Algorithm R tells us how to add 1 to N in this system.

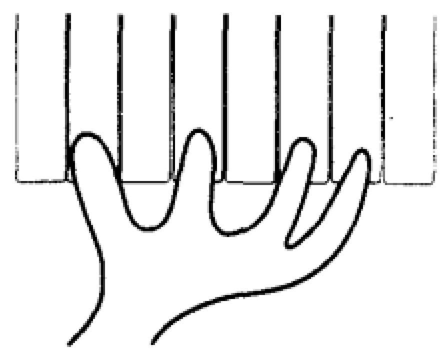
Although the strings of (26) and (27) are not in lexicographic order, they are examples of a more general concept called *genlex order*, a name coined by Timothy Walsh. A sequence of strings $\alpha_1, \ldots, \alpha_N$ is said to be in genlex order when all strings with a common prefix occur consecutively. For example, all 3-combinations that begin with 53 appear together in (27).

Genlex order means that the strings can be arranged in a trie structure, as in Fig. 31 of Section 6.3, but with the children of each node ordered arbitrarily. When a trie is traversed in any order such that each node is visited just before or just after its descendants, all nodes with a common prefix — that is, all nodes of a subtrie — appear consecutively. This principle makes genlex order convenient, because it corresponds to recursive generation schemes. Many of the algorithms we have seen for generating n-tuples have therefore produced their results in some version of genlex order; similarly, the method of "plain changes" (Algorithm 7.2.1.2P) visits permutations in a genlex order of the corresponding inversion tables.

The revolving-door method of Algorithm R is a genlex routine that changes only one element of the combination at each step. But it isn't totally satisfactory, because it frequently must change two of the indices c_j simultaneously, in order to preserve the condition $c_t > \cdots > c_2 > c_1$. For example, Algorithm R changes 210 into 320, and (27) includes nine such "crossing" moves.

The source of this defect can be traced to our proof that (25) satisfies the revolving-door property: We observed that the string $010^{s-1}11^{t-2}$ is followed by $110^{s-1}01^{t-2}$ when $t \geq 2$. Hence the recursive construction Γ_{st} involves transitions of the form $110^a0 \leftrightarrow 010^a1$, when a substring like 11000 is changed to 01001 or vice versa; the two 1s cross each other.

A Gray path for combinations is said to be *homogeneous* if it changes only one of the indices c_j at each step. A homogeneous scheme is characterized in bitstring form by having only transitions of the forms $10^a \leftrightarrow 0^a1$ within strings, for $a \geq 1$, when we pass from one string to the next. With a homogeneous scheme we can, for example, play all t-note chords on an n-note keyboard by moving only one finger at a time.



A slight modification of (25) yields a genlex scheme for (s,t)-combinations that is pleasantly homogeneous. The basic idea is to construct a sequence that begins with 0^s1^t and ends with 1^t0^s, and the following recursion suggests itself almost immediately: Let $K_{s0} = 0^s$, $K_{0t} = 1^t$, $K_{s(-1)} = \emptyset$, and

$$K_{st} = 0K_{(s-1)t},\ 10K^R_{(s-1)(t-1)},\ 11K_{s(t-2)} \quad \text{for } st > 0. \qquad (31)$$

At the commas of this sequence we have 01^t0^{s-1} followed by $101^{t-1}0^{s-1}$, and 10^s1^{t-1} followed by 110^s1^{t-2}; both of these transitions are homogeneous, although the second one requires the 1 to jump across s 0s. The combinations K_{33} for $s = t = 3$ are

000111	010101	101100	100011	
001011	010011	101001	110001	
001101	011001	101010	110010	(32)
001110	011010	100110	110100	
010110	011100	100101	111000	

in bitstring form, and the corresponding "finger patterns" are

210	420	532	510	
310	410	530	540	
320	430	531	541	(33)
321	431	521	542	
421	432	520	543.	

When a homogeneous scheme for ordinary combinations $c_t \ldots c_1$ is converted to the corresponding scheme (6) for combinations with repetitions $d_t \ldots d_1$, it retains the property that only one of the indices d_j changes at each step. And when it is converted to the corresponding schemes (9) or (11) for compositions $p_t \ldots p_0$ or $q_t \ldots q_0$, only two (adjacent) parts change when c_j changes.

Near-perfect schemes. But we can do even better! All (s,t)-combinations can be generated by a sequence of strongly homogeneous transitions that are either $01 \leftrightarrow 10$ or $001 \leftrightarrow 100$. In other words, we can insist that each step causes a single index c_j to change by at most 2. Let's call such generation schemes *near-perfect.*

Imposing such strong conditions actually makes it fairly easy to discover near-perfect schemes, because comparatively few choices are available. Indeed, if we restrict ourselves to genlex methods that are near-perfect on n-bit strings, T. A. Jenkyns and D. McCarthy observed that all such methods can be easily characterized [*Ars Combinatoria* **40** (1995), 153–159]:

Theorem N. *If $st > 0$, there are exactly $2s$ near-perfect ways to list all (s,t)-combinations in a genlex order. In fact, when $1 \le a \le s$, there is exactly one such listing, N_{sta}, that begins with 1^t0^s and ends with $0^a1^t0^{s-a}$; the other s possibilities are the reverse lists, N^R_{sta}.*

Proof. The result certainly holds when $s = t = 1$; otherwise we use induction on $s+t$. The listing N_{sta}, if it exists, must have the form $1X_{s(t-1)}$, $0Y_{(s-1)t}$ for some near-perfect genlex listings $X_{s(t-1)}$ and $Y_{(s-1)t}$. If $t = 1$, $X_{s(t-1)}$ is the single string 0^s; hence $Y_{(s-1)t}$ must be $N_{(s-1)1(a-1)}$ if $a > 1$, and it must be $N^R_{(s-1)11}$ if $a = 1$. On the other hand if $t > 1$, the near-perfect condition implies that the last string of $X_{s(t-1)}$ cannot begin with 1; hence $X_{s(t-1)} = N_{s(t-1)b}$ for some b. If $a > 1$, $Y_{(s-1)t}$ must be $N_{(s-1)t(a-1)}$, hence b must be 1; similarly, b must be 1 if $s = 1$. Otherwise we have $a = 1 < s$, and this forces $Y_{(s-1)t} = N^R_{(s-1)tc}$ for some c. The transition from $10^b1^{t-1}0^{s-b}$ to $0^{c+1}1^t0^{s-1-c}$ is near-perfect only if $c = 1$ and $b = 2$. ▌

The proof of Theorem N yields the following recursive formulas when $st > 0$:

$$N_{sta} = \begin{cases} 1N_{s(t-1)1},\ 0N_{(s-1)t(a-1)}, & \text{if } 1 < a \le s; \\ 1N_{s(t-1)2},\ 0N^R_{(s-1)t1}, & \text{if } 1 = a < s; \\ 1N_{1(t-1)1},\ 01^t, & \text{if } 1 = a = s. \end{cases} \qquad (34)$$

Also, of course, $N_{s0a} = 0^s$.

Let us set $A_{st} = N_{st1}$ and $B_{st} = N_{st2}$. These near-perfect listings, discovered by Phillip J. Chase in 1976, have the net effect of shifting a leftmost block of 1s to the right by one or two positions, respectively, and they satisfy the following mutual recursions:

$$A_{st} = 1B_{s(t-1)},\ 0A^R_{(s-1)t}; \qquad B_{st} = 1A_{s(t-1)},\ 0A_{(s-1)t}. \qquad (35)$$

"To take one step forward, take two steps forward, then one step backward; to take two steps forward, take one step forward, then another." These equations

Table 2

CHASE'S SEQUENCES FOR (3,3)-COMBINATIONS

$A_{33} = \widehat{C}^R_{33}$				$B_{33} = C_{33}$			
543	531	321	420	543	520	432	410
541	530	320	421	542	510	430	210
540	510	310	431	540	530	431	310
542	520	210	430	541	531	421	320
532	521	410	432	521	532	420	321

hold for all integer values of s and t, if we define A_{st} and B_{st} to be $\emptyset$ when s or t is negative, except that $A_{00} = B_{00} = \epsilon$ (the empty string). Thus A_{st} actually takes $\min(s,1)$ forward steps, and B_{st} actually takes $\min(s,2)$. For example, Table 2 shows the relevant listings for $s = t = 3$, using an equivalent index-list form $c_3c_2c_1$ instead of the bit strings $a_5a_4a_3a_2a_1a_0$.

Chase noticed that a computer implementation of these sequences becomes simpler if we define

$$C_{st} = \begin{cases} A_{st}, & \text{if } s+t \text{ is odd;} \\ B_{st}, & \text{if } s+t \text{ is even;} \end{cases} \qquad \widehat{C}_{st} = \begin{cases} A^R_{st}, & \text{if } s+t \text{ is even;} \\ B^R_{st}, & \text{if } s+t \text{ is odd.} \end{cases} \tag{36}$$

[See *Congressus Numerantium* **69** (1989), 215–242.] Then we have

$$C_{st} = \begin{cases} 1C_{s(t-1)},\ 0\widehat{C}_{(s-1)t}, & \text{if } s+t \text{ is odd;} \\ 1C_{s(t-1)},\ 0C_{(s-1)t}, & \text{if } s+t \text{ is even;} \end{cases} \tag{37}$$

$$\widehat{C}_{st} = \begin{cases} 0C_{(s-1)t},\ 1\widehat{C}_{s(t-1)}, & \text{if } s+t \text{ is even;} \\ 0\widehat{C}_{(s-1)t},\ 1\widehat{C}_{s(t-1)}, & \text{if } s+t \text{ is odd.} \end{cases} \tag{38}$$

When bit a_j is ready to change, we can tell where we are in the recursion by testing whether j is even or odd.

Indeed, the sequence C_{st} can be generated by a surprisingly simple algorithm, based on general ideas that apply to *any* genlex scheme. Let us say that bit a_j is *active* in a genlex algorithm if it is supposed to change before anything to its left is altered. (In other words, the node for an active bit in the corresponding trie is not the rightmost child of its parent.) Suppose we have an auxiliary table $w_n \ldots w_1w_0$, where $w_j = 1$ if and only if either a_j is active or $j < r$, where r is the least subscript such that $a_r \neq a_0$; we also let $w_n = 1$. Then the following method will find the successor of $a_{n-1} \ldots a_1a_0$:

> Set $j \leftarrow r$. If $w_j = 0$, set $w_j \leftarrow 1$, $j \leftarrow j+1$, and repeat until $w_j = 1$. Terminate if $j = n$; otherwise set $w_j \leftarrow 0$. Change a_j to $1 - a_j$, and make any other changes to $a_{j-1} \ldots a_0$ and r that apply to the particular genlex scheme being used. (39)

The beauty of this approach comes from the fact that the loop is guaranteed to be efficient: We can prove that the operation $j \leftarrow j+1$ will be performed less than once per generation step, on the average (see exercise 36).

By analyzing the transitions that occur when bits change in (37) and (38), we can readily flesh out the remaining details:

Algorithm C (*Chase's sequence*). This algorithm visits all (s,t)-combinations $a_{n-1}\dots a_1a_0$, where $n=s+t$, in the near-perfect order of Chase's sequence C_{st}.

C1. [Initialize.] Set $a_j \leftarrow 0$ for $0 \le j < s$, $a_j \leftarrow 1$ for $s \le j < n$, and $w_j \leftarrow 1$ for $0 \le j \le n$. If $s > 0$, set $r \leftarrow s$; otherwise set $r \leftarrow t$.

C2. [Visit.] Visit the combination $a_{n-1}\dots a_1a_0$.

C3. [Find j and branch.] Set $j \leftarrow r$. If $w_j = 0$, set $w_j \leftarrow 1$, $j \leftarrow j+1$, and repeat until $w_j = 1$. Terminate if $j = n$; otherwise set $w_j \leftarrow 0$ and make a four-way branch: Go to C4 if j is odd and $a_j \ne 0$, to C5 if j is even and $a_j \ne 0$, to C6 if j is even and $a_j = 0$, to C7 if j is odd and $a_j = 0$.

C4. [Move right one.] Set $a_{j-1} \leftarrow 1$, $a_j \leftarrow 0$. If $r = j > 1$, set $r \leftarrow j-1$; otherwise if $r = j-1$ set $r \leftarrow j$. Return to C2.

C5. [Move right two.] If $a_{j-2} \ne 0$, go to C4. Otherwise set $a_{j-2} \leftarrow 1$, $a_j \leftarrow 0$. If $r = j$, set $r \leftarrow \max(j-2, 1)$; otherwise if $r = j-2$, set $r \leftarrow j-1$. Return to C2.

C6. [Move left one.] Set $a_j \leftarrow 1$, $a_{j-1} \leftarrow 0$. If $r = j > 1$, set $r \leftarrow j-1$; otherwise if $r = j-1$ set $r \leftarrow j$. Return to C2.

C7. [Move left two.] If $a_{j-1} \ne 0$, go to C6. Otherwise set $a_j \leftarrow 1$, $a_{j-2} \leftarrow 0$. If $r = j-2$, set $r \leftarrow j$; otherwise if $r = j-1$, set $r \leftarrow j-2$. Return to C2. ▮

***Analysis of Chase's sequence.** The magical properties of Algorithm C cry out for further exploration, and a closer look turns out to be quite instructive. Given a bit string $a_{n-1}\dots a_1a_0$, let us define $a_n = 1$, $u_n = n \bmod 2$, and

$$u_j = (1-u_{j+1})a_{j+1}, \quad v_j = (u_j + j) \bmod 2, \quad w_j = (v_j + a_j) \bmod 2, \qquad (40)$$

for $n > j \ge 0$. For example, we might have $n = 26$ and

$$\begin{aligned} a_{25}\dots a_1a_0 &= 11001001000011111101101010,\\ u_{25}\dots u_1u_0 &= 10100100100001010100100101,\\ v_{25}\dots v_1v_0 &= 00001110001011111110001111,\\ w_{25}\dots w_1w_0 &= 11000111001000000011100101. \end{aligned} \qquad (41)$$

With these definitions we can prove by induction that $v_j = 0$ if and only if bit a_j is being "controlled" by C rather than by $\widehat{C}$ in the recursions (37)–(38) that generate $a_{n-1}\dots a_1a_0$, except when a_j is part of the final run of 0s or 1s at the right end. Therefore w_j agrees with the value computed by Algorithm C at the moment when $a_{n-1}\dots a_1a_0$ is visited, for $r \le j < n$. These formulas can be used to determine exactly where a given combination appears in Chase's sequence (see exercise 39).

If we want to work with the index-list form $c_t\dots c_2c_1$ instead of the bit strings $a_{n-1}\dots a_1a_0$, it is convenient to change the notation slightly, writing

$C_t(n)$ for C_{st} and $\widehat{C}_t(n)$ for $\widehat{C}_{st}$ when $s+t=n$. Then $C_0(n)=\widehat{C}_0(n)=\epsilon$, and the recursions for $t\ge 0$ take the form

$$C_{t+1}(n+1)=\begin{cases} nC_t(n),\ \widehat{C}_{t+1}(n), & \text{if } n \text{ is even;}\\ nC_t(n),\ C_{t+1}(n), & \text{if } n \text{ is odd;}\end{cases} \tag{42}$$

$$\widehat{C}_{t+1}(n+1)=\begin{cases} C_{t+1}(n),\ n\widehat{C}_t(n), & \text{if } n \text{ is odd;}\\ \widehat{C}_{t+1}(n),\ n\widehat{C}_t(n), & \text{if } n \text{ is even.}\end{cases} \tag{43}$$

These new equations can be expanded to tell us, for example, that

$$\begin{aligned} C_{t+1}(9) &= 8C_t(8),\ 6C_t(6),\ 4C_t(4),\ \ldots,\ 3\widehat{C}_t(3),\ 5\widehat{C}_t(5),\ 7\widehat{C}_t(7);\\ C_{t+1}(8) &= 7C_t(7),\ 6C_t(6),\ 4C_t(4),\ \ldots,\ 3\widehat{C}_t(3),\ 5\widehat{C}_t(5);\\ \widehat{C}_{t+1}(9) &= \qquad\quad 6C_t(6),\ 4C_t(4),\ \ldots,\ 3\widehat{C}_t(3),\ 5\widehat{C}_t(5),\ 7\widehat{C}_t(7),\ 8\widehat{C}_t(8);\\ \widehat{C}_{t+1}(8) &= \qquad\quad 6C_t(6),\ 4C_t(4),\ \ldots,\ 3\widehat{C}_t(3),\ 5\widehat{C}_t(5),\ 7\widehat{C}_t(7); \end{aligned} \tag{44}$$

notice that the same pattern predominates in all four sequences. The meaning of "..." in the middle depends on the value of t: We simply omit all terms $nC_t(n)$ and $n\widehat{C}_t(n)$ where $n<t$.

Except for edge effects at the very beginning or end, all of the expansions in (44) are based on the infinite progression

$$\ldots,\ 10,\ 8,\ 6,\ 4,\ 2,\ 0,\ 1,\ 3,\ 5,\ 7,\ 9,\ \ldots, \tag{45}$$

which is a natural way to arrange the nonnegative integers into a doubly infinite sequence. If we omit all terms of (45) that are $<t$, given any integer $t\ge 0$, the remaining terms retain the property that adjacent elements differ by either 1 or 2. Richard Stanley has suggested the name *endo-order* for this sequence, because we can remember it by thinking "even numbers decreasing, odd ...". (Notice that if we retain only the terms less than N and complement with respect to N, endo-order becomes organ-pipe order; see exercise 6.1–18.)

We could program the recursions of (42) and (43) directly, but it is interesting to unwind them using (44), thus obtaining an iterative algorithm analogous to Algorithm C. The result needs only $O(t)$ memory locations, and it is especially efficient when t is relatively small compared to n. Exercise 45 contains the details.

***Near-perfect multiset permutations.** Chase's sequences lead in a natural way to an algorithm that will generate permutations of any desired multiset $\{s_0\cdot 0, s_1\cdot 1,\ldots,s_d\cdot d\}$ in a near-perfect manner, meaning that

i) every transition is either $a_{j+1}a_j \leftrightarrow a_ja_{j+1}$ or $a_{j+1}a_ja_{j-1} \leftrightarrow a_{j-1}a_ja_{j+1}$;

ii) transitions of the second kind have $a_j=\min(a_{j-1},a_{j+1})$.

Algorithm C tells us how to do this when $d=1$, and we can extend it to larger values of d by the following recursive construction [*CACM* **13** (1970), 368–369, 376]: Suppose

$$\alpha_0,\ \alpha_1,\ \ldots,\ \alpha_{N-1}$$

is any near-perfect listing of the permutations of $\{s_1 \cdot 1, \ldots, s_d \cdot d\}$. Then Algorithm C, with $s = s_0$ and $t = s_1 + \cdots + s_d$, tells us how to generate a listing

$$\Lambda_j \;=\; \alpha_j 0^s,\ \ldots,\ 0^a \alpha_j 0^{s-a} \tag{46}$$

in which all transitions are $0x \leftrightarrow x0$ or $00x \leftrightarrow x00$; the final entry has $a = 1$ or 2 leading zeros, depending on s and t. Therefore all transitions of the sequence

$$\Lambda_0,\ \Lambda_1^R,\ \Lambda_2,\ \ldots,\ (\Lambda_{N-1} \text{ or } \Lambda_{N-1}^R) \tag{47}$$

are near-perfect; and this list clearly contains all the permutations.

For example, the permutations of $\{0,0,0,1,1,2\}$ generated in this way are

211000, 210100, 210001, 210010, 200110, 200101, 200011, 201001, 201010, 201100,
021100, 021001, 021010, 020110, 020101, 020011, 000211, 002011, 002101, 002110,
001120, 001102, 001012, 000112, 010012, 010102, 010120, 011020, 011002, 011200,
101200, 101020, 101002, 100012, 100102, 100120, 110020, 110002, 110200, 112000,
121000, 120100, 120001, 120010, 100210, 100201, 100021, 102001, 102010, 102100,
012100, 012001, 012010, 010210, 010201, 010021, 000121, 001021, 001201, 001210.

***Perfect schemes.** Why should we settle for a near-perfect generator like C_{st}, instead of insisting that all transitions have the simplest possible form $01 \leftrightarrow 10$?

One reason is that perfect schemes don't always exist. For example, we observed in 7.2.1.2–(2) that there is no way to generate all six permutations of $\{1,1,2,2\}$ with adjacent interchanges; thus there is no perfect scheme for $(2,2)$-combinations. In fact, our chances of achieving perfection are only about 1 in 4:

Theorem P. *The generation of all (s,t)-combinations $a_{s+t-1} \ldots a_1 a_0$ by adjacent interchanges $01 \leftrightarrow 10$ is possible if and only if $s \le 1$ or $t \le 1$ or st is odd.*

Proof. Consider all permutations of the multiset $\{s \cdot 0, t \cdot 1\}$. We learned in exercise 5.1.2–16 that the number m_k of such permutations having k inversions is the coefficient of z^k in the z-nomial coefficient

$$\binom{s+t}{t}_z \;=\; \prod_{k=s+1}^{s+t} (1 + z + \cdots + z^{k-1}) \Big/ \prod_{k=1}^{t} (1 + z + \cdots + z^{k-1}). \tag{48}$$

Every adjacent interchange changes the number of inversions by ± 1, so a perfect generation scheme is possible only if approximately half of all the permutations have an odd number of inversions. More precisely, the value of $\binom{s+t}{t}_{-1} = m_0 - m_1 + m_2 - \cdots$ must be 0 or ± 1. But exercise 49 shows that

$$\binom{s+t}{t}_{-1} \;=\; \binom{\lfloor (s+t)/2 \rfloor}{\lfloor t/2 \rfloor} [st \text{ is even}], \tag{49}$$

and this quantity exceeds 1 unless $s \le 1$ or $t \le 1$ or st is odd.

Conversely, perfect schemes are easy with $s \le 1$ or $t \le 1$, and they turn out to be possible also whenever st is odd. The first nontrivial case occurs for $s = t = 3$, when there are four essentially different solutions; the most symmetrical of these is

$$\begin{aligned}&210 - 310 - 410 - 510 - 520 - 521 - 531 - 532 - 432 - 431 -\\ &\quad 421 - 321 - 320 - 420 - 430 - 530 - 540 - 541 - 542 - 543\end{aligned} \tag{50}$$

(see exercise 51). Several authors have constructed Hamiltonian paths in the relevant graph for arbitrary odd numbers s and t; for example, the method of Eades, Hickey, and Read [*JACM* **31** (1984), 19–29] makes an interesting exercise in programming with recursive coroutines. Unfortunately, however, none of the known constructions are sufficiently simple to describe in a short space, or to implement with reasonable efficiency. Perfect combination generators have therefore not yet proved to be of practical importance. ▌

In summary, then, we have seen that the study of (s,t)-combinations leads to many fascinating patterns, some of which are of great practical importance and some of which are merely elegant and/or beautiful. Figure 26 illustrates the principal options that are available in the case $s=t=5$, when $\binom{10}{5}=252$ combinations arise. Lexicographic order (Algorithm L), the revolving-door Gray code (Algorithm R), the homogeneous scheme K_{55} of (31), and Chase's near-perfect scheme (Algorithm C) are shown in parts (a), (b), (c), and (d) of the illustration. Part (e) shows the near-perfect scheme that is as close to perfection as possible while still being in genlex order of the c array (see exercise 34), while part (f) is the perfect scheme of Eades, Hickey, and Read. Finally, Figs. 26(g) and 26(h) are listings that proceed by rotating $a_j a_{j-1} \ldots a_0 \leftarrow a_{j-1} \ldots a_0 a_j$ or by swapping $a_j \leftrightarrow a_0$, akin to Algorithms 7.2.1.2C and 7.2.1.2E (see exercises 55 and 56).

***Combinations of a multiset.** If multisets can have permutations, they can have combinations too. For example, consider the multiset $\{b,b,b,b,g,g,g,r,r,r,w,w\}$, representing a sack that contains four blue balls and three that are green, three red, two white. There are 37 ways to choose five balls from this sack; in lexicographic order (but descending in each combination) they are

$$\begin{aligned}&gbbbb,\ ggbbb,\ gggbb,\ rbbbb,\ rgbbb,\ rggbb,\ rgggb,\ rrbbb,\ rrgbb,\ rrggb,\\ &rrggg,\ rrrbb,\ rrrgb,\ rrrgg,\ wbbbb,\ wgbbb,\ wggbb,\ wgggb,\ wrbbb,\ wrgbb,\\ &wrggb,\ wrggg,\ wrrbb,\ wrrgb,\ wrrgg,\ wrrrb,\ wrrrg,\ wwbbb,\ wwgbb,\ wwggb,\\ &wwggg,\ wwrbb,\ wwrgb,\ wwrgg,\ wwrrb,\ wwrrg,\ wwrrr.\end{aligned} \qquad (51)$$

This fact might seem frivolous and/or esoteric, yet we will see in Theorem W below that the lexicographic generation of multiset combinations yields optimal solutions to significant combinatorial problems.

James Bernoulli observed in his *Ars Conjectandi* (1713), 119–123, that we can enumerate such combinations by looking at the coefficient of z^5 in the product $(1+z+z^2)(1+z+z^2+z^3)^2(1+z+z^2+z^3+z^4)$. Indeed, his observation is easy to understand, because we get all possible selections from the sack if we multiply out the polynomials

$$(1+w+ww)(1+r+rr+rrr)(1+g+gg+ggg)(1+b+bb+bbb+bbbb).$$

Multiset combinations are also equivalent to *bounded compositions*, namely to compositions in which the individual parts are bounded. For example, the 37 multicombinations listed in (51) correspond to 37 solutions of

$$5=r_3+r_2+r_1+r_0,\quad 0\le r_3\le 2,\quad 0\le r_2,r_1\le 3,\quad 0\le r_0\le 4,$$

namely $5=0+0+1+4=0+0+2+3=0+0+3+2=0+1+0+4=\cdots=2+3+0+0$.

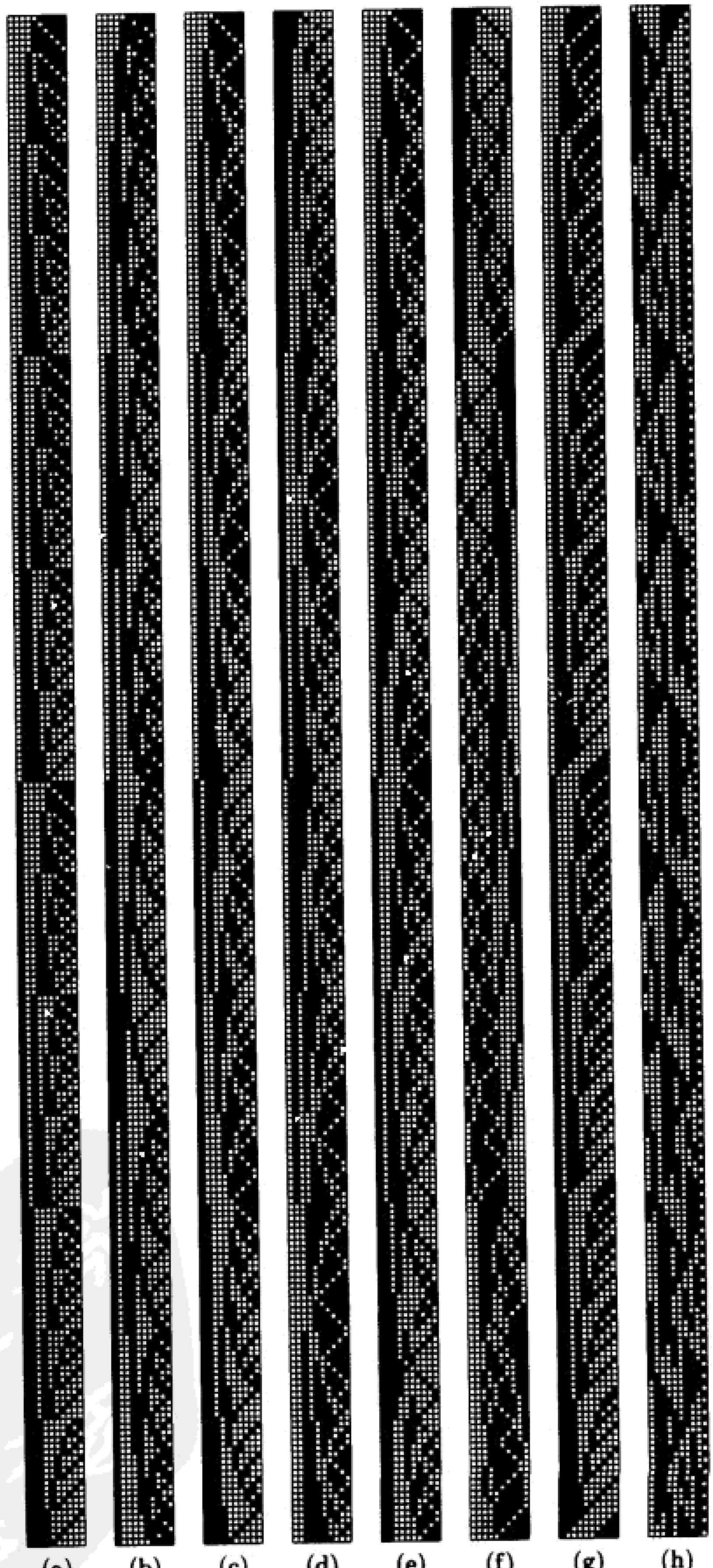


Fig. 26. Examples of (5, 5)-combinations:

a) lexicographic;
b) revolving-door;
c) homogeneous;
d) near-perfect;
e) nearer-perfect;
f) perfect;
g) suffix-rotated;
h) right-swapped.

Bounded compositions, in turn, are special cases of *contingency tables*, which are of great importance in statistics. And all of these combinatorial configurations can be generated with Gray-like codes as well as in lexicographic order. Exercises 60–63 explore some of the basic ideas involved.

***Shadows.** Sets of combinations appear frequently in mathematics. For example, a set of 2-combinations (namely a set of pairs) is essentially a graph, and a set of t-combinations for general t is called a uniform hypergraph. If the vertices of a convex polyhedron are perturbed slightly, so that no three are collinear, no four lie in a plane, and in general no $t+1$ lie in a $(t-1)$-dimensional hyperplane, the resulting $(t-1)$-dimensional faces are "simplexes" whose vertices have great significance in computer applications. Researchers have learned that such sets of combinations have important properties related to lexicographic generation.

If α is any t-combination $c_t \ldots c_2 c_1$, its *shadow* $\partial\alpha$ is the set of all its $(t-1)$-element subsets $c_{t-1} \ldots c_2 c_1$, ..., $c_t \ldots c_3 c_1$, $c_t \ldots c_3 c_2$. For example, $\partial 5310 = \{310, 510, 530, 531\}$. We can also represent a t-combination as a bit string $a_{n-1} \ldots a_1 a_0$, in which case $\partial\alpha$ is the set of all strings obtained by changing a 1 to a 0: $\partial 101011 = \{001011, 100011, 101001, 101010\}$. If A is any set of t-combinations, we define its shadow

$$\partial A = \bigcup \{ \partial\alpha \mid \alpha \in A \} \qquad (52)$$

to be the set of all $(t-1)$-combinations in the shadows of its members. For example, $\partial\partial 5310 = \{10, 30, 31, 50, 51, 53\}$.

These definitions apply also to combinations with repetitions, namely to multicombinations: $\partial 5330 = \{330, 530, 533\}$ and $\partial\partial 5330 = \{30, 33, 50, 53\}$. In general, when A is a set of t-element multisets, ∂A is a set of $(t-1)$-element multisets. Notice, however, that ∂A never has repeated elements itself.

The *upper shadow* $\varrho\alpha$ with respect to a universe U is defined similarly, but it goes from t-combinations to $(t+1)$-combinations:

$$\varrho\alpha = \{ \beta \subseteq U \mid \alpha \in \partial\beta \}, \qquad \text{for } \alpha \in U; \qquad (53)$$

$$\varrho A = \bigcup \{ \varrho\alpha \mid \alpha \in A \}, \qquad \text{for } A \subseteq U. \qquad (54)$$

If, for example, $U = \{0, 1, 2, 3, 4, 5, 6\}$, we have $\varrho 5310 = \{53210, 54310, 65310\}$; on the other hand, if $U = \{\infty\cdot 0, \infty\cdot 1, \ldots, \infty\cdot 6\}$, we have $\varrho 5310 = \{53100, 53110, 53210, 53310, 54310, 55310, 65310\}$.

The following fundamental theorems, which have many applications in various branches of mathematics and computer science, tell us how small a set's shadows can be:

Theorem K. *If A is a set of N t-combinations contained in $U = \{0, 1, \ldots, n-1\}$, then*

$$|\partial A| \ge |\partial P_{Nt}| \qquad \text{and} \qquad |\varrho A| \ge |\varrho Q_{Nnt}|, \qquad (55)$$

where P_{Nt} denotes the first N combinations generated by Algorithm L, namely the N lexicographically smallest combinations $c_t \ldots c_2 c_1$ that satisfy (3), and Q_{Nnt} denotes the N lexicographically largest. ▌

Theorem M. *If A is a set of N t-multicombinations contained in the multiset $U = \{\infty\cdot 0, \infty\cdot 1, \ldots, \infty\cdot s\}$, then*

$$|\partial A| \ge |\partial \widehat{P}_{Nt}| \qquad \text{and} \qquad |\varrho A| \ge |\varrho \widehat{Q}_{Nst}|, \tag{56}$$

where $\widehat{P}_{Nt}$ denotes the N lexicographically smallest multicombinations $d_t \ldots d_2 d_1$ that satisfy (6), *and $\widehat{Q}_{Nst}$ denotes the N lexicographically largest.* ∎

Both of these theorems are consequences of a stronger result that we shall prove later. Theorem K is generally called the Kruskal–Katona theorem, because it was discovered by J. B. Kruskal [*Math. Optimization Techniques*, edited by R. Bellman (1963), 251–278] and rediscovered by G. Katona [*Theory of Graphs*, Tihany 1966, edited by Erdős and Katona (Academic Press, 1968), 187–207]; M. P. Schützenberger had previously stated it in a less-well-known publication, with incomplete proof [*RLE Quarterly Progress Report* **55** (1959), 117–118]. Theorem M goes back to F. S. Macaulay, many years earlier [*Proc. London Math. Soc.* (2) **26** (1927), 531–555].

Before proving (55) and (56), let's take a closer look at what those formulas mean. We know from Theorem L that the first N of all t-combinations visited by Algorithm L are those that precede $n_t \ldots n_2 n_1$, where

$$N = \binom{n_t}{t} + \cdots + \binom{n_2}{2} + \binom{n_1}{1}, \qquad n_t > \cdots > n_2 > n_1 \ge 0$$

is the degree-t combinatorial representation of N. Sometimes this representation has fewer than t nonzero terms, because n_j can be equal to $j-1$; let's suppress the zeros, and write

$$N = \binom{n_t}{t} + \binom{n_{t-1}}{t-1} + \cdots + \binom{n_v}{v}, \qquad n_t > n_{t-1} > \cdots > n_v \ge v \ge 1. \tag{57}$$

Now the first $\binom{n_t}{t}$ combinations $c_t \ldots c_1$ are the t-combinations of $\{0, \ldots, n_t - 1\}$; the next $\binom{n_{t-1}}{t-1}$ are those in which $c_t = n_t$ and $c_{t-1} \ldots c_1$ is a $(t-1)$-combination of $\{0, \ldots, n_{t-1} - 1\}$; and so on. For example, if $t = 5$ and $N = \binom{9}{5} + \binom{7}{4} + \binom{4}{3}$, the first N combinations are

$$P_{N5} = \{43210, \ldots, 87654\} \cup \{93210, \ldots, 96543\} \cup \{97210, \ldots, 97321\}. \tag{58}$$

The shadow of this set P_{N5} is, fortunately, easy to understand: It is

$$\partial P_{N5} = \{3210, \ldots, 8765\} \cup \{9210, \ldots, 9654\} \cup \{9710, \ldots, 9732\}, \tag{59}$$

namely the first $\binom{9}{4} + \binom{7}{3} + \binom{4}{2}$ combinations in lexicographic order when $t = 4$.

In other words, if we define Kruskal's function κ_t by the formula

$$\kappa_t N = \binom{n_t}{t-1} + \binom{n_{t-1}}{t-2} + \cdots + \binom{n_v}{v-1} \tag{60}$$

when N has the unique representation (57), we have

$$\partial P_{Nt} = P_{(\kappa_t N)(t-1)}. \tag{61}$$

Theorem K tells us, for example, that a graph with a million edges can contain at most

$$\binom{1414}{3} + \binom{1009}{2} = 470{,}700{,}300$$

triangles, that is, at most 470,700,300 sets of vertices $\{u, v, w\}$ with $u — v — w — u$. The reason is that $1000000 = \binom{1414}{2} + \binom{1009}{1}$ by exercise 17, and the edges $P_{(1000000)2}$ do support $\binom{1414}{3} + \binom{1009}{2}$ triangles; but if there were more, the graph would necessarily have at least $\kappa_3 470700301 = \binom{1414}{2} + \binom{1009}{1} + \binom{1}{0} = 1000001$ edges in their shadow.

Kruskal defined the companion function

$$\lambda_t N = \binom{n_t}{t+1} + \binom{n_{t-1}}{t} + \cdots + \binom{n_v}{v+1} \tag{62}$$

to deal with questions such as this. The κ and λ functions are related by an interesting law proved in exercise 72:

$$M + N = \binom{s+t}{t} \quad \text{implies} \quad \kappa_s M + \lambda_t N = \binom{s+t}{t+1}, \quad \text{if } st > 0. \tag{63}$$

Turning to Theorem M, the sizes of $\partial\widehat{P}_{Nt}$ and $\varrho\widehat{Q}_{Nst}$ turn out to be

$$|\partial\widehat{P}_{Nt}| = \mu_t N \qquad \text{and} \qquad |\varrho\widehat{Q}_{Nst}| = N + \kappa_s N \tag{64}$$

(see exercise 81), where the function μ_t satisfies

$$\mu_t N = \binom{n_t - 1}{t-1} + \binom{n_{t-1} - 1}{t-2} + \cdots + \binom{n_v - 1}{v-1} \tag{65}$$

when N has the combinatorial representation (57).

Table 3 shows how these functions $\kappa_t N$, $\lambda_t N$, and $\mu_t N$ behave for small values of t and N. When t and N are large, they can be well approximated in terms of a remarkable function $\tau(x)$ introduced by Teiji Takagi in 1903; see Fig. 27 and exercises 82–85.

Theorems K and M are corollaries of a much more general theorem of discrete geometry, discovered by Da-Lun Wang and Ping Wang [*SIAM J. Applied Math.* **33** (1977), 55–59], which we shall now proceed to investigate. Consider the *discrete n-dimensional torus* $T(m_1, \ldots, m_n)$ whose elements are integer vectors $x = (x_1, \ldots, x_n)$ with $0 \le x_1 < m_1$, …, $0 \le x_n < m_n$. We define the sum and difference of two such vectors x and y as in Eqs. 4.3.2–(2) and 4.3.2–(3):

$$x + y = \big((x_1 + y_1) \bmod m_1, \ldots, (x_n + y_n) \bmod m_n\big), \tag{66}$$
$$x - y = \big((x_1 - y_1) \bmod m_1, \ldots, (x_n - y_n) \bmod m_n\big). \tag{67}$$

We also define the so-called *cross order* on such vectors by saying that $x \preceq y$ if and only if

$$\nu x < \nu y \quad \text{or} \quad (\nu x = \nu y \text{ and } x \ge y \text{ lexicographically}); \tag{68}$$

here, as usual, $\nu(x_1, \ldots, x_n) = x_1 + \cdots + x_n$. For example, when $m_1 = m_2 = 2$ and $m_3 = 3$, the 12 vectors $x_1x_2x_3$ in increasing cross order are

$$000,\ 100,\ 010,\ 001,\ 110,\ 101,\ 011,\ 002,\ 111,\ 102,\ 012,\ 112, \tag{69}$$

Table 3

EXAMPLES OF THE KRUSKAL–MACAULAY FUNCTIONS κ, λ, AND μ

$N =$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$\kappa_1 N =$	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$\kappa_2 N =$	0	2	3	3	4	4	4	5	5	5	5	6	6	6	6	6	7	7	7	7	7
$\kappa_3 N =$	0	3	5	6	6	8	9	9	10	10	10	12	13	13	14	14	14	15	15	15	15
$\kappa_4 N =$	0	4	7	9	10	10	13	15	16	16	18	19	19	20	20	20	23	25	26	26	28
$\kappa_5 N =$	0	5	9	12	14	15	15	19	22	24	25	25	28	30	31	31	33	34	34	35	35
$\lambda_1 N =$	0	0	1	3	6	10	15	21	28	36	45	55	66	78	91	105	120	136	153	171	190
$\lambda_2 N =$	0	0	0	1	1	2	4	4	5	7	10	10	11	13	16	20	20	21	23	26	30
$\lambda_3 N =$	0	0	0	0	1	1	1	2	2	3	5	5	5	6	6	7	9	9	10	12	15
$\lambda_4 N =$	0	0	0	0	0	1	1	1	1	2	2	2	3	3	4	6	6	6	6	7	7
$\lambda_5 N =$	0	0	0	0	0	0	1	1	1	1	1	2	2	2	2	3	3	3	4	4	5
$\mu_1 N =$	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$\mu_2 N =$	0	1	2	2	3	3	3	4	4	4	4	5	5	5	5	5	6	6	6	6	6
$\mu_3 N =$	0	1	2	3	3	4	5	5	6	6	6	7	8	8	9	9	9	10	10	10	10
$\mu_4 N =$	0	1	2	3	4	4	5	6	7	7	8	9	9	10	10	10	11	12	13	13	14
$\mu_5 N =$	0	1	2	3	4	5	5	6	7	8	9	9	10	11	12	12	13	14	14	15	15

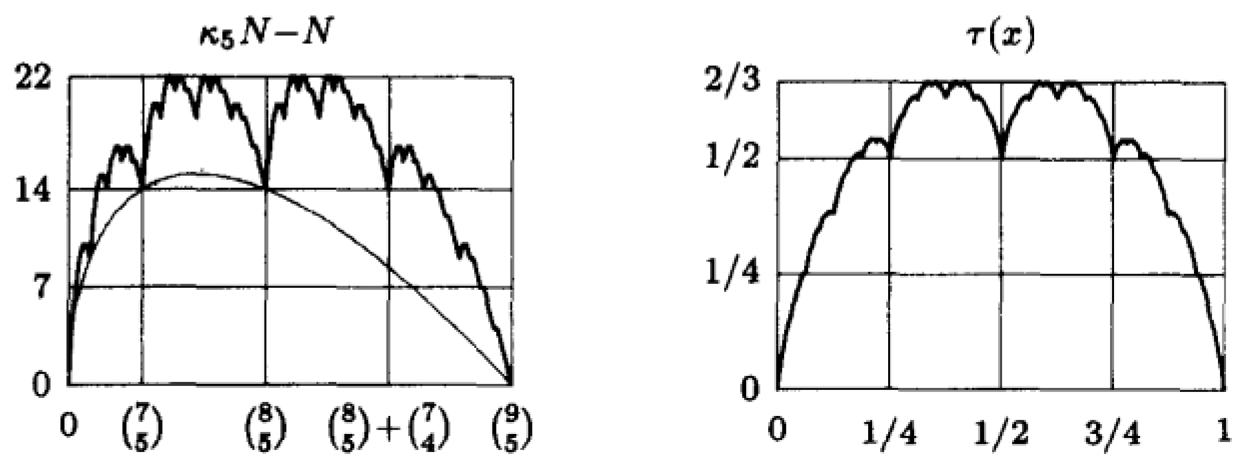


Fig. 27. Approximating a Kruskal function with the Takagi function. (The smooth curve in the left-hand graph is the lower bound $\underline{\kappa}_5 N - N$ of exercise 80.)

omitting parentheses and commas for convenience. The *complement* of a vector in $T(m_1, \ldots, m_n)$ is

$$\overline{x} = (m_1 - 1 - x_1, \ldots, m_n - 1 - x_n). \qquad (70)$$

Notice that $x \preceq y$ holds if and only if $\overline{x} \succeq \overline{y}$. Therefore we have

$$\text{rank}(x) + \text{rank}(\overline{x}) = T - 1, \qquad \text{where } T = m_1 \ldots m_n, \qquad (71)$$

if rank(x) denotes the number of vectors that precede x in cross order.

We will find it convenient to call the vectors "points" and to name the points $e_0, e_1, \ldots, e_{T-1}$ in increasing cross order. Thus we have $e_7 = 002$ in (69), and $\overline{e}_r = e_{T-1-r}$ in general. Notice that

$$e_1 = 100\ldots00, \quad e_2 = 010\ldots00, \quad \ldots, \quad e_n = 000\ldots01; \qquad (72)$$

these are the so-called *unit vectors.* The set

$$S_N \;=\; \{e_0, e_1, \ldots, e_{N-1}\} \tag{73}$$

consisting of the smallest N points is called a *standard set*, and in the special case $N = n+1$ we write

$$E = \{e_0, e_1, \ldots, e_n\} = \{000\ldots00, 100\ldots00, 010\ldots00, \ldots, 000\ldots01\}. \tag{74}$$

Any set of points X has a *spread* X^+, a *core* X°, and a *dual* $X^\sim$, defined by the rules

$$X^+ \;=\; \{\, x \in S_T \mid x \in X \text{ or } x - e_1 \in X \text{ or } \cdots \text{ or } x - e_n \in X \,\}; \tag{75}$$

$$X^\circ \;=\; \{\, x \in S_T \mid x \in X \text{ and } x + e_1 \in X \text{ and } \cdots \text{ and } x + e_n \in X \,\}; \tag{76}$$

$$X^\sim \;=\; \{\, x \in S_T \mid \overline{x} \notin X \,\}. \tag{77}$$

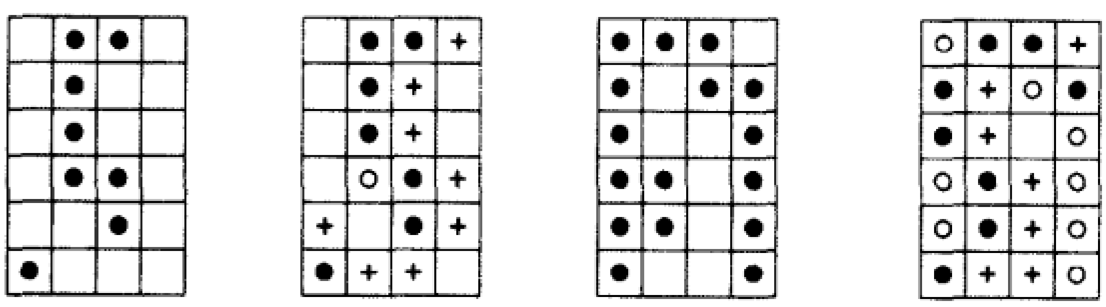
We can also define the spread of X algebraically, writing

$$X^+ \;=\; X + E, \tag{78}$$

where $X+Y$ denotes $\{\, x+y \mid x \in X \text{ and } y \in Y \,\}$. Clearly

$$X^+ \subseteq Y \qquad \text{if and only if} \qquad X \subseteq Y^\circ. \tag{79}$$

These notions can be illustrated in the two-dimensional case $m_1 = 4$, $m_2 = 6$, by the more-or-less random toroidal arrangement $X = \{00, 12, 13, 14, 15, 21, 22, 25\}$ for which we have, pictorially,



X $\qquad$ X° and X^+ $\qquad$ $X^\sim$ $\qquad$ $X^{\sim\circ}$ and $X^{\sim+}$; (80)

here X in the first two diagrams consists of points marked • or ∘, X° comprises just the ∘s, and X^+ consists of +s plus •s plus ∘s. Notice that if we rotate the diagram for $X^{\sim\circ}$ and $X^{\sim+}$ by 180°, we obtain the diagram for X° and X^+, but with (•, ∘, +,) respectively changed to (+, , •, ∘); and in fact the identities

$$X^\circ \;=\; X^{\sim+\sim}, \qquad X^+ \;=\; X^{\sim\circ\sim} \tag{81}$$

hold in general (see exercise 86).

Now we are ready to state the theorem of Wang and Wang:

Theorem W. *Let X be any set of N points in the discrete torus $T(m_1, \ldots, m_n)$, where $m_1 \le \cdots \le m_n$. Then $|X^+| \ge |S_N^+|$ and $|X^\circ| \le |S_N^\circ|$.*

In other words, the standard sets S_N have the smallest spread and largest core, among all N-point sets. We will prove this result by following a general approach first used by F. W. J. Whipple to prove Theorem M [*Proc. London Math. Soc.* (2) **28** (1928), 431–437]. The first step is to prove that the spread and the core of standard sets are standard:

Lemma S. *There are functions* α *and* β *such that* $S_N^+ = S_{\alpha N}$ *and* $S_N^\circ = S_{\beta N}$.

Proof. We may assume that $N > 0$. Let r be maximum with $e_r \in S_N^+$, and let $\alpha N = r + 1$; we must prove that $e_q \in S_N^+$ for $0 \le q < r$. Suppose $e_q = x = (x_1, \ldots, x_n)$ and $e_r = y = (y_1, \ldots, y_n)$, and let k be the largest subscript with $x_k > 0$. Since $y \in S_N^+$, there is a subscript j such that $y - e_j \in S_N$. It suffices to prove that $x - e_k \preceq y - e_j$, and exercise 88 does this.

The second part follows from (81), with $\beta N = T - \alpha(T - N)$, because $S_N^\sim = S_{T-N}$. ∎

Theorem W is obviously true when $n = 1$, so we assume by induction that it has been proved in $n - 1$ dimensions. The next step is to *compress* the given set X in the kth coordinate position, by partitioning it into disjoint sets

$$X_k(a) = \{ x \in X \mid x_k = a \} \tag{82}$$

for $0 \le a < m_k$ and replacing each $X_k(a)$ by

$$X'_k(a) = \{ (s_1, \ldots, s_{k-1}, a, s_k, \ldots, s_{n-1}) \mid (s_1, \ldots, s_{n-1}) \in S_{|X_k(a)|} \}, \tag{83}$$

a set with the same number of elements. The sets S used in (83) are standard in the $(n-1)$-dimensional torus $T(m_1, \ldots, m_{k-1}, m_{k+1}, \ldots, m_n)$. Notice that we have $(x_1, \ldots, x_{k-1}, a, x_{k+1}, \ldots, x_n) \preceq (y_1, \ldots, y_{k-1}, a, y_{k+1}, \ldots, y_n)$ if and only if $(x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n) \preceq (y_1, \ldots, y_{k-1}, y_{k+1}, \ldots, y_n)$; therefore $X'_k(a) = X_k(a)$ if and only if the $(n-1)$-dimensional points $(x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n)$ with $(x_1, \ldots, x_{k-1}, a, x_{k+1}, \ldots, x_n) \in X$ are as small as possible when projected onto the $(n-1)$-dimensional torus. We let

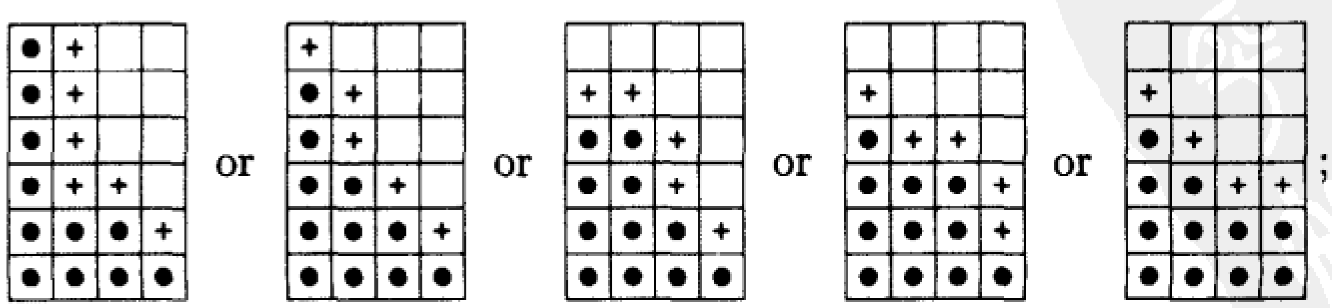
$$C_k X = X'_k(0) \cup X'_k(1) \cup \cdots \cup X'_k(m_k - 1) \tag{84}$$

be the compression of X in position k. Exercise 90 proves the basic fact that compression does not increase the size of the spread:

$$|X^+| \ge |(C_k X)^+|, \qquad \text{for } 1 \le k \le n. \tag{85}$$

Furthermore, if compression changes X, it replaces some of the elements by other elements of lower rank. Therefore we need to prove Theorem W only for sets X that are totally compressed, having $X = C_k X$ for all k.

Consider, for example, the case $n = 2$. A totally compressed set in two dimensions has all points moved to the left of their rows and the bottom of their columns, as in the eleven-point sets



the rightmost of these is standard, and has the smallest spread. Exercise 91 completes the proof of Theorem W in two dimensions.

When $n > 2$, suppose $x = (x_1, \ldots, x_n) \in X$ and $x_j > 0$. The condition $C_k X = X$ implies that, if $0 \le i < j$ and $i \ne k \ne j$, we have $x + e_i - e_j \in X$. Applying this fact for three values of k tells us that $x + e_i - e_j \in X$ whenever $0 \le i < j$. Consequently

$$X_n(a) + E_n(0) \subseteq X_n(a-1) + e_n \quad \text{for } 0 < a < m, \tag{86}$$

where $m = m_n$ and $E_n(0)$ is a clever abbreviation for the set $\{e_0, \ldots, e_{n-1}\}$.

Let $X_n(a)$ have N_a elements, so that $N = |X| = N_0 + N_1 + \cdots + N_{m-1}$, and let $Y = X^+$. Then

$$Y_n(a) \;=\; \bigl(X_n\bigl((a-1) \bmod m\bigr) + e_n\bigr) \cup \bigl(X_n(a) + E_n(0)\bigr)$$

is standard in $n-1$ dimensions, and (86) tells us that

$$N_{m-1} \le \beta N_{m-2} \le N_{m-2} \le \cdots \le N_1 \le \beta N_0 \le N_0 \le \alpha N_0,$$

where α and β refer to coordinates 1 through $n-1$. Therefore

$$\begin{aligned}|Y| &= |Y_n(0)| + |Y_n(1)| + |Y_n(2)| + \cdots + |Y_n(m-1)| \\ &= \alpha N_0 + N_0 + N_1 + \cdots + N_{m-2} = \alpha N_0 + N - N_{m-1}.\end{aligned}$$

The proof of Theorem W now has a beautiful conclusion. Let $Z = S_N$, and suppose $|Z_n(a)| = M_a$. We want to prove that $|X^+| \ge |Z^+|$, namely that

$$\alpha N_0 + N - N_{m-1} \;\ge\; \alpha M_0 + N - M_{m-1}, \tag{87}$$

because the arguments of the previous paragraph apply to Z as well as to X. We will prove (87) by showing that $N_{m-1} \le M_{m-1}$ and $N_0 \ge M_0$.

Using the $(n-1)$-dimensional α and β functions, let us define

$$N'_{m-1} = N_{m-1},\ N'_{m-2} = \alpha N'_{m-1},\ \ldots,\ N'_1 = \alpha N'_2,\ N'_0 = \alpha N'_1; \tag{88}$$

$$N''_0 = N_0,\ N''_1 = \beta N''_0,\ N''_2 = \beta N''_1,\ \ldots,\ N''_{m-1} = \beta N''_{m-2}. \tag{89}$$

Then we have $N'_a \le N_a \le N''_a$ for $0 \le a < m$, and it follows that

$$N' = N'_0 + N'_1 + \cdots + N'_{m-1} \;\le\; N \;\le\; N'' = N''_0 + N''_1 + \cdots + N''_{m-1}. \tag{90}$$

Exercise 92 proves that the standard set $Z' = S_{N'}$ has exactly N'_a elements with nth coordinate equal to a, for each a; and by the duality between α and β, the standard set $Z'' = S_{N''}$ likewise has exactly N''_a elements with nth coordinate a. Finally, therefore,

$$\begin{aligned}M_{m-1} = |Z_n(m-1)| \ge |Z'_n(m-1)| = N_{m-1},\\ M_0 = |Z_n(0)| \le |Z''_n(0)| = N_0,\end{aligned}$$

because $Z' \subseteq Z \subseteq Z''$ by (90). By (81) we also have $|X^\circ| \le |Z^\circ|$. ∎

Now we are ready to prove Theorems K and M, which are in fact special cases of a substantially more general theorem of Clements and Lindström that applies to arbitrary multisets [*J. Combinatorial Theory* **7** (1969), 230–238]:

Corollary C. *If A is a set of N t-multicombinations contained in the multiset $U = \{s_0 \cdot 0, s_1 \cdot 1, \ldots, s_d \cdot d\}$, where $s_0 \ge s_1 \ge \cdots \ge s_d$, then*

$$|\partial A| \ge |\partial P_{Nt}| \qquad \text{and} \qquad |\varrho A| \ge |\varrho Q_{Nt}|, \tag{91}$$

where P_{Nt} denotes the N lexicographically smallest multicombinations $d_t \ldots d_2 d_1$ of U, and Q_{Nt} denotes the N lexicographically largest.

Proof. Multicombinations of U can be represented as points $x_1 \ldots x_n$ of the torus $T(m_1, \ldots, m_n)$, where $n = d+1$ and $m_j = s_{n-j} + 1$; we let x_j be the number of occurrences of $n-j$. This correspondence preserves lexicographic order. For example, if $U = \{0,0,0,1,1,2,3\}$, its 3-multicombinations are

$$000,\ 100,\ 110,\ 200,\ 210,\ 211,\ 300,\ 310,\ 311,\ 320,\ 321, \tag{92}$$

in lexicographic order, and the corresponding points $x_1x_2x_3x_4$ are

$$0003, 0012, 0021, 0102, 0111, 0120, 1002, 1011, 1020, 1101, 1110. \tag{93}$$

Let T_w be the points of the torus that have weight $x_1 + \cdots + x_n = w$. Then every allowable set A of t-multicombinations is a subset of T_t. Furthermore — and this is the main point — the spread of $T_0 \cup T_1 \cup \cdots \cup T_{t-1} \cup A$ is

$$\begin{aligned}(T_0 \cup T_1 \cup \cdots \cup T_{t-1} \cup A)^+ &= T_0^+ \cup T_1^+ \cup \cdots \cup T_{t-1}^+ \cup A^+ \\ &= T_0 \cup T_1 \cup \cdots \cup T_t \cup \varrho A.\end{aligned} \tag{94}$$

Thus the upper shadow ϱA is simply $(T_0 \cup T_1 \cup \cdots \cup T_{t-1} \cup A)^+ \cap T_{t+1}$, and Theorem W tells us in essence that $|A| = N$ implies $|\varrho A| \ge |\varrho(S_{M+N} \cap T_t)|$, where $M = |T_0 \cup \cdots \cup T_{t-1}|$. Hence, by the definition of cross order, $S_{M+N} \cap T_t$ consists of the lexicographically largest N t-multicombinations, namely Q_{Nt}.

The proof that $|\partial A| \ge |\partial P_{Nt}|$ now follows by complementation (see exercise 94). ▌

EXERCISES

1. [*M23*] Explain why Golomb's rule (8) makes all sets $\{c_1, \ldots, c_t\} \subseteq \{0, \ldots, n-1\}$ correspond uniquely to multisets $\{e_1, \ldots, e_t\} \subseteq \{\infty \cdot 0, \ldots, \infty \cdot n-t\}$.

2. [*16*] What path in an 11×13 grid corresponds to the bit string (13)?

▸ **3.** [*21*] (R. R. Fenichel, 1968.) Show that the compositions $q_t + \cdots + q_1 + q_0$ of s into $t+1$ nonnegative parts can be generated in lexicographic order by a simple loopless algorithm.

4. [*16*] Show that every composition $q_t \ldots q_0$ of s into $t+1$ nonnegative parts corresponds to a composition $r_s \ldots r_0$ of t into $s+1$ nonnegative parts. What composition corresponds to 10224000001010 under this correspondence?

▸ **5.** [*20*] What is a good way to generate all of the integer solutions to the following systems of inequalities?

a) $n > x_t \ge x_{t-1} > x_{t-2} \ge x_{t-3} > \cdots > x_1 \ge 0$, when t is odd.

b) $n \gg x_t \gg x_{t-1} \gg \cdots \gg x_2 \gg x_1 \gg 0$, where $a \gg b$ means $a \ge b+2$.

6. [*M22*] How often is each step of Algorithm T performed?

7. [*22*] Design an algorithm that runs through the "dual" combinations $b_s \ldots b_2 b_1$ in *decreasing* lexicographic order (see (5) and Table 1). Like Algorithm T, your algorithm should avoid redundant assignments and unnecessary searching.

8. [*M23*] Design an algorithm that generates all (s,t)-combinations $a_{n-1} \ldots a_1 a_0$ lexicographically in bitstring form. The total running time should be $O(\binom{n}{t})$, assuming that $st > 0$.

9. [*M26*] When all (s,t)-combinations $a_{n-1} \ldots a_1 a_0$ are listed in lexicographic order, let $2A_{st}$ be the total number of bit changes between adjacent strings. For example, $A_{33} = 25$ because there are respectively

$$2+2+2+4+2+2+4+2+2+6+2+2+4+2+2+4+2+2+2=50$$

bit changes between the 20 strings in Table 1.

a) Show that $A_{st} = \min(s,t) + A_{(s-1)t} + A_{s(t-1)}$ when $st > 0$; $A_{st} = 0$ when $st = 0$.

b) Prove that $A_{st} < 2\binom{s+t}{t}$.

▶ **10.** [*21*] The "World Series" of baseball is traditionally a competition in which the American League champion (A) plays the National League champion (N) until one of them has beaten the other four times. What is a good way to list all possible scenarios AAAA, AAANA, AAANNA, ..., NNNN? What is a simple way to assign consecutive integers to those scenarios?

11. [*19*] Which of the scenarios in exercise 10 occurred most often during the 1900s? Which of them never occurred? [*Hint:* World Series scores are easily found on the Internet.]

12. [*HM32*] A set V of n-bit vectors that is closed under addition modulo 2 is called a *binary vector space*.

a) Prove that every such V contains 2^t elements, for some integer t, and can be represented as the set $\{x_1\alpha_1 \oplus \cdots \oplus x_t\alpha_t \mid 0 \le x_1, \ldots, x_t \le 1\}$ where the vectors $\alpha_1, \ldots, \alpha_t$ form a "canonical basis" with the following property: There is a t-combination $c_t \ldots c_2 c_1$ of $\{0,1,\ldots,n-1\}$ such that, if α_k is the binary vector $a_{k(n-1)} \ldots a_{k1} a_{k0}$, we have

$$a_{kc_j} = [j=k] \quad \text{for } 1 \le j,k \le t; \qquad a_{kl} = 0 \quad \text{for } 0 \le l < c_k,\ 1 \le k \le t.$$

For example, the canonical bases with $n = 9$, $t = 4$, and $c_4c_3c_2c_1 = 7641$ have the general form

$$\begin{aligned} \alpha_1 &= *\,0\,0\,*\,0\,*\,*\,1\,0, \\ \alpha_2 &= *\,0\,0\,*\,1\,0\,0\,0\,0, \\ \alpha_3 &= *\,0\,1\,0\,0\,0\,0\,0\,0, \\ \alpha_4 &= *\,1\,0\,0\,0\,0\,0\,0\,0; \end{aligned}$$

there are 2^8 ways to replace the eight asterisks by 0s and/or 1s, and each of these defines a canonical basis. We call t the dimension of V.

b) How many t-dimensional spaces are possible with n-bit vectors?

c) Design an algorithm to generate all canonical bases $(\alpha_1, \ldots, \alpha_t)$ of dimension t. *Hint:* Let the associated combinations $c_t \ldots c_1$ increase lexicographically as in Algorithm L.

d) What is the 1000000th basis visited by your algorithm when $n = 9$ and $t = 4$?

13. [*25*] A one-dimensional *Ising configuration* of length n, weight t, and energy r, is a binary string $a_{n-1} \ldots a_0$ such that $\sum_{j=0}^{n-1} a_j = t$ and $\sum_{j=1}^{n-1} b_j = r$, where $b_j =$

$a_j \oplus a_{j-1}$. For example, $a_{12} \ldots a_0 = 1100100100011$ has weight 6 and energy 6, since $b_{12} \ldots b_1 = 010110110010$.

Design an algorithm to generate all such configurations, given n, t, and r.

14. [*26*] When the binary strings $a_{n-1} \ldots a_1 a_0$ of (s,t)-combinations are generated in lexicographic order, we sometimes need to change $2\min(s,t)$ bits to get from one combination to the next. For example, 011100 is followed by 100011 in Table 1. Therefore we apparently cannot hope to generate all combinations with a loopless algorithm unless we visit them in some other order.

Show, however, that there actually is a way to compute the lexicographic successor of a given combination in $O(1)$ steps, if each combination is represented indirectly in a doubly linked list as follows: There are arrays $l[0], \ldots, l[n]$ and $r[0], \ldots, r[n]$ such that $l[r[j]] = j$ for $0 \le j \le n$. If $x_0 = l[0]$ and $x_j = l[x_{j-1}]$ for $0 < j < n$, then $a_j = [x_j > s]$ for $0 \le j < n$.

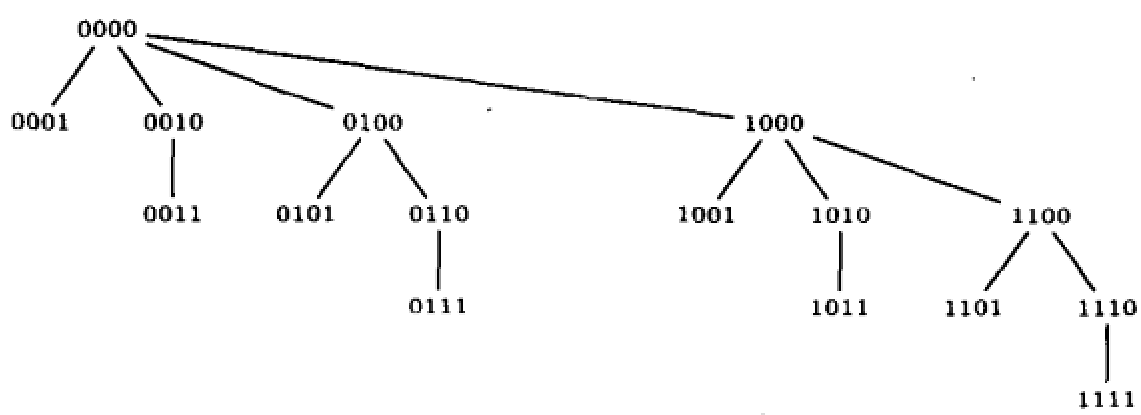
15. [*M22*] Use the fact that dual combinations $b_s \ldots b_2 b_1$ occur in reverse lexicographic order to prove that the sum $\binom{b_s}{s} + \cdots + \binom{b_2}{2} + \binom{b_1}{1}$ has a simple relation to the sum $\binom{c_t}{t} + \cdots + \binom{c_2}{2} + \binom{c_1}{1}$.

16. [*M21*] What is the millionth combination generated by Algorithm L when t is (a) 2? (b) 3? (c) 4? (d) 5? (e) 1000000?

17. [*HM25*] Given N and t, what is a good way to compute the combinatorial representation (20)?

▶ **18.** [*20*] What binary tree do we get when the binomial tree T_n is represented by "right child" and "left sibling" pointers as in exercise 2.3.2–5?

19. [*21*] Instead of labeling the branches of the binomial tree T_4 as shown in (22), we could label each node with the bit string of its corresponding combination:



If T_∞ has been labeled in this way, suppressing leading zeros, preorder is the same as the ordinary increasing order of binary notation; so the millionth node turns out to be 11110100001000111111. But what is the millionth node of T_∞ in *postorder*?

20. [*M20*] Find generating functions g and h such that Algorithm F finds exactly $[z^N]\,g(z)$ feasible combinations and sets $t \leftarrow t+1$ exactly $[z^N]\,h(z)$ times.

21. [*M22*] Prove the alternating combination law (30).

22. [*M23*] What is the millionth revolving-door combination visited by Algorithm R when t is (a) 2? (b) 3? (c) 4? (d) 5? (e) 1000000?

23. [*M23*] Suppose we augment Algorithm R by setting $j \leftarrow t+1$ in step R1, and $j \leftarrow 1$ if R3 goes directly to R2. Find the probability distribution of j, and its average value. What does this imply about the running time of the algorithm?

▸ **24.** [*M25*] (W. H. Payne, 1974.) Continuing the previous exercise, let j_k be the value of j on the kth visit by Algorithm R. Show that $|j_{k+1} - j_k| \le 2$, and explain how to make the algorithm loopless by exploiting this property.

25. [*M35*] Let $c_t \ldots c_2 c_1$ and $c'_t \ldots c'_2 c'_1$ be the Nth and N'th combinations generated by the revolving-door method, Algorithm R. If the set $C = \{c_t, \ldots, c_2, c_1\}$ has $m > 0$ elements not in $C' = \{c'_t, \ldots, c'_2, c'_1\}$, prove that $|N - N'| > \sum_{k=1}^{m-1} \binom{2k}{k-1}$.

26. [*26*] Do elements of the *ternary* reflected Gray code have properties similar to the revolving-door Gray code Γ_{st}, if we extract only the n-tuples $a_{n-1} \ldots a_1 a_0$ such that (a) $a_{n-1} + \cdots + a_1 + a_0 = t$? (b) $\{a_{n-1}, \ldots, a_1, a_0\} = \{r \cdot 0, s \cdot 1, t \cdot 2\}$?

▸ **27.** [*25*] Show that there is a simple way to generate all combinations of *at most* t elements of $\{0, 1, \ldots, n-1\}$, using only Gray-code-like transitions $0 \leftrightarrow 1$ and $01 \leftrightarrow 10$. (In other words, each step should either insert a new element, delete an element, or shift an element by ± 1.) For example,

0000, 0001, 0011, 0010, 0110, 0101, 0100, 1100, 1010, 1001, 1000

is one such sequence when $n = 4$ and $t = 2$. *Hint:* Think of Chinese rings.

28. [*M21*] True or false: A listing of (s,t)-combinations $a_{n-1} \ldots a_1 a_0$ in bitstring form is in genlex order if and only if the corresponding index-form listings $b_s \ldots b_2 b_1$ (for the 0s) and $c_t \ldots c_2 c_1$ (for the 1s) are both in genlex order.

▸ **29.** [*M28*] (P. J. Chase.) Given a string on the symbols +, -, and 0, say that an *R-block* is a substring of the form $-^{k+1}$ that is preceded by 0 and not followed by -; an *L-block* is a substring of the form $+-^k$ that is followed by 0; in both cases $k \ge 0$. For example, the string +00++-+++-000- has two L-blocks and one R-block, shown in gray. Notice that blocks cannot overlap.

We form the *successor* of such a string as follows, whenever at least one block is present: Replace the rightmost $0-^{k+1}$ by $-+^k0$, if the rightmost block is an R-block; otherwise replace the rightmost $+-^k0$ by $0+^{k+1}$. Also negate the first sign, if any, that appears to the right of the block that has been changed. For example,

-+00++- → -0+0-+- → -0+-0-- → -0+--+0 → -0+--0+ → -00+++-,

where the notation $\alpha \to \beta$ means that β is the successor of α.

a) What strings have no blocks (and therefore no successor)?
b) Can there be a cycle of strings with $\alpha_0 \to \alpha_1 \to \cdots \to \alpha_{k-1} \to \alpha_0$?
c) Prove that if $\alpha \to \beta$ then $-\beta \to -\alpha$, where "$-$" means "negate all the signs." (Therefore every string has at most one predecessor.)
d) Show that if $\alpha_0 \to \alpha_1 \to \cdots \to \alpha_k$ and $k > 0$, the strings α_0 and α_k do not have all their 0s in the same positions. (Therefore, if α_0 has s signs and t zeros, k must be less than $\binom{s+t}{t}$.)
e) Prove that every string α with s signs and t zeros belongs to exactly one chain $\alpha_0 \to \alpha_1 \to \cdots \to \alpha_{\binom{s+t}{t}-1}$.

30. [*M32*] The previous exercise defines 2^s ways to generate all combinations of s 0s and t 1s, via the mapping + $\mapsto$ 0, - $\mapsto$ 0, and 0 $\mapsto$ 1. Show that each of these ways is a homogeneous genlex sequence, definable by an appropriate recurrence. Is Chase's sequence (37) a special case of this general construction?

31. [*M23*] How many genlex listings of (s,t)-combinations are possible in (a) bitstring form $a_{n-1} \ldots a_1 a_0$? (b) index-list form $c_t \ldots c_2 c_1$?

▶ **32.** [*M32*] How many of the genlex listings of (s,t)-combination strings $a_{n-1}\ldots a_1a_0$ (a) have the revolving-door property? (b) are homogeneous?

33. [*HM33*] How many of the genlex listings in exercise 31(b) are near-perfect?

34. [*M32*] Continuing exercise 33, explain how to find such schemes that are as near as possible to perfection, in the sense that the number of "imperfect" transitions $c_j \leftarrow c_j \pm 2$ is minimized, when s and t are not too large.

35. [*M26*] How many steps of Chase's sequence C_{st} use an imperfect transition?

▶ **36.** [*M21*] Prove that method (39) performs the operation $j \leftarrow j+1$ a total of exactly $\binom{s+t}{t}-1$ times as it generates all (s,t)-combinations $a_{n-1}\ldots a_1a_0$, given any genlex scheme for combinations in bitstring form.

▶ **37.** [*27*] What algorithm results when the general genlex method (39) is used to produce (s,t)-combinations $a_{n-1}\ldots a_1a_0$ in (a) lexicographic order? (b) the revolving-door order of Algorithm R? (c) the homogeneous order of (31)?

38. [*26*] Design a genlex algorithm like Algorithm C for the *reverse* sequence C_{st}^R.

39. [*M21*] When $s=12$ and $t=14$, how many combinations precede the bit string 11001001000011111101101010 in Chase's sequence C_{st}? (See (41).)

40. [*M22*] What is the millionth combination in Chase's sequence C_{st}, when $s=12$ and $t=14$?

41. [*M27*] Show that there is a permutation $c(0)$, $c(1)$, $c(2)$, ... of the nonnegative integers such that the elements of Chase's sequence C_{st} are obtained by complementing the least significant $s+t$ bits of the elements $c(k)$ for $0 \le k < 2^{s+t}$ that have weight $\nu(c(k)) = s$. (Thus the sequence $\bar{c}(0)$, ..., $\bar{c}(2^n-1)$ contains, as subsequences, all of the C_{st} for which $s+t=n$, just as Gray binary code $g(0)$, ..., $g(2^n-1)$ contains all the revolving-door sequences Γ_{st}.) Explain how to compute the binary representation $c(k) = (\ldots a_2a_1a_0)_2$ from the binary representation $k = (\ldots b_2b_1b_0)_2$.

42. [*HM34*] Use generating functions of the form $\sum_{s,t} g_{st}w^sz^t$ to analyze each step of Algorithm C.

43. [*20*] Prove or disprove: If $s(x)$ and $p(x)$ denote respectively the successor and predecessor of x in endo-order, then $s(x+1) = p(x)+1$.

▶ **44.** [*M21*] Let $C_t(n)-1$ denote the sequence obtained from $C_t(n)$ by striking out all combinations with $c_1 = 0$, then replacing $c_t\ldots c_1$ by $(c_t-1)\ldots(c_1-1)$ in the combinations that remain. Show that $C_t(n)-1$ is near-perfect.

45. [*32*] Exploit endo-order and the expansions sketched in (44) to generate the combinations $c_t\ldots c_2c_1$ of Chase's sequence $C_t(n)$ with a nonrecursive procedure.

▶ **46.** [*33*] Construct a nonrecursive algorithm for the dual combinations $b_s\ldots b_2b_1$ of Chase's sequence C_{st}, namely for the positions of the zeros in $a_{n-1}\ldots a_1a_0$.

47. [*26*] Implement the near-perfect multiset permutation method of (46) and (47).

48. [*M21*] Suppose $\alpha_0, \alpha_1, \ldots, \alpha_{N-1}$ is any listing of the permutations of the multiset $\{s_1\cdot 1,\ldots,s_d\cdot d\}$, where α_k differs from α_{k+1} by the interchange of two elements. Let $\beta_0, \ldots, \beta_{M-1}$ be any revolving-door listing for (s,t)-combinations, where $s=s_0$, $t=s_1+\cdots+s_d$, and $M=\binom{s+t}{t}$. Then let Λ_j be the list of M elements obtained by starting with $\alpha_j \uparrow \beta_0$ and applying the revolving-door exchanges; here $\alpha \uparrow \beta$ denotes the string obtained by substituting the elements of α for the 1s in β, preserving left-right order. For example, if $\beta_0, \ldots, \beta_{M-1}$ is 0110, 0101, 1100, 1001, 0011, 1010, and if $\alpha_j = 12$, then Λ_j is 0120, 0102, 1200, 1002, 0012, 1020. (The revolving-door listing need *not* be homogeneous.)

Prove that the list (47) contains all permutations of $\{s_0 \cdot 0, s_1 \cdot 1, \ldots, s_d \cdot d\}$, and that adjacent permutations differ from each other by the interchange of two elements.

49. [*HM23*] If q is a primitive mth root of unity, such as $e^{2\pi i/m}$, show that

$$\binom{n}{k}_q = \binom{\lfloor n/m \rfloor}{\lfloor k/m \rfloor}\binom{n \bmod m}{k \bmod m}_q.$$

▶ **50.** [*HM25*] Extend the formula of the previous exercise to *q-multinomial* coefficients

$$\binom{n_1 + \cdots + n_t}{n_1, \ldots, n_t}_q.$$

51. [*25*] Find all Hamiltonian paths in the graph whose vertices are permutations of $\{0,0,0,1,1,1\}$ related by adjacent transposition. Which of those paths are equivalent under the operations of interchanging 0s with 1s and/or left-right reflection?

52. [*M37*] Generalizing Theorem P, find a necessary and sufficient condition that all permutations of the multiset $\{s_0 \cdot 0, \ldots, s_d \cdot d\}$ can be generated by adjacent transpositions $a_j a_{j-1} \leftrightarrow a_{j-1} a_j$.

53. [*M46*] (D. H. Lehmer, 1965.) Suppose the N permutations of $\{s_0 \cdot 0, \ldots, s_d \cdot d\}$ cannot be generated by a perfect scheme, because $(N + x)/2$ of them have an even number of inversions, where $x \geq 2$. Is it possible to generate them all with a sequence of $N + x - 2$ adjacent interchanges $a_{\delta_k} \leftrightarrow a_{\delta_k - 1}$ for $1 \leq k < N + x - 1$, where $x - 1$ cases are "spurs" with $\delta_k = \delta_{k-1}$ that take us back to the permutation we've just seen? For example, a suitable sequence $\delta_1 \ldots \delta_{94}$ for the 90 permutations of $\{0,0,1,1,2,2\}$, where $x = \binom{2+2+2}{2,2,2}_{-1} = 6$, is $234535432523451\alpha42\alpha^R51\alpha42\alpha^R51\alpha4$, where $\alpha = 45352542345355$, if we start with $a_5a_4a_3a_2a_1a_0 = 221100$.

54. [*M40*] For what values of s and t can all (s,t)-combinations be generated if we allow end-around swaps $a_{n-1} \leftrightarrow a_0$ in addition to adjacent interchanges $a_j \leftrightarrow a_{j-1}$?

▶ **55.** [*33*] (Frank Ruskey, 2004.) (a) Show that all (s,t)-combinations $a_{s+t-1} \ldots a_1 a_0$ can be generated efficiently by doing successive rotations $a_j a_{j-1} \ldots a_0 \leftarrow a_{j-1} \ldots a_0 a_j$. (b) What MMIX instructions will take $(a_{s+t-1} \ldots a_1 a_0)_2$ to its successor, when $s+t < 64$?

56. [*M49*] (Buck and Wiedemann, 1984.) Can all (t,t)-combinations $a_{2t-1} \ldots a_1 a_0$ be generated by repeatedly swapping a_0 with some other element?

▶ **57.** [*22*] (Frank Ruskey.) Can a piano player run through all possible 4-note chords that span at most one octave, changing only one finger at a time? This is the problem of generating all combinations $c_t \ldots c_1$ such that $n > c_t > \cdots > c_1 \geq 0$ and $c_t - c_1 < m$, where $t = 4$ and (a) $m = 8$, $n = 52$ if we consider only the white notes of a piano keyboard; (b) $m = 13$, $n = 88$ if we consider also the black notes.

58. [*20*] Consider the piano player's problem of exercise 57 with the additional condition that the chords don't involve adjacent notes. (In other words, $c_{j+1} > c_j + 1$ for $t > j \geq 1$. Such chords tend to be more harmonious.)

59. [*M25*] Is there a *perfect* solution to the 4-note piano player's problem, in which each step moves a finger to an *adjacent* key?

60. [*23*] Design an algorithm to generate all *bounded* compositions

$$t = r_s + \cdots + r_1 + r_0, \qquad \text{where } 0 \leq r_j \leq m_j \text{ for } s \geq j \geq 0.$$

61. [*32*] Show that all bounded compositions can be generated by changing only two of the parts at each step.

▶ **62.** [*M27*] A *contingency table* is an $m \times n$ matrix of nonnegative integers (a_{ij}) having given row sums $r_i = \sum_{j=1}^{n} a_{ij}$ and column sums $c_j = \sum_{i=1}^{m} a_{ij}$, where $r_1 + \cdots + r_m = c_1 + \cdots + c_n$.

a) Show that $2 \times n$ contingency tables are equivalent to bounded compositions.

b) What is the lexicographically largest contingency table for $(r_1, \ldots, r_m; c_1, \ldots, c_n)$, when matrix entries are read row-wise from left to right and top to bottom, namely in the order $(a_{11}, a_{12}, \ldots, a_{1n}, a_{21}, \ldots, a_{mn})$?

c) What is the lexicographically largest contingency table for $(r_1, \ldots, r_m; c_1, \ldots, c_n)$, when matrix entries are read column-wise from top to bottom and left to right, namely in the order $(a_{11}, a_{21}, \ldots, a_{m1}, a_{12}, \ldots, a_{mn})$?

d) What is the lexicographically smallest contingency table for $(r_1, \ldots, r_m; c_1, \ldots, c_n)$, in the row-wise and column-wise senses?

e) Explain how to generate all contingency tables for $(r_1, \ldots, r_m; c_1, \ldots, c_n)$ in lexicographic order.

63. [*M41*] Show that all contingency tables for $(r_1, \ldots, r_m; c_1, \ldots, c_n)$ can be generated by changing exactly four entries of the matrix at each step.

▶ **64.** [*M30*] Construct a genlex Gray cycle for all of the $2^s\binom{s+t}{t}$ *subcubes* that have s digits and t asterisks, using only the transformations $*0 \leftrightarrow 0*$, $*1 \leftrightarrow 1*$, $0 \leftrightarrow 1$. For example, one such cycle when $s = t = 2$ is

(00**, 01**, 0*1*, 0**1, 0**0, 0*0*, *00*, *01*, *0*1, *0*0, **00, **01,
11, **10, *1*0, *1*1, *11*, *10*, 1*0*, 10, 1**1, 1*1*, 11**, 10**).

65. [*M40*] Enumerate the total number of genlex Gray paths on subcubes that use only the transformations allowed in exercise 64. How many of those paths are cycles?

▶ **66.** [*22*] Given $n \geq t \geq 0$, show that there is a Gray path through all of the canonical bases $(\alpha_1, \ldots, \alpha_t)$ of exercise 12, changing just one bit at each step. For example, one such path when $n = 3$ and $t = 2$ is

$$\begin{matrix} 001 \\ 010 \end{matrix},\ \begin{matrix} 101 \\ 010 \end{matrix},\ \begin{matrix} 101 \\ 110 \end{matrix},\ \begin{matrix} 001 \\ 110 \end{matrix},\ \begin{matrix} 001 \\ 100 \end{matrix},\ \begin{matrix} 011 \\ 100 \end{matrix},\ \begin{matrix} 010 \\ 100 \end{matrix}.$$

67. [*46*] Consider the Ising configurations of exercise 13 for which $a_0 = 0$. Given n, t, and r, is there a Gray cycle for these configurations in which all transitions have the forms $0^k1 \leftrightarrow 10^k$ or $01^k \leftrightarrow 1^k0$? For example, in the case $n = 9$, $t = 5$, $r = 6$, there is a unique cycle

(010101110, 010110110, 011010110, 011011010, 011101010, 010111010).

68. [*M01*] If α is a t-combination, what is (a) $\partial^t \alpha$? (b) $\partial^{t+1}\alpha$?

▶ **69.** [*M22*] How large is the smallest set A of t-combinations for which $|\partial A| < |A|$?

70. [*M25*] What is the maximum value of $\kappa_t N - N$, for $N \geq 0$?

71. [*M20*] How many t-cliques can a million-edge graph have?

▶ **72.** [*M22*] Show that if N has the degree-t combinatorial representation (57), there is an easy way to find the degree-s combinatorial representation of the complementary number $M = \binom{s+t}{t} - N$, whenever $N < \binom{s+t}{t}$. Derive (63) as a consequence.

73. [*M23*] (A. J. W. Hilton, 1976.) Let A be a set of s-combinations and B a set of t-combinations, both contained in $U = \{0, \ldots, n-1\}$ where $n \geq s + t$. Show that if A and B are *cross-intersecting*, in the sense that $\alpha \cap \beta \neq \emptyset$ for all $\alpha \in A$ and $\beta \in B$, then so are the sets Q_{Mns} and Q_{Nnt} defined in Theorem K, where $M = |A|$ and $N = |B|$.

74. [*M21*] What are $|\varrho P_{Nt}|$ and $|\varrho Q_{Nnt}|$ in Theorem K?

75. [*M20*] The right-hand side of (60) is not always the degree-$(t-1)$ combinatorial representation of $\kappa_t N$, because $v-1$ might be zero. Show, however, that a positive integer N has at most two representations if we allow $v=0$ in (57), and both of them yield the same value $\kappa_t N$ according to (60). Therefore

$$\kappa_k\kappa_{k+1}\ldots\kappa_t N = \binom{n_t}{k-1} + \binom{n_{t-1}}{k-2} + \cdots + \binom{n_v}{k-1+v-t} \qquad \text{for } 1 \le k \le t.$$

76. [*M20*] Find a simple formula for $\kappa_t(N+1) - \kappa_t N$.

▶ **77.** [*M26*] Prove the following properties of the κ functions by manipulating binomial coefficients, without assuming Theorem K:

a) $\kappa_t(M+N) \le \kappa_t M + \kappa_t N$.

b) $\kappa_t(M+N) \le \max(\kappa_t M, N) + \kappa_{t-1} N$.

Hint: $\binom{m_t}{t} + \cdots + \binom{m_1}{1} + \binom{n_t}{t} + \cdots + \binom{n_1}{1}$ is equal to $\binom{m_t \vee n_t}{t} + \cdots + \binom{m_1 \vee n_1}{1} + \binom{m_t \wedge n_t}{t} + \cdots + \binom{m_1 \wedge n_1}{1}$, where $\vee$ and $\wedge$ denote max and min.

78. [*M22*] Show that Theorem K follows easily from inequality (b) in the previous exercise. Conversely, both inequalities are simple consequences of Theorem K. *Hint:* Any set A of t-combinations can be written $A = A_1 + A_0 0$, where $A_1 = \{\alpha \in A \mid 0 \notin \alpha\}$.

79. [*M23*] Prove that if $t \ge 2$, we have $M \ge \mu_t N$ if and only if $M + \lambda_{t-1} M \ge N$.

80. [*HM26*] (L. Lovász, 1979.) The function $\binom{x}{t}$ increases monotonically from 0 to ∞ as x increases from $t-1$ to ∞; hence we can define

$$\underline{\kappa}_t N = \binom{x}{t-1}, \qquad \text{if } N = \binom{x}{t} \text{ and } x \ge t-1.$$

Prove that $\kappa_t N \ge \underline{\kappa}_t N$ for all integers $t \ge 1$ and $N \ge 0$. *Hint:* Equality holds when x is an integer.

▶ **81.** [*M27*] Show that the minimum shadow sizes in Theorem M are given by (64).

82. [*HM31*] The Takagi function of Fig. 27 is defined for $0 \le x \le 1$ by the formula

$$\tau(x) = \sum_{k=1}^{\infty} \int_0^x r_k(t)\,dt,$$

where $r_k(t) = (-1)^{\lfloor 2^k t \rfloor}$ is the Rademacher function of Eq. 7.2.1.1–(16).

a) Prove that $\tau(x)$ is continuous in the interval $[0\,.\,.\,1]$, but its derivative does not exist at any point.

b) Show that $\tau(x)$ is the only continuous function that satisfies

$$\tau(\tfrac{1}{2}x) \;=\; \tau(1-\tfrac{1}{2}x) \;=\; \tfrac{1}{2}x + \tfrac{1}{2}\tau(x) \qquad \text{for } 0 \le x \le 1.$$

c) What is the asymptotic value of $\tau(\epsilon)$ when ϵ is small?

d) Prove that $\tau(x)$ is rational when x is rational.

e) Find all roots of the equation $\tau(x) = 1/2$.

f) Find all roots of the equation $\tau(x) = \max_{0 \le x \le 1} \tau(x)$.

83. [*HM46*] Determine the set R of all rational numbers r such that the equation $\tau(x) = r$ has uncountably many solutions. If $\tau(x)$ is rational and x is irrational, is it true that $\tau(x) \in R$? (*Warning:* This problem can be addictive.)

84. [*HM27*] If $T = \binom{2t-1}{t}$, prove the asymptotic formula

$$\kappa_t N - N = \frac{T}{t}\left(\tau\left(\frac{N}{T}\right) + O\left(\frac{(\log t)^3}{t}\right)\right) \qquad \text{for } 0 \le N \le T.$$

85. [*HM21*] Relate the functions $\lambda_t N$ and $\mu_t N$ to the Takagi function $\tau(x)$.

86. [*M20*] Prove the law of spread/core duality, $X^{\sim +} = X^{\circ\sim}$.

87. [*M21*] True or false: (a) $X \subseteq Y^\circ$ if and only if $Y^\sim \subseteq X^{\sim\circ}$; (b) $X^{\circ+\circ} = X^\circ$; (c) $\alpha M \le N$ if and only if $M \le \beta N$.

88. [*M20*] Explain why cross order is useful, by completing the proof of Lemma S.

89. [*16*] Compute the α and β functions for the $2 \times 2 \times 3$ torus (69).

90. [*M22*] Prove the basic compression lemma, (85).

91. [*M24*] Prove Theorem W for two-dimensional toruses $T(l, m)$, $l \le m$.

92. [*M28*] Let $x = x_1 \ldots x_{n-1}$ be the Nth element of the torus $T(m_1, \ldots, m_{n-1})$, and let S be the set of all elements of $T(m_1, \ldots, m_{n-1}, m)$ that are $\preceq x_1 \ldots x_{n-1}(m-1)$ in cross order. If N_a elements of S have final component a, for $0 \le a < m$, prove that $N_{m-1} = N$ and $N_{a-1} = \alpha N_a$ for $1 \le a < m$, where α is the spread function for standard sets in $T(m_1, \ldots, m_{n-1})$.

93. [*M25*] (a) Find an N for which the conclusion of Theorem W is false when the parameters $m_1, m_2, \ldots, m_n$ have not been sorted into nondecreasing order. (b) Where does the proof of that theorem use the hypothesis that $m_1 \le m_2 \le \cdots \le m_n$?

94. [*M20*] Show that the ∂ half of Corollary C follows from the ϱ half. *Hint:* The complements of the multicombinations (92) with respect to U are 3211, 3210, 3200, 3110, 3100, 3000, 2110, 2100, 2000, 1100, 1000.

95. [*17*] Explain why Theorems K and M follow from Corollary C.

▶ **96.** [*M22*] If S is an infinite sequence $(s_0, s_1, s_2, \ldots)$ of positive integers, let

$$\binom{S(n)}{k} = [z^k] \prod_{j=0}^{n-1} (1 + z + \cdots + z^{s_j});$$

thus $\binom{S(n)}{k}$ is the ordinary binomial coefficient $\binom{n}{k}$ if $s_0 = s_1 = s_2 = \cdots = 1$.

Generalizing the combinatorial number system, show that every nonnegative integer N has a unique representation

$$N = \binom{S(n_t)}{t} + \binom{S(n_{t-1})}{t-1} + \cdots + \binom{S(n_1)}{1}$$

where $n_t \ge n_{t-1} \ge \cdots \ge n_1 \ge 0$ and $\{n_t, n_{t-1}, \ldots, n_1\} \subseteq \{s_0 \cdot 0, s_1 \cdot 1, s_2 \cdot 2, \ldots\}$. Use this representation to give a simple formula for the numbers $|\partial P_{Nt}|$ in Corollary C.

▶ **97.** [*M26*] The text remarked that the vertices of a convex polyhedron can be perturbed slightly so that all of its faces are simplexes. In general, any set of combinations that contains the shadows of all its elements is called a *simplicial complex*; thus C is a simplicial complex if and only if $\alpha \subseteq \beta$ and $\beta \in C$ implies that $\alpha \in C$, if and only if C is an order ideal with respect to set inclusion.

The *size vector* of a simplicial complex C on n vertices is $(N_0, N_1, \ldots, N_n)$ when C contains exactly N_t combinations of size t.

a) What are the size vectors of the five regular solids (the tetrahedron, cube, octahedron, dodecahedron, and icosahedron), when their vertices are slightly tweaked?

b) Construct a simplicial complex with size vector $(1,4,5,2,0)$.

c) Find a necessary and sufficient condition that a given size vector $(N_0, N_1, \ldots, N_n)$ is feasible.

d) Prove that $(N_0, \ldots, N_n)$ is feasible if and only its "dual" vector $(\overline{N}_0, \ldots, \overline{N}_n)$ is feasible, where we define $\overline{N}_t = \binom{n}{t} - N_{n-t}$.

e) List all feasible size vectors $(N_0, N_1, N_2, N_3, N_4)$ and their duals. Which of them are self-dual?

98. [*30*] Continuing exercise 97, find an efficient way to count the feasible size vectors $(N_0, N_1, \ldots, N_n)$ when $n \le 100$.

99. [*M25*] A *clutter* is a set C of combinations that are incomparable, in the sense that $\alpha \subseteq \beta$ and $\alpha, \beta \in C$ implies $\alpha = \beta$. The size vector of a clutter is defined as in exercise 97.

a) Find a necessary and sufficient condition that $(M_0, M_1, \ldots, M_n)$ is the size vector of a clutter.

b) List all such size vectors in the case $n = 4$.

▸ **100.** [*M30*] (Clements and Lindström.) Let A be a "simplicial multicomplex," a set of submultisets of the multiset U in Corollary C with the property that $\partial A \subseteq A$. How large can the total weight $\nu A = \sum\{|\alpha| \mid \alpha \in A\}$ be when $|A| = N$?

101. [*M25*] If $f(x_1, \ldots, x_n)$ is a Boolean formula, let $F(p)$ be the probability that $f(x_1, \ldots, x_n) = 1$ when each variable x_j independently is 1 with probability p.

a) Calculate $G(p)$ and $H(p)$ for the Boolean formulas $g(w,x,y,z) = wxz \vee wyz \vee xy\bar{z}$, $h(w,x,y,z) = \bar{w}yz \vee xyz$.

b) Show that there is a *monotone* Boolean function $f(w,x,y,z)$ such that $F(p) = G(p)$, but there is no such function with $F(p) = H(p)$. Explain how to test this condition in general.

102. [*HM35*] (F. S. Macaulay, 1927.) A *polynomial ideal* I in the variables $\{x_1 \ldots, x_s\}$ is a set of polynomials closed under the operations of addition, multiplication by a constant, and multiplication by any of the variables. It is called *homogeneous* if it consists of all linear combinations of a set of homogeneous polynomials, namely of polynomials like $xy + z^2$ whose terms all have the same degree. Let N_t be the maximum number of linearly independent elements of degree t in I. For example, if $s = 2$, the set of all $\alpha(x_0, x_1, x_2)(x_0 x_1^2 - 2x_1 x_2^2) + \beta(x_0, x_1, x_2) x_0 x_1 x_2^2$, where α and β run through all possible polynomials in $\{x_0, x_1, x_2\}$, is a homogeneous polynomial ideal with $N_0 = N_1 = N_2 = 0$, $N_3 = 1$, $N_4 = 4$, $N_5 = 9$, $N_6 = 15$, ...

a) Prove that for any such ideal I there is another ideal I' in which all homogeneous polynomials of degree t are linear combinations of N_t independent *monomials*. (A monomial is a product of variables, like $x_1^3 x_2 x_5^4$.)

b) Use Theorem M and (64) to prove that $N_{t+1} \ge N_t + \kappa_s N_t$ for all $t \ge 0$.

c) Show that $N_{t+1} > N_t + \kappa_s N_t$ occurs for only finitely many t. (This statement is equivalent to "Hilbert's basis theorem," proved by David Hilbert in *Göttinger Nachrichten* (1888), 450–457; *Math. Annalen* **36** (1890), 473–534.)

▸ **103.** [*M38*] The shadow of a subcube $a_1 \ldots a_n$, where each a_j is either 0 or 1 or $*$, is obtained by replacing some $*$ by 0 or 1. For example,

$$\partial 0{*}11{*}0 = \{0011{*}0,\ 0111{*}0,\ 0{*}1100,\ 0{*}1110\}.$$

Find a set P_{Nst} such that, if A is any set of N subcubes $a_1 \ldots a_n$ having s digits and t asterisks, $|\partial A| \ge |P_{Nst}|$.

104. [*M41*] The shadow of a binary string $a_1 \ldots a_n$ is obtained by deleting one of its bits. For example,

$$\partial 110010010 = \{10010010, 11010010, 11000010, 11001000, 11001001\}.$$

Find a set P_{Nn} such that, if A is any set of N binary strings $a_1 \ldots a_n$, $|\partial A| \geq |P_{Nn}|$.

105. [*M20*] A *universal cycle of t-combinations* for $\{0, 1, \ldots, n-1\}$ is a cycle of $\binom{n}{t}$ numbers whose blocks of t consecutive elements run through every t-combination $\{c_1, \ldots, c_t\}$. For example,

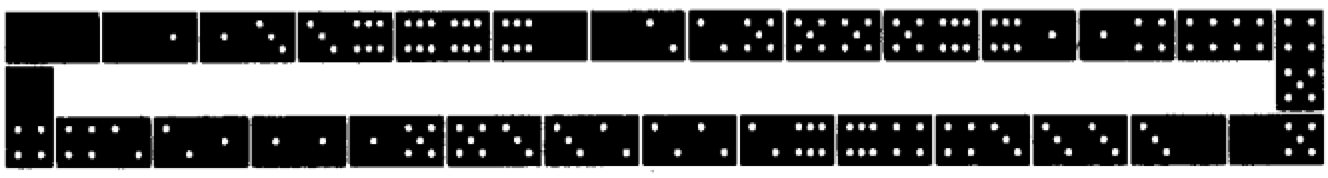
$$(02145061320516243152630425364103546)$$

is a universal cycle when $t = 3$ and $n = 7$.

Prove that no such cycle is possible unless $\binom{n}{t}$ is a multiple of n.

106. [*M21*] (L. Poinsot, 1809.) Find a "nice" universal cycle of 2-combinations for $\{0, 1, \ldots, 2m\}$. *Hint:* Consider the differences of consecutive elements, mod $(2m + 1)$.

107. [*22*] (O. Terquem, 1849.) Poinsot's theorem implies that all 28 dominoes of a traditional "double-six" set can be arranged in a cycle so that the spots of adjacent dominoes match each other:



How many such cycles are possible?

108. [*M31*] Find universal cycles of 3-combinations for the sets $\{0, \ldots, n-1\}$ when $n \bmod 3 \neq 0$.

109. [*M31*] Find universal cycles of 3-*multicombinations* for $\{0, 1, \ldots, n-1\}$ when $n \bmod 3 \neq 0$ (namely for combinations $d_1 d_2 d_3$ with repetitions permitted). For example,

$$(00012241112330222344133340024440113)$$

is such a cycle when $n = 5$.

▸ **110.** [*26*] *Cribbage* is a game played with 52 cards, where each card has a suit (♣, ♢, ♡, or ♠) and a face value (A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, or K). One feature of the game is to compute the score of a 5-card combination $C = \{c_1, c_2, c_3, c_4, c_5\}$, where one card c_k is called the *starter*. The score is the sum of points computed as follows, for each subset S of C and each choice of k: Let $|S| = s$.

i) Fifteens: If $\sum\{v(c) \mid c \in S\} = 15$, where $(v(\mathtt{A}), v(2), v(3), \ldots, v(9), v(10), v(\mathtt{J}), v(\mathtt{Q}), v(\mathtt{K})) = (1, 2, 3, \ldots, 9, 10, 10, 10, 10)$, score two points.

ii) Pairs: If $s = 2$ and both cards have the same face value, score two points.

iii) Runs: If $s \geq 3$ and the face values are consecutive, and if C does not contain a run of length $s + 1$, score s points.

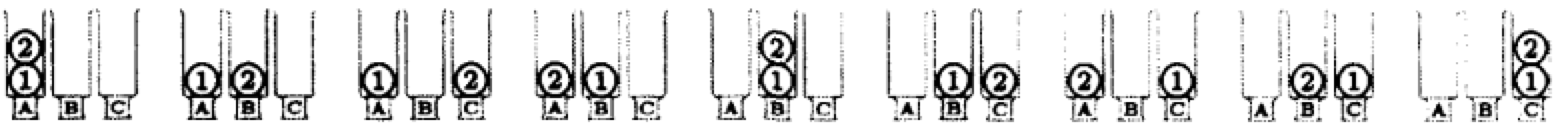
iv) Flushes: If $s = 4$ and all cards of S have the same suit, and if $c_k \notin S$, score $4 + [c_k$ has the same suit as the others$]$.

v) Nobs: If $s = 1$ and $c_k \notin S$, score 1 if the card is J of the same suit as c_k.

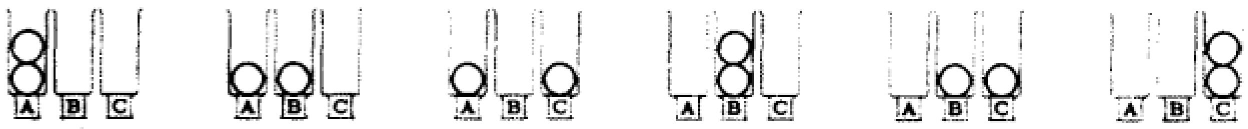
For example, if you hold {J♣, 5♣, 5♢, 6♡} and if 4♣ is the starter, you score 4×2 for fifteens, 2 for a pair, 2×3 for runs, plus 1 for nobs, totalling 17.

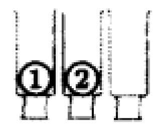
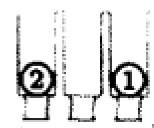
Exactly how many combinations and starter choices lead to a score of x points, for $x = 0, 1, 2, \ldots$?

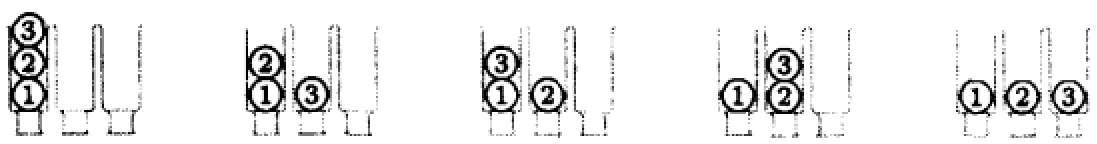
7.2.1.4. Generating all partitions. Richard Stanley's magnificent book *Enumerative Combinatorics* (1986) begins by discussing The Twelvefold Way, a $2 \times 2 \times 3$ array of basic combinatorial problems that arise frequently in practice (see Table 1). All twelve of Stanley's basic problems can be described in terms of the ways that a given number of balls can be placed into a given number of urns. For example, there are nine ways to put 2 balls into 3 urns if the balls and urns are labeled:



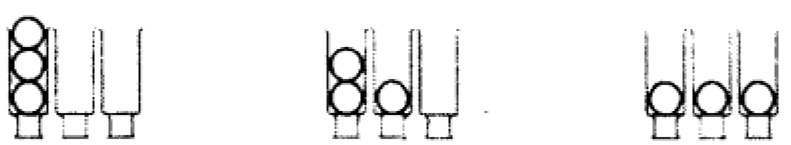
(The order of balls *within* an urn is ignored.) But if the balls are unlabeled, some of these arrangements are indistinguishable, so only six different ways are possible:

 (1)

If the urns are unlabeled, arrangements like  and  are essentially the same, hence only two of the original nine arrangements are distinguishable. And if we have three labeled balls, the only distinct ways to place them into three unlabeled urns are

 (2)

Finally, if neither balls nor urns are labeled, these five possibilities reduce to only three:

 (3)

The Twelvefold Way considers all arrangements that are possible when balls and urns are labeled or unlabeled, and when the urns may optionally be required to contain at least one ball or at most one ball.

Table 1

THE TWELVEFOLD WAY

balls per urn	unrestricted	≤ 1	≥ 1
n labeled balls, m labeled urns	n-tuples of m things	n-permutations of m things	partitions of $\{1,\ldots,n\}$ into m ordered parts
n unlabeled balls, m labeled urns	n-multicombinations of m things	n-combinations of m things	compositions of n into m parts
n labeled balls, m unlabeled urns	partitions of $\{1,\ldots,n\}$ into $\le m$ parts	n pigeons into m holes	partitions of $\{1,\ldots,n\}$ into m parts
n unlabeled balls, m unlabeled urns	partitions of n into $\le m$ parts	n pigeons into m holes	partitions of n into m parts

We've learned about n-tuples, permutations, combinations, and compositions in previous sections of this chapter; and two of the twelve entries in Table 1 are trivial (namely the ones related to "pigeons"). So we can complete our study of classical combinatorial mathematics by learning about the remaining five entries in the table, which all involve *partitions.*

Let us begin by acknowledging that the word "partition"
has numerous meanings in mathematics.
Any time a division of some object into subobjects is undertaken,
the word partition is likely to pop up.

— GEORGE ANDREWS, *The Theory of Partitions* (1976)

Two quite different concepts share the same name: The *partitions of a set* are the ways to subdivide it into disjoint subsets; thus (2) illustrates the five partitions of $\{1,2,3\}$, namely

$$\{1,2,3\}, \qquad \{1,2\}\{3\}, \qquad \{1,3\}\{2\}, \qquad \{1\}\{2,3\}, \qquad \{1\}\{2\}\{3\}. \tag{4}$$

And the *partitions of an integer* are the ways to write it as a sum of positive integers, disregarding order; thus (3) illustrates the three partitions of 3, namely

$$3, \qquad 2+1, \qquad 1+1+1. \tag{5}$$

We shall follow the common practice of referring to integer partitions as simply "partitions," without any qualifying adjective; the other kind will be called "set partitions" in what follows, to make the distinction clear. Both kinds of partitions are important, so we'll study each of them in turn.

Generating all partitions of an integer. A partition of n can be defined formally as a sequence of nonnegative integers $a_1 \ge a_2 \ge \cdots$ such that $n = a_1 + a_2 + \cdots$; for example, one partition of 7 has $a_1 = a_2 = 3$, $a_3 = 1$, and $a_4 = a_5 = \cdots = 0$. The number of nonzero terms is called the number of *parts*, and the zero terms are usually suppressed. Thus we write $7 = 3 + 3 + 1$, or simply 331 to save space when the context is clear.

The simplest way to generate all partitions, and one of the fastest, is to visit them in reverse lexicographic order, starting with 'n' and ending with '$11\ldots1$'. For example, the partitions of 8 are

$$\begin{matrix}8,\ 71,\ 62,\ 611,\ 53,\ 521,\ 5111,\ 44,\ 431,\ 422,\ 4211,\ 41111,\ 332,\ 3311,\\ 3221,\ 32111,\ 311111,\ 2222,\ 22211,\ 221111,\ 2111111,\ 11111111,\end{matrix} \tag{6}$$

when listed in this order.

If a partition isn't all 1s, it ends with $(x+1)$ followed by zero or more 1s, for some $x \ge 1$; therefore the next smallest partition in lexicographic order is obtained by replacing the suffix $(x+1)1\ldots1$ by $x\ldots xr$ for some appropriate remainder $r \le x$. The process is quite efficient if we keep track of the largest subscript q such that $a_q \ne 1$, as suggested by J. K. S. McKay [*CACM* **13** (1970), 52]:

Algorithm P (*Partitions in reverse lexicographic order*). This algorithm generates all partitions $a_1 \geq a_2 \geq \cdots \geq a_m \geq 1$ with $a_1 + a_2 + \cdots + a_m = n$ and $1 \leq m \leq n$, assuming that $n \geq 1$.

P1. [Initialize.] Set $a_m \leftarrow 1$ for $n \geq m > 1$. Then set $a_0 \leftarrow 0$ and $m \leftarrow 1$.

P2. [Store the final part.] Set $a_m \leftarrow n$ and $q \leftarrow m - [n=1]$.

P3. [Visit.] Visit the partition $a_1 a_2 \ldots a_m$. Then go to P5 if $a_q \neq 2$.

P4. [Change 2 to 1+1.] Set $a_q \leftarrow 1$, $q \leftarrow q-1$, $m \leftarrow m+1$, and return to P3. (At this point we have $a_k=1$ for $q<k\leq n$)

P5. [Decrease a_q.] Terminate the algorithm if $q = 0$. Otherwise set $x \leftarrow a_q - 1$, $a_q \leftarrow x$, $n \leftarrow m - q + 1$, and $m \leftarrow q + 1$.

P6. [Copy x if necessary.] If $n \leq x$, return to step P2. Otherwise set $a_m \leftarrow x$, $m \leftarrow m + 1$, $n \leftarrow n - x$, and repeat this step. ▮

Notice that the operation of going from one partition to the next is particularly easy if a 2 is present; then step P4 simply changes the rightmost 2 to a 1 and appends another 1 at the right. This happy situation is, fortunately, the most common case. For example, nearly 79% of all partitions contain a 2 when $n = 100$.

Another simple algorithm is available when we want to generate all partitions of n into a fixed number of parts. The following method, which was featured in C. F. Hindenburg's 18th-century dissertation [*Infinitinomii Dignitatum Exponentis Indeterminati* (Göttingen, 1779), 73–91], visits the partitions in *colex* order, namely in lexicographic order of the reflected sequence $a_m \ldots a_2 a_1$:

Algorithm H (*Partitions into m parts*). This algorithm generates all integer m-tuples $a_1 \ldots a_m$ such that $a_1 \geq \cdots \geq a_m \geq 1$ and $a_1 + \cdots + a_m = n$, assuming that $n \geq m \geq 2$.

H1. [Initialize.] Set $a_1 \leftarrow n - m + 1$ and $a_j \leftarrow 1$ for $1 < j \leq m$. Also set $a_{m+1} \leftarrow -1$.

H2. [Visit.] Visit the partition $a_1 \ldots a_m$. Then go to H4 if $a_2 \geq a_1 - 1$.

H3. [Tweak a_1 and a_2.] Set $a_1 \leftarrow a_1 - 1$, $a_2 \leftarrow a_2 + 1$, and return to H2.

H4. [Find j.] Set $j \leftarrow 3$ and $s \leftarrow a_1 + a_2 - 1$. Then, if $a_j \geq a_1 - 1$, set $s \leftarrow s + a_j$, $j \leftarrow j + 1$, and repeat until $a_j < a_1 - 1$. (Now $s = a_1 + \cdots + a_{j-1} - 1$.)

H5. [Increase a_j.] Terminate if $j > m$. Otherwise set $x \leftarrow a_j + 1$, $a_j \leftarrow x$, $j \leftarrow j - 1$.

H6. [Tweak $a_1 \ldots a_j$.] While $j > 1$, set $a_j \leftarrow x$, $s \leftarrow s - x$, and $j \leftarrow j - 1$. Finally set $a_1 \leftarrow s$ and return to H2. ▮

For example, when $n = 11$ and $m = 4$ the successive partitions visited are

$$8111,\ 7211,\ 6311,\ 5411,\ 6221,\ 5321,\ 4421,\ 4331,\ 5222,\ 4322,\ 3332. \qquad (7)$$

The basic idea is that colex order goes from one partition $a_1 \ldots a_m$ to the next by finding the smallest j such that a_j can be increased without changing $a_{j+1} \ldots a_m$. The new partition $a'_1 \ldots a'_m$ will have $a'_1 \geq \cdots \geq a'_j = a_j + 1$ and $a'_1 + \cdots + a'_j =$

$a_1 + \cdots + a_j$, and these conditions are achievable if and only if $a_j < a_1 - 1$. Furthermore, the smallest such partition $a'_1 \ldots a'_m$ in colex order has $a'_2 = \cdots = a'_j = a_j + 1$.

Step H3 handles the simple case $j = 2$, which is by far the most common. And indeed, the value of j almost always turns out to be quite small; we will prove later that the total running time of Algorithm H is at most a small constant times the number of partitions visited, plus $O(m)$.

Other representations of partitions. We've defined a partition as a sequence of nonnegative integers $a_1a_2\ldots$ with $a_1 \geq a_2 \geq \cdots$ and $a_1 + a_2 + \cdots = n$, but we can also regard it as an n-tuple of nonnegative integers $c_1c_2\ldots c_n$ such that

$$c_1 + 2c_2 + \cdots + nc_n \;=\; n. \tag{8}$$

Here c_j is the number of times the integer j appears in the sequence $a_1a_2\ldots$; for example, the partition 331 corresponds to the counts $c_1 = 1$, $c_2 = 0$, $c_3 = 2$, $c_4 = c_5 = c_6 = c_7 = 0$. The number of parts is then $c_1+c_2+\cdots+c_n$. A procedure analogous to Algorithm P can readily be devised to generate partitions in part-count form; see exercise 5.

We have already seen the part-count representation implicitly in formulas like Eq. 1.2.9–(38), which expresses the symmetric function

$$h_n = \sum_{N\geq d_n\geq\cdots\geq d_2\geq d_1\geq 1} x_{d_1}x_{d_2}\ldots x_{d_n} \tag{9}$$

as

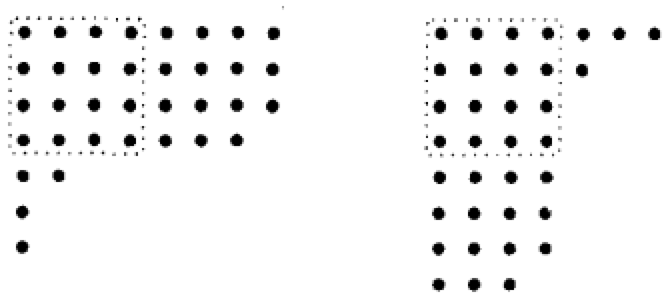
$$\sum_{\substack{c_1,c_2,\ldots,c_n\geq 0\\ c_1+2c_2+\cdots+nc_n=n}} \frac{S_1^{c_1}}{1^{c_1}c_1!}\,\frac{S_2^{c_2}}{2^{c_2}c_2!}\cdots\frac{S_n^{c_n}}{n^{c_n}c_n!}, \tag{10}$$

where S_j is the symmetric function $x_1^j + x_2^j + \cdots + x_N^j$. The sum in (9) is essentially taken over all n-multicombinations of N, while the sum in (10) is taken over all partitions of n. Thus, for example, $h_3 = \frac{1}{6}S_1^3 + \frac{1}{2}S_1S_2 + \frac{1}{3}S_3$, and when $N = 2$ we have

$$x^3 + x^2y + xy^2 + y^3 = \tfrac{1}{6}(x+y)^3 + \tfrac{1}{2}(x+y)(x^2+y^2) + \tfrac{1}{3}(x^3+y^3).$$

Other sums over partitions appear in exercises 1.2.5–21, 1.2.9–10, 1.2.9–11, 1.2.10–12, etc.; for this reason partitions are of central importance in the study of symmetric functions, a class of functions that pervades mathematics in general. [Chapter 7 of Richard Stanley's *Enumerative Combinatorics* **2** (1999) is an excellent introduction to advanced aspects of symmetric function theory.]

Partitions can be visualized in an appealing way by considering an array of n dots, having a_1 dots in the top row and a_2 in the next row, etc. Such an arrangement of dots is called the *Ferrers diagram* of the partition, in honor of N. M. Ferrers [see *Philosophical Mag.* **5** (1853), 199–202]; and the largest square subarray of dots that it contains is called the *Durfee square*, after W. P. Durfee [see *Johns Hopkins Univ. Circular* **2** (December 1882), 23]. For example, the Ferrers diagram of 8887211 is shown with its 4 × 4 Durfee square in Fig. 28(a).



(a) 8887211 (b) 75444443

Fig. 28. The Ferrers diagrams and Durfee squares of two conjugate partitions.

The Durfee square contains k^2 dots when k is the largest subscript such that $a_k \geq k$; we may call k the *trace* of the partition.

If α is any partition $a_1a_2\ldots$, its *conjugate* $\alpha^T = b_1b_2\ldots$ is obtained by transposing the rows and columns of the corresponding Ferrers diagram. For example, Fig. 28(b) shows that $(8887211)^T = 75444443$. When $\beta = \alpha^T$ we obviously have $\alpha = \beta^T$; the partition β has a_1 parts and α has b_1 parts. Indeed, there's a simple relation between the part-count representation $c_1\ldots c_n$ of α and the conjugate partition $b_1b_2\ldots$, namely

$$b_j - b_{j+1} = c_j \qquad \text{for all } j \geq 1. \tag{11}$$

This relation makes it easy to compute the conjugate of a given partition, or to write it down by inspection (see exercise 6).

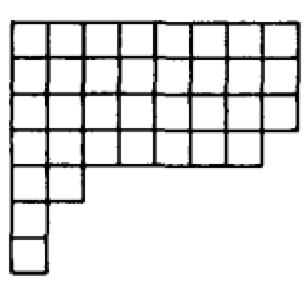
The notion of conjugation often explains properties of partitions that would otherwise be quite mysterious. For example, now that we know the definition of α^T, we can easily see that the value of $j-1$ in step H5 of Algorithm H is just the second-smallest part of the conjugate partition $(a_1\ldots a_m)^T$. Therefore the average amount of work that needs to be done in steps H4 and H6 is essentially proportional to the average size of the second-smallest part of a random partition whose largest part is m. And we will see below that the second-smallest part is almost always quite small.

Moreover, *Algorithm H produces partitions in lexicographic order of their conjugates.* For example, the respective conjugates of (7) are

$$\begin{gathered}41111111,\ 4211111,\ 422111,\ 42221,\ 431111,\\ 43211,\ 4322,\ 4331,\ 44111,\ 4421,\ 443;\end{gathered} \tag{12}$$

these are the partitions of $n = 11$ with largest part 4. One way to generate all partitions of n is to start with the trivial partition 'n', then run Algorithm H for $m = 2, 3, \ldots, n$ in turn; this process yields all α in lexicographic order of α^T (see exercise 7). Thus Algorithm H can be regarded as a dual of Algorithm P.

There is at least one more useful way to represent partitions, called the *rim representation.* Suppose we replace the dots of a Ferrers diagram by boxes, thereby obtaining a tableau shape as we did in Section 5.1.4; for example, the partition 8887211 of Fig. 28(a) becomes

 . (13)

The right-hand boundary of this shape can be regarded as a path from the lower left corner to the upper right corner of an $n \times n$ square, and we know from Table 7.2.1.3–1 that such a path corresponds to an (n, n)-combination.

For example, (13) corresponds to the 70-bit string

$$0\ldots0100101111101000 1\ldots1 \;=\; 0^{28}1^{1}0^{2}1^{1}0^{1}1^{5}0^{1}1^{1}0^{3}1^{27}, \tag{14}$$

where we place enough 0s at the beginning and 1s at the end to make exactly n of each. The 0s represent upward steps of the path, and the 1s represent rightward steps. It is easy to see that the bit string defined in this way has exactly n inversions; conversely, every permutation of the multiset $\{n \cdot 0,\ n \cdot 1\}$ that has exactly n inversions corresponds to a partition of n. When the partition has t different parts, its bit string can be written in the form

$$0^{n-q_1-q_2-\cdots-q_t}1^{p_1}0^{q_1}1^{p_2}0^{q_2}\ldots1^{p_t}0^{q_t}1^{n-p_1-p_2-\cdots-p_t}, \tag{15}$$

where the exponents p_j and q_j are positive integers. Then the partition's standard representation is

$$a_1a_2\ldots \;=\; (p_1+\cdots+p_t)^{q_t}(p_1+\cdots+p_{t-1})^{q_{t-1}}\ldots(p_1)^{q_1}, \tag{16}$$

namely $(1{+}1{+}5{+}1)^3(1{+}1{+}5)^1(1{+}1)^1(1)^2 = 8887211$ in our example.

The number of partitions. Inspired by a question that was posed to him by Philipp Naudé in 1740, Leonhard Euler wrote two fundamental papers in which he counted partitions of various kinds by studying their generating functions [*Commentarii Academiæ Scientiarum Petropolitanæ* **13** (1741), 64–93; *Novi Comment. Acad. Sci. Pet.* **3** (1750), 125–169]. He observed that the coefficient of z^n in the infinite product

$$(1{+}z{+}z^2{+}\cdots{+}z^j{+}\cdots)(1{+}z^2{+}z^4{+}\cdots{+}z^{2k}{+}\cdots)(1{+}z^3{+}z^6{+}\cdots{+}z^{3l}{+}\cdots)\ldots$$

is the number of nonnegative integer solutions to the equation $j{+}2k{+}3l{+}\cdots = n$; and $1 + z^m + z^{2m} + \cdots$ is $1/(1-z^m)$. Therefore if we write

$$P(z) \;=\; \prod_{m=1}^{\infty}\frac{1}{1-z^m} \;=\; \sum_{n=0}^{\infty}p(n)z^n, \tag{17}$$

the number of partitions of n is $p(n)$. This function $P(z)$ turns out to have an amazing number of subtle mathematical properties.

For example, Euler discovered that massive cancellation occurs when the denominator of $P(z)$ is multiplied out:

$$\begin{aligned}(1{-}z)(1{-}z^2)(1{-}z^3)\ldots &= 1 - z - z^2 + z^5 + z^7 - z^{12} - z^{15} + z^{22} + z^{26} - \cdots\\ &= \sum_{-\infty<n<\infty}(-1)^n z^{(3n^2+n)/2}.\end{aligned} \tag{18}$$

A combinatorial proof of this remarkable identity, based on Ferrers diagrams, appears in exercise 5.1.1–14; we can also prove it by setting $u = z$ and $v = z^2$ in the even more remarkable identity of Jacobi,

$$\prod_{k=1}^{\infty}(1-u^k v^{k-1})(1-u^{k-1}v^k)(1-u^k v^k) \;=\; \sum_{n=-\infty}^{\infty}(-1)^n u^{\binom{n}{2}} v^{\binom{-n}{2}}, \qquad (19)$$

because the left-hand side becomes $\prod_{k=1}^{\infty}(1-z^{3k-2})(1-z^{3k-1})(1-z^{3k})$; see exercise 5.1.1–20. Euler's identity (18) implies that the partition numbers satisfy the recurrence

$$p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7) + p(n-12) + p(n-15) - \cdots, \qquad (20)$$

from which we can compute their values more rapidly than by performing the power series calculations in (17):

$n =$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$p(n) =$	1	1	2	3	5	7	11	15	22	30	42	56	77	101	135	176

We know from Section 1.2.8 that solutions to the Fibonacci recurrence $f(n) = f(n-1) + f(n-2)$ grow exponentially, with $f(n) = \Theta(\phi^n)$ when $f(0)$ and $f(1)$ are positive. The additional terms '$-\,p(n-5) - p(n-7)$' in (20) have a dampening effect on partition numbers, however; in fact, if we were to stop the recurrence there, the resulting sequence would oscillate between positive and negative values. Further terms '$+\,p(n-12) + p(n-15)$' reinstate exponential growth.

The actual growth rate of $p(n)$ turns out to be of order $A^{\sqrt{n}}/n$ for a certain constant A. For example, exercise 33 proves directly that $p(n)$ grows at least as fast as $e^{2\sqrt{n}}/n$. And one fairly easy way to obtain a decent *upper* bound is to take logarithms in (17),

$$\ln P(z) \;=\; \sum_{m=1}^{\infty} \ln\frac{1}{1-z^m} \;=\; \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{z^{mn}}{n}, \qquad (21)$$

and then to look at the behavior near $z = 1$ by setting $z = e^{-t}$:

$$\ln P(e^{-t}) = \sum_{m,n\ge 1}\frac{e^{-mnt}}{n} = \sum_{n\ge 1}\frac{1}{n}\,\frac{1}{e^{tn}-1} < \sum_{n\ge 1}\frac{1}{n^2 t} = \frac{\zeta(2)}{t}. \qquad (22)$$

Consequently, since $p(n) \le p(n+1) < p(n+2) < \cdots$ and $e^t > 1$, we have

$$\frac{p(n)}{1-e^{-t}} \;<\; \sum_{k=0}^{\infty} p(k)e^{(n-k)t} \;=\; e^{nt}P(e^{-t}) \;<\; e^{nt+\zeta(2)/t} \qquad (23)$$

for all $t > 0$. Setting $t = \sqrt{\zeta(2)/n}$ gives

$$p(n) \;<\; Ce^{2C\sqrt{n}}/\sqrt{n}, \qquad \text{where } C = \sqrt{\zeta(2)} = \pi/\sqrt{6}. \qquad (24)$$

We can obtain more accurate information about the size of $\ln P(e^{-t})$ by using Euler's summation formula (Section 1.2.11.2) or Mellin transforms (Section 5.2.2); see exercise 25. But the methods we have seen so far aren't powerful enough to deduce the precise behavior of $P(e^{-t})$, so it is time for us to add a new weapon to our arsenal of techniques.

Euler's generating function $P(z)$ is ideally suited to the *Poisson summation formula* [*J. École Royale Polytechnique* **12** (1823), 404–509, §63], according to which

$$\sum_{n=-\infty}^{\infty} f(n+\theta) \;=\; \lim_{M\to\infty} \sum_{m=-M}^{M} e^{2\pi mi\theta}\int_{-\infty}^{\infty} e^{-2\pi miy} f(y)\,dy, \qquad (25)$$

whenever f is a "well-behaved" function. This formula is based on the fact that the left-hand side is a periodic function of θ, and the right-hand side is the expansion of that function as a Fourier series. The function f is sufficiently nice if, for example, $\int_{-\infty}^{\infty}\bigl|f(y)\bigr|\,dy < \infty$ and either

i) $f(n+\theta)$ is an analytic function of the complex variable θ in the region $|\Im\theta| \le \epsilon$ for some $\epsilon > 0$ and $0 \le \Re\theta \le 1$, and the left-hand side converges uniformly in that rectangle; or

ii) $f(\theta) = \frac{1}{2}\lim_{\epsilon\to 0}\bigl(f(\theta-\epsilon)+f(\theta+\epsilon)\bigr) = g(\theta) - h(\theta)$ for all real numbers θ, where g and h are monotone increasing and $g(\pm\infty)$, $h(\pm\infty)$ are finite.

[See Peter Henrici, *Applied and Computational Complex Analysis* **2** (New York: Wiley, 1977), Theorem 10.6.2.] Poisson's formula is not a panacea for summation problems of every kind; but when it does apply the results can be spectacular, as we will see.

Let us multiply Euler's formula (18) by $z^{1/24}$ in order to "complete the square":

$$\frac{z^{1/24}}{P(z)} \;=\; \sum_{n=-\infty}^{\infty} (-1)^n\, z^{\frac{3}{2}(n+\frac{1}{6})^2}. \qquad (26)$$

Then for all $t > 0$ we have $e^{-t/24}/P(e^{-t}) = \sum_{n=-\infty}^{\infty} f(n)$, where

$$f(y) \;=\; e^{-\frac{3}{2}t(y+\frac{1}{6})^2}\cos \pi y; \qquad (27)$$

and this function f qualifies for Poisson's summation formula under both of the criteria (i) and (ii) stated above. Therefore we can try to integrate $e^{-2\pi miy}f(y)$, and for $m = 0$ the result is

$$\int_{-\infty}^{\infty} f(y)\,dy \;=\; \sqrt{\frac{\pi}{2t}}\,e^{-\pi^2/6t}. \qquad (28)$$

To this we must add

$$\sum_{m=1}^{\infty}\int_{-\infty}^{\infty}(e^{2\pi miy}+e^{-2\pi miy})\,f(y)\,dy \;=\; 2\sum_{m=1}^{\infty}\int_{-\infty}^{\infty} f(y)\cos 2\pi my\,dy; \qquad (29)$$

again the integral turns out to be doable. And the results (see exercise 27) fit together quite beautifully, giving

$$\frac{e^{-t/24}}{P(e^{-t})} \;=\; \sqrt{\frac{2\pi}{t}}\sum_{n=-\infty}^{\infty}(-1)^n e^{-6\pi^2(n+\frac{1}{6})^2/t} \;=\; \sqrt{\frac{2\pi}{t}}\,\frac{e^{-\pi^2/6t}}{P(e^{-4\pi^2/t})}. \qquad (30)$$

Surprise! We have proved another remarkable fact about $P(z)$:

Theorem D. *The generating function* (17) *for partitions satisfies the functional relation*

$$\ln P(e^{-t}) \;=\; \frac{\zeta(2)}{t}+\frac{1}{2}\ln\frac{t}{2\pi}-\frac{t}{24}+\ln P(e^{-4\pi^2/t}) \tag{31}$$

when $\Re t > 0$. ▌

This theorem was discovered by Richard Dedekind [*Crelle* **83** (1877), 265–292, §6], who wrote $\eta(\tau)$ for the function $z^{1/24}/P(z)$ when $z = e^{2\pi i\tau}$; his proof was based on a much more complicated theory of elliptic functions. Notice that when t is a small positive number, $\ln P(e^{-4\pi^2/t})$ is *extremely* tiny; for example, when $t = 0.1$ we have $\exp(-4\pi^2/t) \approx 3.5\times 10^{-172}$. Therefore Theorem D tells us essentially everything we need to know about the value of $P(z)$ when z is near 1.

G. H. Hardy and S. Ramanujan used this knowledge to deduce the asymptotic behavior of $p(n)$ for large n, and their work was extended many years later by Hans Rademacher, who discovered a series that is not only asymptotic but convergent [*Proc. London Math. Soc.* (2) **17** (1918), 75–115; **43** (1937), 241–254]. The Hardy–Ramanujan–Rademacher formula for $p(n)$ is surely one of the most astonishing identities ever discovered; it states that

$$p(n) \;=\; \frac{\pi}{2^{5/4}3^{3/4}(n-1/24)^{3/4}}\sum_{k=1}^{\infty}\frac{A_k(n)}{k}\,I_{3/2}\biggl(\sqrt{\frac{2}{3}}\,\frac{\pi}{k}\sqrt{n-1/24}\biggr). \tag{32}$$

Here $I_{3/2}$ denotes the modified spherical Bessel function

$$I_{3/2}(z) \;=\; \Bigl(\frac{z}{2}\Bigr)^{3/2}\sum_{k=0}^{\infty}\frac{1}{\Gamma(k+5/2)}\frac{(z^2/4)^k}{k!} \;=\; \sqrt{\frac{2z}{\pi}}\Bigl(\frac{\cosh z}{z}-\frac{\sinh z}{z^2}\Bigr); \tag{33}$$

and the coefficient $A_k(n)$ is defined by the formula

$$A_k(n) \;=\; \sum_{h=0}^{k-1}[h\perp k]\exp\Bigl(2\pi i\Bigl(\frac{\sigma(h,k,0)}{24}-\frac{nh}{k}\Bigr)\Bigr) \tag{34}$$

where $\sigma(h,k,0)$ is the Dedekind sum defined in Eq. 3.3.3–(16). We have

$$A_1(n)=1, \qquad A_2(n)=(-1)^n, \qquad A_3(n)=2\cos\frac{(24n+1)\pi}{18}, \tag{35}$$

and in general $A_k(n)$ lies between $-k$ and k.

A proof of (32) would take us far afield, but the basic idea is to use the "saddle point method" discussed in Section 7.2.1.5. The term for $k=1$ is derived from the behavior of $P(z)$ when z is near 1; and the next term is derived from the behavior when z is near -1, where a transformation similar to (31) can be applied. In general, the kth term of (32) takes account of the way $P(z)$ behaves when z approaches $e^{2\pi ih/k}$ for irreducible fractions h/k with denominator k; every kth root of unity is a pole of each of the factors $1/(1-z^k)$, $1/(1-z^{2k})$, $1/(1-z^{3k})$, … in the infinite product for $P(z)$.

The leading term of (32) can be simplified greatly, if we merely want a rough approximation:

$$p(n) = \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\bigl(1 + O(n^{-1/2})\bigr). \tag{36}$$

Or, if we choose to retain a few more details,

$$p(n) = \frac{e^{\pi\sqrt{2n'/3}}}{4n'\sqrt{3}}\left(1 - \frac{1}{\pi}\sqrt{\frac{3}{2n'}}\right)\left(1 + O\bigl(e^{-\pi\sqrt{n/6}}\bigr)\right), \quad n' = n - \frac{1}{24}. \tag{37}$$

For example, $p(100)$ has the exact value 190,569,292; formula (36) tells us that $p(100) \approx 1.993 \times 10^8$, while (37) gives the far better estimate 190,568,944.783.

Andrew Odlyzko has observed that, when n is large, the Hardy–Ramanujan–Rademacher formula actually gives a near-optimum way to compute the precise value of $p(n)$, because the arithmetic operations can be carried out in nearly $O(\log p(n)) = O(n^{1/2})$ steps. The first few terms of (32) give the main contribution; then the series settles down to terms that are of order $k^{-3/2}$ and usually of order k^{-2}. Furthermore, about half of the coefficients $A_k(n)$ turn out to be zero (see exercise 28). For example, when $n = 10^6$, the terms for $k = 1$, 2, and 3 are $\approx 1.47 \times 10^{1107}$, 1.23×10^{550}, and -1.23×10^{364}, respectively. The sum of the first 250 terms is $\approx$ 1471684986...73818.01, while the true value is 1471684986...73818; and 123 of those 250 terms are zero.

The number of parts. It is convenient to introduce the notation

$$\left|{n \atop m}\right| \tag{38}$$

for the number of partitions of n that have exactly m parts. Then the recurrence

$$\left|{n \atop m}\right| = \left|{n-1 \atop m-1}\right| + \left|{n-m \atop m}\right| \tag{39}$$

holds for all integers m and n, because $\left|{n-1 \atop m-1}\right|$ counts the partitions whose smallest part is 1 and $\left|{n-m \atop m}\right|$ counts the others. (If the smallest part is 2 or more, we can subtract 1 from each part and get a partition of $n-m$ into m parts.) By similar reasoning we can conclude that $\left|{m+n \atop m}\right|$ is the number of partitions of n into *at most* m parts, namely into m nonnegative summands. We also know, by considering Ferrers diagrams, that $\left|{n \atop m}\right|$ is the number of partitions of n whose *largest* part is m. Thus $\left|{n \atop m}\right|$ is a good number to know. The boundary conditions

$$\left|{n \atop 0}\right| = \delta_{n0} \qquad \text{and} \qquad \left|{n \atop m}\right| = 0 \quad \text{for } m < 0 \text{ or } n < 0 \tag{40}$$

make it easy to tabulate $\left|{n \atop m}\right|$ for small values of the parameters, and we obtain an array of numbers analogous to the familiar triangles for $\binom{n}{m}$, $\left[{n \atop m}\right]$, $\left\{{n \atop m}\right\}$, and $\left\langle{n \atop m}\right\rangle$ that we've seen before; see Table 2. The generating function is

$$\sum_n \left|{n \atop m}\right| z^n = \frac{z^m}{(1-z)(1-z^2)\ldots(1-z^m)}. \tag{41}$$

Table 2

PARTITION NUMBERS

n	$\left\|{n \atop 0}\right\|$	$\left\|{n \atop 1}\right\|$	$\left\|{n \atop 2}\right\|$	$\left\|{n \atop 3}\right\|$	$\left\|{n \atop 4}\right\|$	$\left\|{n \atop 5}\right\|$	$\left\|{n \atop 6}\right\|$	$\left\|{n \atop 7}\right\|$	$\left\|{n \atop 8}\right\|$	$\left\|{n \atop 9}\right\|$	$\left\|{n \atop 10}\right\|$	$\left\|{n \atop 11}\right\|$
0	1	0	0	0	0	0	0	0	0	0	0	0
1	0	1	0	0	0	0	0	0	0	0	0	0
2	0	1	1	0	0	0	0	0	0	0	0	0
3	0	1	1	1	0	0	0	0	0	0	0	0
4	0	1	2	1	1	0	0	0	0	0	0	0
5	0	1	2	2	1	1	0	0	0	0	0	0
6	0	1	3	3	2	1	1	0	0	0	0	0
7	0	1	3	4	3	2	1	1	0	0	0	0
8	0	1	4	5	5	3	2	1	1	0	0	0
9	0	1	4	7	6	5	3	2	1	1	0	0
10	0	1	5	8	9	7	5	3	2	1	1	0
11	0	1	5	10	11	10	7	5	3	2	1	1

Almost all partitions of n have $\Theta(\sqrt{n}\log n)$ parts. This fact, discovered by P. Erdős and J. Lehner [*Duke Math. J.* **8** (1941), 335–345], has a very instructive proof:

Theorem E. *Let $C = \pi/\sqrt{6}$ and $m = \frac{1}{2C}\sqrt{n}\ln n + x\sqrt{n} + O(1)$. Then*

$$\frac{1}{p(n)}\left|{m+n \atop m}\right| = F(x)\bigl(1 + O(n^{-1/2+\epsilon})\bigr) \tag{42}$$

for all $\epsilon > 0$ and all fixed x as $n \to \infty$, where

$$F(x) = e^{-e^{-Cx}/C}. \tag{43}$$

This function $F(x)$ approaches 0 quite rapidly when $x \to -\infty$, and it rapidly increases to 1 when $x \to +\infty$; so it is a probability distribution function. Figure 29(b) shows that the corresponding density function $f(x) = F'(x)$ is largely concentrated in the region $-2 \le x \le 4$. The values of $\left|{n \atop m}\right| = \left|{m+n \atop m}\right| - \left|{m-1+n \atop m-1}\right|$ are shown in Fig. 29(a) for comparison when $n = 100$; in this case $\frac{1}{2C}\sqrt{n}\ln n \approx 18$.

Proof. We will use the fact that $\left|{m+n \atop m}\right|$ is the number of partitions of n whose largest part is $\le m$. Then, by the principle of inclusion and exclusion, Eq. 1.3.3–(29), we have

$$\left|{m+n \atop m}\right| = p(n) - \sum_{j>m} p(n-j) + \sum_{j_2>j_1>m} p(n-j_1-j_2) - \sum_{j_3>j_2>j_1>m} p(n-j_1-j_2-j_3) + \cdots,$$

because $p(n-j_1-\cdots-j_r)$ is the number of partitions of n that use each of the parts $\{j_1,\ldots,j_r\}$ at least once. Let us write this as

$$\frac{1}{p(n)}\left|{m+n \atop m}\right| = 1 - \Sigma_1 + \Sigma_2 - \Sigma_3 + \cdots, \quad \Sigma_r = \sum_{j_r>\cdots>j_1>m} \frac{p(n-j_1-\cdots-j_r)}{p(n)}. \tag{44}$$

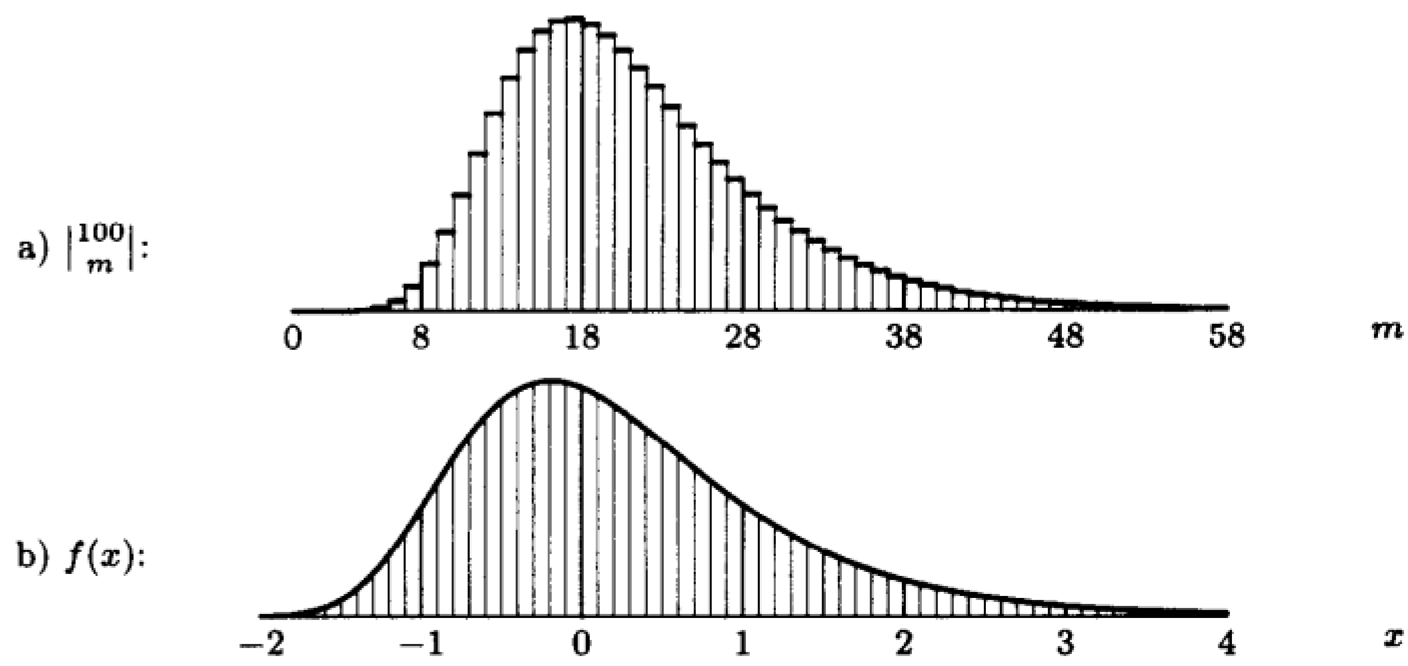


Fig. 29. Partitions of n with m parts, when (a) $n = 100$; (b) $n \to \infty$. (See Theorem E.)

In order to evaluate Σ_r we need to have a good estimate of the ratio $p(n-t)/p(n)$. And we're in luck, because Eq. (36) implies that

$$\begin{aligned}\frac{p(n-t)}{p(n)} &= \exp\bigl(2C\sqrt{n-t} - \ln(n-t) + O\bigl((n-t)^{-1/2}\bigr) - 2C\sqrt{n} + \ln n\bigr)\\ &= \exp\bigl(-Ctn^{-1/2} + O(n^{-1/2+2\epsilon})\bigr) \qquad \text{if } 0 \le t \le n^{1/2+\epsilon}. \qquad (45)\end{aligned}$$

Furthermore, if $t \ge n^{1/2+\epsilon}$ we have $p(n-t)/p(n) \le p(n - n^{1/2+\epsilon})/p(n) \approx \exp(-Cn^{\epsilon})$, a value that is asymptotically smaller than any power of n. Therefore we may safely use the approximation

$$\frac{p(n-t)}{p(n)} \approx \alpha^t, \qquad \alpha = \exp(-Cn^{-1/2}), \qquad (46)$$

for all values of $t \ge 0$. For example, we have

$$\begin{aligned}\Sigma_1 = \sum_{j>m} \frac{p(n-j)}{p(n)} &= \frac{\alpha^{m+1}}{1-\alpha}\bigl(1 + O(n^{-1/2+2\epsilon})\bigr) + \sum_{n \ge j > n^{1/2+\epsilon}} \frac{p(n-j)}{p(n)}\\ &= \frac{e^{-Cx}}{C}\bigl(1 + O(n^{-1/2+2\epsilon})\bigr) + O(ne^{-Cn^{\epsilon}}),\end{aligned}$$

because $\alpha/(1-\alpha) = n^{1/2}/C + O(1)$ and $\alpha^m = n^{-1/2}e^{-Cx}$. A similar argument (see exercise 36) proves that, if $r = O(\log n)$,

$$\Sigma_r = \frac{e^{-Crx}}{C^r r!}\bigl(1 + O(n^{-1/2+2\epsilon})\bigr) + O(e^{-n^{\epsilon/2}}). \qquad (47)$$

Finally — and this is a wonderful property of the inclusion-exclusion principle in general — the partial sums of (44) always "bracket" the true value, in the sense that

$$1 - \Sigma_1 + \Sigma_2 - \cdots - \Sigma_{2r-1} \le \frac{1}{p(n)}\left|{m+n \atop m}\right| \le 1 - \Sigma_1 + \Sigma_2 - \cdots - \Sigma_{2r-1} + \Sigma_{2r} \quad (48)$$

for all r. (See exercise 37.) When $2r$ is near $\ln n$ and n is large, the term Σ_{2r} is extremely tiny; therefore we obtain (42), except with 2ϵ in place of ϵ. ∎

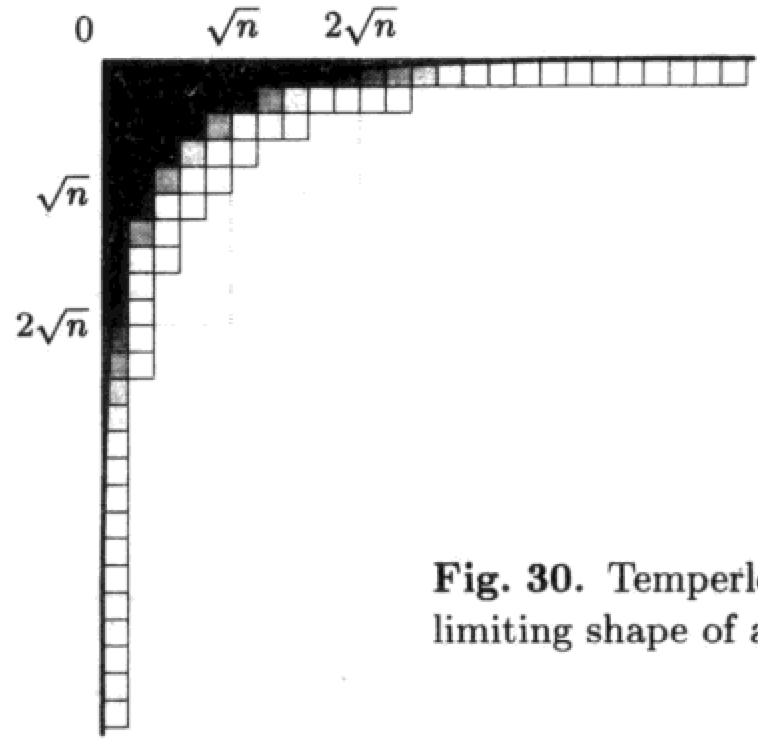


Fig. 30. Temperley's curve (49) for the limiting shape of a random partition.

Theorem E tells us that the largest part of a random partition almost always is $\frac{1}{2C}\sqrt{n}\ln n + O(\sqrt{n}\,)$, and when n is reasonably large the other parts tend to be predictable as well. Suppose, for example, that we take all the partitions of 25 and superimpose their Ferrers diagrams, changing dots to boxes as in the rim representation. Which cells are occupied most often? Figure 30 shows the result: A random partition tends to have a typical shape that approaches a limiting curve as $n \to \infty$.

H. N. V. Temperley [*Proc. Cambridge Philos. Soc.* **48** (1952), 683–697] gave heuristic reasons to believe that most parts a_k of a large random partition $a_1 \ldots a_m$ will satisfy the approximate law

$$e^{-Ck/\sqrt{n}} + e^{-Ca_k/\sqrt{n}} \approx 1, \tag{49}$$

and his formula has subsequently been verified in a strong form. For example, a theorem of Boris Pittel [*Advances in Applied Math.* **18** (1997), 432–488] allows us to conclude that the trace of a random partition is almost always $\frac{\ln 2}{C}\sqrt{n} \approx 0.54\sqrt{n}$, in accordance with (49), with an error of at most $O(\sqrt{n}\ln n)^{1/2}$; thus about 29% of all the Ferrers dots tend to lie in the Durfee square.

If, on the other hand, we look only at partitions of n with m parts, where m is fixed, the limiting shape is rather different: Almost all such partitions have

$$a_k \approx \frac{n}{m}\ln\frac{m}{k}, \tag{50}$$

if m is reasonably large. Figure 31 illustrates the case $n = 50$, $m = 5$. In fact, the same limit holds when m grows with n, but at a slower rate than $\sqrt{n}$ [see Vershik and Yakubovich, *Moscow Math. J.* **1** (2001), 457–468].

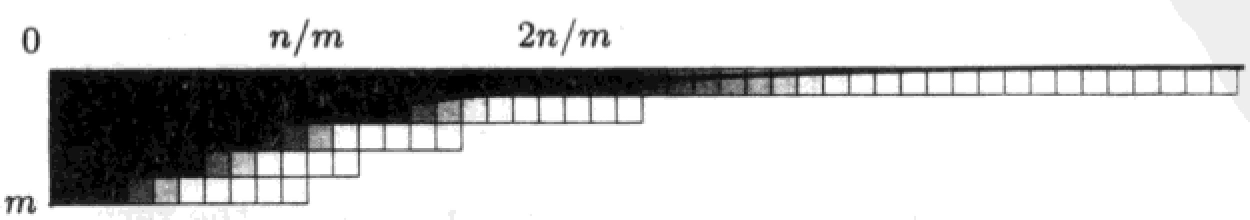


Fig. 31. The limiting shape (50) when there are m parts.

The rim representation of partitions gives us further information about partitions that are *doubly* bounded, in the sense that we not only restrict the number of parts but also the size of each part. A partition that has at most m parts, each of size at most l, fits inside an $m \times l$ box. All such partitions correspond to permutations of the multiset $\{m \cdot 0, l \cdot 1\}$ that have exactly n inversions, and we have studied the inversions of multiset permutations in exercise 5.1.2–16. In particular, that exercise derives a nonobvious formula for the number of ways n inversions can happen:

Theorem C. *The number of partitions of n that have no more than m parts and no part larger than l is*

$$[z^n]\binom{l+m}{m}_z = [z^n]\frac{(1-z^{l+1})}{(1-z)}\frac{(1-z^{l+2})}{(1-z^2)}\cdots\frac{(1-z^{l+m})}{(1-z^m)}. \qquad (51)$$

This result is due to A. Cauchy, *Comptes Rendus Acad. Sci.* **17** (Paris, 1843), 523–531. Notice that when $l \to \infty$ the numerator becomes simply 1. An interesting combinatorial proof of a more general result appears in exercise 39 below. ▌

Analysis of the algorithms. Now we know more than enough about the quantitative aspects of partitions to deduce the behavior of Algorithm P quite precisely. Suppose steps P1, ..., P6 of that algorithm are executed respectively $T_1(n), \ldots, T_6(n)$ times. We obviously have $T_1(n) = 1$ and $T_3(n) = p(n)$; furthermore Kirchhoff's law tells us that $T_2(n) = T_5(n)$ and $T_4(n) + T_5(n) = T_3(n)$. We get to step P4 once for each partition that contains a 2; and this is clearly $p(n-2)$.

Thus the only possible mystery about the running time of Algorithm P is the number of times we must perform step P6, which loops back to itself. A moment's thought, however, reveals that the algorithm stores a value ≥ 2 into the array $a_1 a_2 \ldots$ only in steps P2 and P6; and every such value is eventually decreased by 1, either in step P4 or step P5. Therefore

$$T_2''(n) + T_6(n) = p(n) - 1, \qquad (52)$$

where $T_2''(n)$ is the number of times step P2 sets a_m to a value ≥ 2. Let $T_2(n) = T_2'(n) + T_2''(n)$, so that $T_2'(n)$ is the number of times step P2 sets $a_m \leftarrow 1$. Then $T_2'(n) + T_4(n)$ is the number of partitions that end in 1, hence

$$T_2'(n) + T_4(n) = p(n-1). \qquad (53)$$

Aha! We've found enough equations to determine all of the required quantities:

$$(T_1(n), \ldots, T_6(n)) = \\ (1,\ p(n)-p(n-2),\ p(n),\ p(n-2),\ p(n)-p(n-2),\ p(n-1)-1). \qquad (54)$$

And from the asymptotics of $p(n)$ we also know the average amount of computation per partition:

$$\left(\frac{T_1(n)}{p(n)}, \ldots, \frac{T_6(n)}{p(n)}\right) = \left(0, \frac{2C}{\sqrt{n}}, 1, 1-\frac{2C}{\sqrt{n}}, \frac{2C}{\sqrt{n}}, 1-\frac{C}{\sqrt{n}}\right) + O\left(\frac{1}{n}\right), \qquad (55)$$

where $C = \pi/\sqrt{6} \approx 1.283$. (See exercise 45.) The total number of memory accesses per partition therefore comes to only $4 - 3C/\sqrt{n} + O(1/n)$.

Whoever wants to go about generating all partitions
not only immerses himself in immense labor,
but also must take pains to keep fully attentive,
so as not to be grossly deceived.

— LEONHARD EULER, *De Partitione Numerorum* (1750)

Algorithm H is more difficult to analyze, but we can at least prove a decent upper bound on its running time. The key quantity is the value of j, the smallest subscript for which $a_j < a_1 - 1$. The successive values of j when $m = 4$ and $n = 11$ are $(2, 2, 2, 3, 2, 2, 3, 4, 2, 3, 5)$, and we have observed that $j = b_{l-1} + 1$ when $b_1 \ldots b_l$ is the conjugate partition $(a_1 \ldots a_m)^T$. (See (7) and (12).) Step H3 singles out the case $j = 2$, because this case is not only the most common, it is also especially easy to handle.

Let $c_m(n)$ be the accumulated total value of $j - 1$, summed over all of the $\left|{n \atop m}\right|$ partitions generated by Algorithm H. For example, $c_4(11) = 1 + 1 + 1 + 2 + 1 + 1 + 2 + 3 + 1 + 2 + 4 = 19$. We can regard $c_m(n)/\left|{n \atop m}\right|$ as a good indication of the running time per partition, because the time to perform the most costly steps, H4 and H6, is roughly proportional to $j - 2$. This ratio $c_m(n)/\left|{n \atop m}\right|$ is *not* bounded, because $c_m(m) = m$ while $\left|{m \atop m}\right| = 1$. But the following theorem shows that Algorithm H is efficient nonetheless:

Theorem H. *The cost measure $c_m(n)$ for Algorithm H is at most $3\left|{n \atop m}\right| + m$.*

Proof. We can readily verify that $c_m(n)$ satisfies the same recurrence as $\left|{n \atop m}\right|$, namely

$$c_m(n) \;=\; c_{m-1}(n-1) \;+\; c_m(n-m), \qquad \text{for } m, n \ge 1, \tag{56}$$

if we artificially define $c_m(n) = 1$ when $1 \le n < m$; see (39). But the boundary conditions are now different:

$$c_m(0) = [m > 0]; \qquad\qquad c_0(n) = 0. \tag{57}$$

Table 3 shows how $c_m(n)$ behaves when m and n are small.

To prove the theorem, we will actually prove a stronger result,

$$c_m(n) \;\le\; 3\left|{n \atop m}\right| + 2m - n - 1 \qquad \text{for } n \ge m \ge 2. \tag{58}$$

Exercise 50 shows that this inequality holds when $m \le n \le 2m$, so the proof will be complete if we can prove it when $n > 2m$. In the latter case we have

$$\begin{aligned} c_m(n) &= c_1(n-m) + c_2(n-m) + c_3(n-m) + \cdots + c_m(n-m) \\ &\le 1 + \left(3\left|{n-m \atop 2}\right| + 3-n+m\right) + \left(3\left|{n-m \atop 3}\right| + 5-n+m\right) + \cdots \\ &\qquad\qquad + \left(3\left|{n-m \atop m}\right| + 2m-1-n+m\right) \\ &= 3\left|{n-m \atop 1}\right| + 3\left|{n-m \atop 2}\right| + \cdots + 3\left|{n-m \atop m}\right| - 3 + m^2 - (m-1)(n-m) \\ &= 3\left|{n \atop m}\right| + 2m^2 - m - (m-1)n - 3 \end{aligned}$$

by induction; and $2m^2 - m - (m-1)n - 3 \le 2m - n - 1$ because $n \ge 2m + 1$. ∎

Table 3

COSTS IN ALGORITHM H

n	$c_0(n)$	$c_1(n)$	$c_2(n)$	$c_3(n)$	$c_4(n)$	$c_5(n)$	$c_6(n)$	$c_7(n)$	$c_8(n)$	$c_9(n)$	$c_{10}(n)$	$c_{11}(n)$
0	0	1	1	1	1	1	1	1	1	1	1	1
1	0	1	1	1	1	1	1	1	1	1	1	1
2	0	1	2	1	1	1	1	1	1	1	1	1
3	0	1	2	3	1	1	1	1	1	1	1	1
4	0	1	3	3	4	1	1	1	1	1	1	1
5	0	1	3	4	4	5	1	1	1	1	1	1
6	0	1	4	6	5	5	6	1	1	1	1	1
7	0	1	4	7	7	6	6	7	1	1	1	1
8	0	1	5	8	11	8	7	7	8	1	1	1
9	0	1	5	11	12	12	9	8	8	9	1	1
10	0	1	6	12	16	17	13	10	9	9	10	1
11	0	1	6	14	19	21	18	14	11	10	10	11

***A Gray code for partitions.** When partitions are generated in part-count form $c_1 \ldots c_n$ as in exercise 5, at most four of the c_j values change at each step. But we might prefer to minimize the changes to the individual parts, generating partitions in such a way that the successor of $a_1 a_2 \ldots a_n$ is always obtained by simply setting $a_j \leftarrow a_j + 1$ and $a_k \leftarrow a_k - 1$ for some j and k, as in the "revolving door" algorithms of Section 7.2.1.3. It turns out that this is always possible; in fact, there is a unique way to do it when $n = 6$:

$$111111,\ 21111,\ 3111,\ 2211,\ 222,\ 321,\ 33,\ 42,\ 411,\ 51,\ 6. \qquad (59)$$

And in general, the $\left|{m+n \atop m}\right|$ partitions of n into at most m parts can always be generated by a suitable Gray path.

Notice that $\alpha \to \beta$ is an allowable transition from one partition to another if and only if we get the Ferrers diagram for β by moving just one dot in the Ferrers diagram for α. Therefore $\alpha^T \to \beta^T$ is also an allowable transition. It follows that every Gray code for partitions into at most m parts corresponds to a Gray code for partitions into parts that do not exceed m. We shall work with the latter constraint.

The total number of Gray codes for partitions is vast: There are 52 when $n = 7$, and 652 when $n = 8$; there are 298,896 when $n = 9$, and 2,291,100,484 when $n = 10$. But no really simple construction is known. The reason is probably that a few partitions have only two neighbors, namely the partitions $d^{n/d}$ when $1 < d < n$ and d is a divisor of n. Such partitions must be preceded and followed by $\{(d{+}1)d^{n/d-2}(d{-}1),\ d^{n/d-1}(d{-}1)1\}$, and this requirement seems to rule out any simple recursive approach.

Carla D. Savage [*J. Algorithms* **10** (1989), 577–595] found a way to surmount the difficulties with only a modest amount of complexity. Let

$$\mu(m,n) = \overbrace{m\ m\ \ldots\ m}^{\lfloor n/m \rfloor}\ (n \bmod m) \qquad (60)$$

be the lexicographically largest partition of n with parts $\le m$; our goal will be to construct recursively defined Gray paths $L(m,n)$ and $M(m,n)$ from the partition 1^n to $\mu(m,n)$, where $L(m,n)$ runs through all partitions whose parts are bounded by m while $M(m,n)$ runs through those partitions and a few more: $M(m,n)$ also includes partitions whose largest part is $m+1$, provided that the other parts are all strictly less than m. For example, $L(3,8)$ is 11111111, 2111111, 311111, 221111, 22211, 2222, 3221, 32111, 3311, 332, while $M(3,8)$ is

$$\begin{array}{c}11111111,\ 2111111,\ 221111,\ 22211,\ 2222,\ 3221,\\ 3311,\ 32111,\ 311111,\ 41111,\ 4211,\ 422,\ 332;\end{array} \tag{61}$$

the additional partitions starting with 4 will give us "wiggle room" in other parts of the recursion. We will define $L(m,n)$ for all $n \ge 0$, but $M(m,n)$ only for $n > 2m$.

The following construction, illustrated for $m = 5$ to simplify the notation, *almost* works:

$$L(5) = \begin{Bmatrix} L(3) \\ 4L(\infty)^R \\ 5L(\infty) \end{Bmatrix} \text{ if } n \le 7; \quad \begin{Bmatrix} L(3) \\ 4L(2)^R \\ 5L(2) \\ 431 \\ 44 \\ 53 \end{Bmatrix} \text{ if } n = 8; \quad \begin{Bmatrix} M(4) \\ 54L(4)^R \\ 55L(5) \end{Bmatrix} \text{ if } n \ge 9; \tag{62}$$

$$M(5) = \begin{Bmatrix} L(4) \\ 5L(4)^R \\ 6L(3) \\ 64L(\infty)^R \\ 55L(\infty) \end{Bmatrix} \text{ if } 11 \le n \le 13; \qquad \begin{Bmatrix} L(4) \\ 5M(4)^R \\ 6L(4) \\ 554L(4)^R \\ 555L(5) \end{Bmatrix} \text{ if } n \ge 14. \tag{63}$$

Here the parameter n in $L(m,n)$ and $M(m,n)$ has been omitted because it can be deduced from the context; each L or M is supposed to generate partitions of whatever amount remains after previous parts have been subtracted. Thus, for example, (63) specifies that

$$M(5,14) \;=\; L(4,14),\ 5M(4,9)^R,\ 6L(4,8),\ 554L(4,0)^R,\ 555L(5,-1);$$

the sequence $L(5,-1)$ is actually empty, and $L(4,0)$ is the empty string, so the final partition of $M(5,14)$ is $554 = \mu(5,14)$ as it should be. The notation $L(\infty)$ stands for $L(\infty,n) = L(n,n)$, the Gray path of all partitions of n, starting with 1^n and ending with n^1.

In general, $L(m)$ and $M(m)$ are defined for all $m \ge 3$ by essentially the same rules, if we replace the digits 2, 3, 4, 5, and 6 in (62) and (63) by $m-3$, $m-2$, $m-1$, m, and $m+1$, respectively. The ranges $n \le 7$, $n = 8$, $n \ge 9$ become $n \le 2m-3$, $n = 2m-2$, $n \ge 2m-1$; the ranges $11 \le n \le 13$ and $n \ge 14$ become $2m+1 \le n \le 3m-2$ and $n \ge 3m-1$. The sequences $L(0)$, $L(1)$, $L(2)$ have obvious definitions because the paths are unique when $m \le 2$. The sequence $M(2)$ is 1^n, 21^{n-2}, 31^{n-3}, 221^{n-4}, 2221^{n-6}, …, $\mu(2,n)$ for $n \ge 5$.

Theorem S. *Gray paths $L'(m,n)$ for $m,n \geq 0$ and $M'(m,n)$ for $n \geq 2m+1 \geq 5$ exist for all partitions with the properties described above, except in the case $L'(4,6)$. Furthermore, L' and M' obey the mutual recursions (62) and (63) except in a few cases.*

Proof. We noted above that (62) and (63) *almost* work; the reader may verify that the only glitch occurs in the case $L(4,6)$, when (62) gives

$$\begin{aligned} L(4,6) &= L(2,6),\ 3L(1,3)^R,\ 4L(1,2),\ 321,\ 33,\ 42 \\ &= 111111,\ 21111,\ 2211,\ 222,\ 3111,\ 411,\ 321,\ 33,\ 42. \end{aligned} \qquad (64)$$

If $m > 4$, we're OK because the transition from the end of $L(m-2,2m-2)$ to the beginning of $(m-1)L(m-3,m-1)^R$ is from $(m-2)(m-2)2$ to $(m-1)(m-3)2$. There is no satisfactory path $L(4,6)$, because all Gray codes through those nine partitions must end with either 411, 33, 3111, 222, or 2211.

In order to neutralize this anomaly we need to patch the definitions of $L(m,n)$ and $M(m,n)$ at eight places where the "buggy subroutine" $L(4,6)$ is invoked. One simple way is to make the following definitions:

$$\begin{aligned} L'(4,6) &= 111111, 21111, 3111, 411, 321, 33, 42; \\ L'(3,5) &= 11111, 2111, 221, 311, 32. \end{aligned} \qquad (65)$$

Thus, we omit 222 and 2211 from $L(4,6)$; we also reprogram $L(3,5)$ so that 2111 is adjacent to 221. Then exercise 60 shows that it is always easy to "splice in" the two partitions that are missing from $L(4,6)$. ∎

EXERCISES

▶ **1.** [*M21*] Give formulas for the total number of possibilities in each problem of The Twelvefold Way. For example, the number of n-tuples of m things is m^n. (Use the notation (38) when appropriate, and be careful to make your formulas correct even when $m = 0$ or $n = 0$.)

▶ **2.** [*20*] Show that a small change to step H1 yields an algorithm that will generate all partitions of n into *at most* m parts.

3. [*M17*] A partition $a_1 + \cdots + a_m$ of n into m parts $a_1 \geq \cdots \geq a_m$ is *optimally balanced* if $|a_i - a_j| \leq 1$ for $1 \leq i,j \leq m$. Prove that there is exactly one such partition, whenever $n \geq m \geq 1$, and give a simple formula that expresses the jth part a_j as a function of j, m, and n.

4. [*M22*] (Gideon Ehrlich, 1974.) What is the lexicographically smallest partition of n in which all parts are $\geq r$? For example, when $n = 19$ and $r = 5$ the answer is 766.

▶ **5.** [*23*] Design an algorithm that generates all partitions of n in the part-count form $c_1 \ldots c_n$ of (8). Generate them in colex order, namely in the lexicographic order of $c_n \ldots c_1$, which is equivalent to lexicographic order of the corresponding partitions $a_1 a_2 \ldots$. For efficiency, maintain also a table of links $l_0 l_1 \ldots l_n$ so that, if the distinct values of k for which $c_k > 0$ are $k_1 < \cdots < k_t$, we have

$$l_0 = k_1, \quad l_{k_1} = k_2, \quad \ldots, \quad l_{k_{t-1}} = k_t, \quad l_{k_t} = 0.$$

(Thus the partition 331 would be represented by $c_1 \ldots c_7 = 1020000$, $l_0 = 1$, $l_1 = 3$, and $l_3 = 0$; the other links l_2, l_4, l_5, l_7 can be set to any convenient values.)

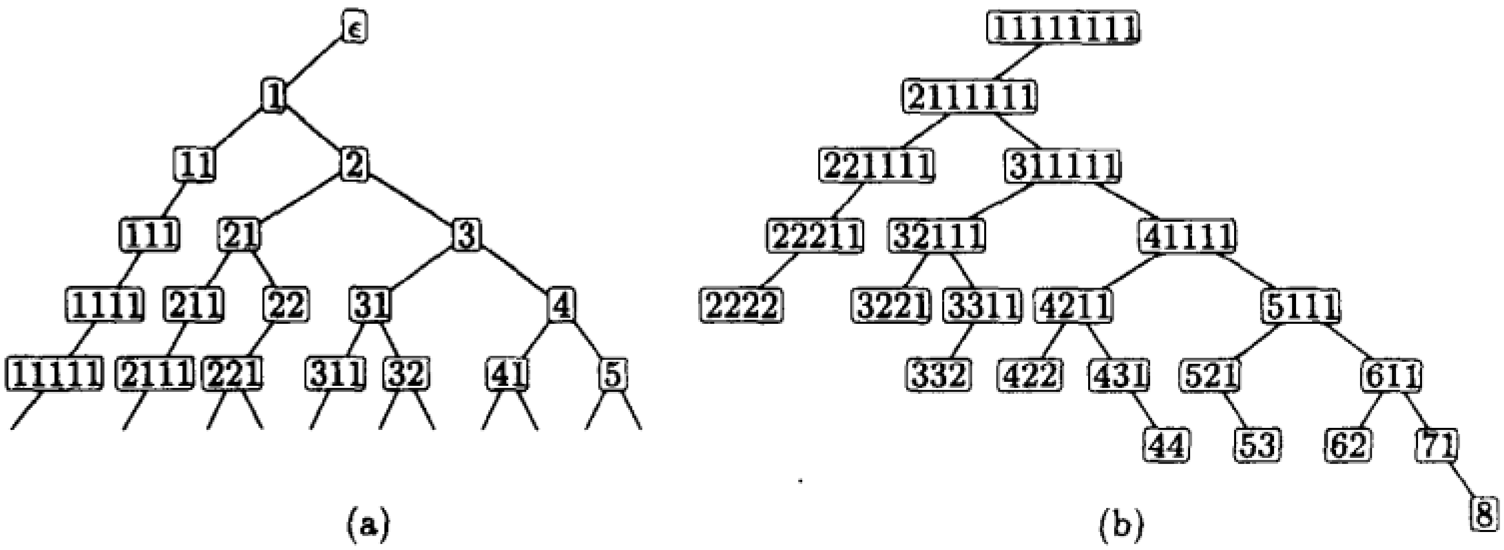
6. [*20*] Design an algorithm to compute $b_1b_2\ldots = (a_1a_2\ldots)^T$, given $a_1a_2\ldots$.

7. [*M20*] Suppose $a_1\ldots a_n$ and $a'_1\ldots a'_n$ are partitions of n with $a_1 \ge \cdots \ge a_n \ge 0$ and $a'_1 \ge \cdots \ge a'_n \ge 0$, and let their respective conjugates be $b_1\ldots b_n = (a_1\ldots a_n)^T$, $b'_1\ldots b'_n = (a'_1\ldots a'_n)^T$. Show that $b_1\ldots b_n < b'_1\ldots b'_n$ if and only if $a_n\ldots a_1 < a'_n\ldots a'_1$.

8. [*15*] When $(p_1\ldots p_t, q_1\ldots q_t)$ is the rim representation of a partition $a_1a_2\ldots$ as in (15) and (16), what is the conjugate partition $(a_1a_2\ldots)^T = b_1b_2\ldots$?

9. [*22*] If $a_1a_2\ldots a_m$ and $b_1b_2\ldots b_m = (a_1a_2\ldots a_m)^T$ are conjugate partitions, show that the multisets $\{a_1+1, a_2+2, \ldots, a_m+m\}$ and $\{b_1+1, b_2+2, \ldots, b_m+m\}$ are equal.

10. [*21*] Two simple kinds of binary trees are sometimes helpful for reasoning about partitions: (a) a tree that includes all partitions of all integers, and (b) a tree that includes all partitions of a given integer n, illustrated here for $n = 8$:



Deduce the general rules underlying these constructions. What order of tree traversal corresponds to lexicographic order of the partitions?

11. [*M22*] How many ways are there to pay one euro, using coins worth 1, 2, 5, 10, 20, 50, and/or 100 cents? What if you are allowed to use at most two of each coin?

▶ **12.** [*M21*] (L. Euler, 1750.) Use generating functions to prove that the number of ways to partition n into *distinct* parts is the number of ways to partition n into *odd* parts. For example, $5 = 4+1 = 3+2$; $5 = 3+1+1 = 1+1+1+1+1$.

[*Note:* The next two exercises use combinatorial techniques to prove extensions of this famous theorem.]

▶ **13.** [*M22*] (F. Franklin, 1882.) Find a one-to-one correspondence between partitions of n that have exactly k parts repeated more than once and partitions of n that have exactly k even parts. (The case $k = 0$ corresponds to Euler's result.)

▶ **14.** [*M28*] (J. J. Sylvester, 1882.) Find a one-to-one correspondence between partitions of n into distinct parts $a_1 > a_2 > \cdots > a_m$ that have exactly k "gaps" where $a_j > a_{j+1} + 1$, and partitions of n into odd parts that have exactly $k+1$ different values. (For example, when $k = 0$ this construction proves that the number of ways to write n as a sum of consecutive integers is the number of odd divisors of n.)

15. [*M20*] (J. J. Sylvester.) Find a generating function for the number of partitions that are *self-conjugate* (namely, partitions such that $\alpha = \alpha^T$).

16. [*M21*] Find the generating function for partitions of trace k, and sum it on k to obtain a nontrivial identity.

17. [*M26*] A *joint partition* of n is a pair of sequences $(a_1, \ldots, a_r;\ b_1, \ldots, b_s)$ of positive integers for which we have

$$a_1 \geq \cdots \geq a_r, \quad b_1 > \cdots > b_s, \quad \text{and} \quad a_1 + \cdots + a_r + b_1 + \cdots + b_s = n.$$

Thus it is an ordinary partition if $s = 0$, and a partition into distinct parts if $r = 0$.

a) Find a simple formula for the generating function $\sum u^{r+s} v^s z^n$, summed over all joint partitions of n with r ordinary parts a_i and s distinct parts b_j.

b) Similarly, find a simple formula for $\sum v^s z^n$ when the sum is over all joint partitions that have exactly $r + s = t$ total parts, given the value of t.

c) What identity do you deduce?

▶ **18.** [*M23*] (Doron Zeilberger.) Show that there is a one-to-one correspondence between pairs of integer sequences $(a_1, a_2, \ldots, a_r;\ b_1, b_2, \ldots, b_s)$ such that

$$a_1 \geq a_2 \geq \cdots \geq a_r, \qquad b_1 > b_2 > \cdots > b_s,$$

and pairs of integer sequences $(c_1, c_2, \ldots, c_{r+s};\ d_1, d_2, \ldots, d_{r+s})$ such that

$$c_1 \geq c_2 \geq \cdots \geq c_{r+s}, \qquad d_j \in \{0,1\} \quad \text{for } 1 \leq j \leq r+s,$$

related by the multiset equations

$$\{a_1, a_2, \ldots, a_r\} = \{c_j \mid d_j = 0\} \quad \text{and} \quad \{b_1, b_2, \ldots, b_s\} = \{c_j + r + s - j \mid d_j = 1\}.$$

Consequently we obtain the interesting identity

$$\sum_{\substack{a_1 \geq \cdots \geq a_r > 0 \\ b_1 > \cdots > b_s > 0}} u^{r+s} v^s z^{a_1 + \cdots + a_r + b_1 + \cdots + b_s} = \sum_{\substack{c_1 \geq \cdots \geq c_t > 0 \\ d_1, \ldots, d_t \in \{0,1\}}} u^t v^{d_1 + \cdots + d_t} z^{c_1 + \cdots + c_t + (t-1)d_1 + \cdots + d_{t-1}}.$$

19. [*M21*] (E. Heine, 1847.) Prove the four-parameter identity

$$\prod_{m=1}^{\infty} \frac{(1-wxz^m)(1-wyz^m)}{(1-wz^m)(1-wxyz^m)} = \sum_{k=0}^{\infty} \frac{w^k (x-1)(x-z)\ldots(x-z^{k-1})(y-1)(y-z)\ldots(y-z^{k-1})z^k}{(1-z)(1-z^2)\ldots(1-z^k)(1-wz)(1-wz^2)\ldots(1-wz^k)}.$$

Hint: Carry out the sum over either k or l in the formula

$$\sum_{k,l \geq 0} u^k v^l z^{kl} \frac{(z-az)(z-az^2)\ldots(z-az^k)}{(1-z)(1-z^2)\ldots(1-z^k)} \frac{(z-bz)(z-bz^2)\ldots(z-bz^l)}{(1-z)(1-z^2)\ldots(1-z^l)}$$

and consider the simplifications that occur when $b = auz$.

▶ **20.** [*M21*] Approximately how long does it take to compute a table of the partition numbers $p(n)$ for $1 \leq n \leq N$, using Euler's recurrence (20)?

21. [*M21*] (L. Euler.) Let $q(n)$ be the number of partitions into distinct parts. What is a good way to compute $q(n)$ if you already know the values of $p(1), \ldots, p(n)$?

22. [*HM21*] (L. Euler.) Let $\sigma(n)$ be the sum of all positive divisors of the positive integer n. Thus, $\sigma(n) = n + 1$ when n is prime, and $\sigma(n)$ can be significantly larger than n when n is highly composite. Prove that, in spite of this rather chaotic behavior, $\sigma(n)$ satisfies almost the same recurrence (20) as the partition numbers:

$$\sigma(n) = \sigma(n-1) + \sigma(n-2) - \sigma(n-5) - \sigma(n-7) + \sigma(n-12) + \sigma(n-15) - \cdots$$

for $n \geq 1$, except that when a term on the right is '$\sigma(0)$' the value 'n' is used instead. For example, $\sigma(11) = 1 + 11 = \sigma(10) + \sigma(9) - \sigma(6) - \sigma(4) = 18 + 13 - 12 - 7$; $\sigma(12) = 1+2+3+4+6+12 = \sigma(11)+\sigma(10)-\sigma(7)-\sigma(5)+12 = 12+18-8-6+12$.

23. [*HM25*] Use Jacobi's triple product identity (19) to prove another formula that he discovered:

$$\prod_{k=1}^{\infty}(1-z^k)^3 \;=\; 1-3z+5z^3-7z^6+9z^{10}-\cdots \;=\; \sum_{n=0}^{\infty}(-1)^n(2n+1)z^{\binom{n+1}{2}}.$$

24. [*M26*] (S. Ramanujan, 1919.) Let $A(z)=\prod_{k=1}^{\infty}(1-z^k)^4$.

a) Prove that $[z^n]\,A(z)$ is a multiple of 5 when $n \bmod 5 = 4$.
b) Prove that $[z^n]\,A(z)B(z)^5$ has the same property, if B is any power series with integer coefficients.
c) Therefore $p(n)$ is a multiple of 5 when $n \bmod 5 = 4$.

25. [*HM27*] Improve on (22) by using (a) Euler's summation formula and (b) Mellin transforms to estimate $\ln P(e^{-t})$. *Hint:* The dilogarithm function $\mathrm{Li}_2(x)=x/1^2+x^2/2^2+x^3/3^2+\cdots$ satisfies $\mathrm{Li}_2(x)+\mathrm{Li}_2(1-x)=\zeta(2)-(\ln x)\ln(1-x)$.

26. [*HM22*] In exercises 5.2.2–44 and 5.2.2–51 we studied two ways to prove that

$$\sum_{k=1}^{\infty}e^{-k^2/n} \;=\; \frac{1}{2}(\sqrt{\pi n}-1)+O(n^{-M}) \qquad \text{for all } M>0.$$

Show that Poisson's summation formula gives a much stronger result.

27. [*HM23*] Evaluate (29) and complete the calculations leading to Theorem D.

28. [*HM42*] (D. H. Lehmer.) Show that the Hardy–Ramanujan–Rademacher coefficients $A_k(n)$ defined in (34) have the following remarkable properties:

a) If k is odd, then $A_{2k}(km+4n+(k^2-1)/8)=A_2(m)A_k(n)$.
b) If p is prime, $p^e>2$, and $k\perp 2p$, then

$$A_{p^e k}(k^2m+p^{2e}n-(k^2+p^{2e}-1)/24) \;=\; (-1)^{[p^e=4]}A_{p^e}(m)A_k(n).$$

In this formula $k^2+p^{2e}-1$ is a multiple of 24 if p or k is divisible by 2 or 3; otherwise division by 24 should be done modulo $p^e k$.

c) If p is prime, $|A_{p^e}(n)|<2^{[p>2]}p^{e/2}$.
d) If p is prime, $A_{p^e}(n)\neq 0$ if and only if $1-24n$ is a quadratic residue modulo p and either $e=1$ or $24n \bmod p \neq 1$.
e) The probability that $A_k(n)=0$, when k is divisible by exactly t primes ≥ 5 and n is a random integer, is approximately $1-2^{-t}$.

▶ **29.** [*M16*] Generalizing (41), evaluate the sum $\sum_{a_1\geq a_2\geq\cdots\geq a_m\geq 1} z_1^{a_1}z_2^{a_2}\ldots z_m^{a_m}$.

30. [*M17*] Find closed forms for the sums

$$\text{(a)}\quad \sum_{k\geq 0}\left|{n-km \atop m-1}\right| \qquad \text{and} \qquad \text{(b)}\quad \sum_{k\geq 0}\left|{n \atop m-k}\right|$$

(which are finite, because the terms being summed are zero when k is large).

31. [*M24*] (A. De Morgan, 1843.) Show that $\left|{n \atop 2}\right|=\lfloor n/2\rfloor$ and $\left|{n \atop 3}\right|=\lfloor (n^2+6)/12\rfloor$; find a similar formula for $\left|{n \atop 4}\right|$.

32. [*M15*] Prove that $\left|{n \atop m}\right|\leq p(n-m)$ for all $m,n\geq 0$. When does equality hold?

33. [*HM20*] Use the fact that there are exactly $\binom{n-1}{m-1}$ *compositions* of n into m parts, Eq. 7.2.1.3–(9), to prove a lower bound on $\left|{n \atop m}\right|$. Then set $m=\lfloor\sqrt{n}\rfloor$ to obtain an elementary lower bound on $p(n)$.

▶ **34.** [*HM21*] Show that $\left|{n-m(m-1)/2 \atop m}\right|$ is the number of partitions of n into m distinct parts. Consequently

$$\left|{n \atop m}\right| = \frac{n^{m-1}}{m!\,(m-1)!}\left(1+O\left(\frac{m^3}{n}\right)\right) \qquad \text{when } m \le n^{1/3}.$$

35. [*HM21*] In the Erdős–Lehner probability distribution (43), what value of x is (a) most probable? (b) the median? (c) the mean? (d) What is the standard deviation?

36. [*HM24*] Prove the key estimate (47) that is needed in Theorem E.

37. [*M22*] Prove the inclusion-exclusion bracketing lemma (48), by analyzing how many times a partition that has exactly q different parts exceeding m is counted in the rth partial sum.

38. [*M20*] What is the generating function for the partitions of n that have exactly m parts, and largest part l?

▶ **39.** [*M25*] (F. Franklin.) Generalizing Theorem C, show that, for $0 \le k \le m$,

$$[z^n]\,\frac{(1-z^{l+1})\ldots(1-z^{l+k})}{(1-z)(1-z^2)\ldots(1-z^m)}$$

is the number of partitions $a_1a_2\ldots$ of n into m or fewer parts with the property that $a_1 \le a_{k+1} + l$.

40. [*M22*] (A. Cauchy.) What is the generating function for partitions into m parts, all *distinct* and less than l?

41. [*HM42*] Extend the Hardy–Ramanujan–Rademacher formula (32) to obtain a convergent series for partitions of n into at most m parts, with no part exceeding l.

42. [*HM42*] Find the limiting shape, analogous to (49), for random partitions of n into at most $\theta\sqrt{n}$ parts, with no part exceeding $\varphi\sqrt{n}$, assuming that $\theta\varphi > 1$.

43. [*M21*] Given n and k, how many partitions of n have $a_1 > a_2 > \cdots > a_k$?

▶ **44.** [*M22*] How many partitions of n have their two smallest parts equal?

45. [*HM21*] Compute the asymptotic value of $p(n-1)/p(n)$, with relative error $O(n^{-2})$.

46. [*M20*] In the text's analysis of Algorithm P, which is larger, $T_2'(n)$ or $T_2''(n)$?

▶ **47.** [*HM22*] (A. Nijenhuis and H. S. Wilf, 1975.) The following simple algorithm, based on a table of the partition numbers $p(0)$, $p(1)$, …, $p(n)$, generates a random partition of n using the part-count representation $c_1\ldots c_n$ of (8). Prove that it produces each partition with equal probability.

N1. [Initialize.] Set $m \leftarrow n$ and $c_1\ldots c_n \leftarrow 0\ldots0$.

N2. [Done?] Terminate if $m = 0$.

N3. [Generate.] Generate a random integer M in the range $0 \le M < mp(m)$.

N4. [Choose parts.] Set $s \leftarrow 0$. Then for $j = 1, 2, \ldots, n$ and for $k = 1, 2, \ldots, \lfloor m/j \rfloor$, repeatedly set $s \leftarrow s + kp(m-jk)$ until $s > M$.

N5. [Update.] Set $c_k \leftarrow c_k + j$, $m \leftarrow m - jk$, and return to N2. ▮

Hint: Step N4, which is based on the identity

$$\sum_{j=1}^{m}\sum_{k=1}^{\lfloor m/j \rfloor} kp(m-jk) \;=\; mp(m),$$

chooses each particular pair of values (j,k) with probability $kp(m-jk)/(mp(m))$.

48. [*HM40*] Analyze the running time of the algorithm in the previous exercise.

▸ **49.** [*HM26*] (a) What is the generating function $F(z)$ for the sum of the smallest parts of all partitions of n? (The series begins $z + 3z^2 + 5z^3 + 9z^4 + 12z^5 + \cdots$.)

(b) Find the asymptotic value of $[z^n]\,F(z)$, with relative error $O(n^{-1})$.

50. [*HM33*] Let $c(m) = c_m(2m)$ in the recurrence (56), (57).

a) Prove that $c_m(m+k) = m - k + c(k)$ for $0 \le k \le m$.
b) Consequently (58) holds for $m \le n \le 2m$ if $c(m) < 3p(m)$ for all m.
c) Show that $c(m) - m$ is the sum of the second-smallest parts of all partitions of m.
d) Find a one-to-one correspondence between all partitions of n with second-smallest part k and all partitions of numbers $\le n$ with smallest part $k+1$.
e) Describe the generating function $\sum_{m\ge 0} c(m)\, z^m$.
f) Conclude that $c(m) < 3p(m)$ for all $m \ge 0$.

51. [*M46*] Make a detailed analysis of Algorithm H.

▸ **52.** [*M21*] What is the millionth partition generated by Algorithm P when $n = 64$? *Hint:* $p(64) = 1741630 = 1000000 + \left|{77 \atop 13}\right| + \left|{60 \atop 10}\right| + \left|{47 \atop 8}\right| + \left|{35 \atop 5}\right| + \left|{27 \atop 3}\right| + \left|{22 \atop 2}\right| + \left|{18 \atop 1}\right| + \left|{15 \atop 0}\right|$.

▸ **53.** [*M21*] What is the millionth partition generated by Algorithm H when $m = 32$ and $n = 100$? *Hint:* $999999 = \left|{80 \atop 12}\right| + \left|{66 \atop 11}\right| + \left|{50 \atop 7}\right| + \left|{41 \atop 6}\right| + \left|{33 \atop 5}\right| + \left|{26 \atop 4}\right| + \left|{21 \atop 4}\right|$.

▸ **54.** [*M30*] The partition $\alpha = a_1 a_2 \ldots$ is said to *majorize* the partition $\beta = b_1 b_2 \ldots$, written $\alpha \succeq \beta$ or $\beta \preceq \alpha$, if $a_1 + \cdots + a_k \ge b_1 + \cdots + b_k$ for all $k \ge 0$.

a) True or false: $\alpha \succeq \beta$ implies $\alpha \ge \beta$ (lexicographically).
b) True or false: $\alpha \succeq \beta$ implies $\beta^T \succeq \alpha^T$.
c) Show that any two partitions of n have a greatest lower bound $\alpha \wedge \beta$ such that $\alpha \succeq \gamma$ and $\beta \succeq \gamma$ if and only if $\alpha \wedge \beta \succeq \gamma$. Explain how to compute $\alpha \wedge \beta$.
d) Similarly, explain how to compute a least upper bound $\alpha \vee \beta$ such that $\gamma \succeq \alpha$ and $\gamma \succeq \beta$ if and only if $\gamma \succeq \alpha \vee \beta$.
e) If α has l parts and β has m parts, how many parts do $\alpha \wedge \beta$ and $\alpha \vee \beta$ have?
f) True or false: If α has distinct parts and β has distinct parts, then so do $\alpha \wedge \beta$ and $\alpha \vee \beta$.

▸ **55.** [*M37*] Continuing the previous exercise, say that α *covers* β if $\alpha \succeq \beta$, $\alpha \ne \beta$, and $\alpha \succeq \gamma \succeq \beta$ implies $\gamma = \alpha$ or $\gamma = \beta$. For example, Fig. 32 illustrates the covering relations between partitions of the number 12.

a) Let us write $\alpha \vdash\!\!> \beta$ if $\alpha = a_1 a_2 \ldots$ and $\beta = b_1 b_2 \ldots$ are partitions for which $b_k = a_k - [k = l] + [k = l+1]$ for all $k \ge 1$ and some $l \ge 1$. Prove that α covers β if and only if $\alpha \vdash\!\!> \beta$ or $\beta^T \vdash\!\!> \alpha^T$.
b) Show that there is an easy way to tell if α covers β by looking at the rim representations of α and β.
c) Let $n = \binom{n_2}{2} + \binom{n_1}{1}$ where $n_2 > n_1 \ge 0$. Show that no partition of n covers more than $n_2 - 2$ partitions.
d) Say that the partition μ is *minimal* if there is no partition λ with $\mu \vdash\!\!> \lambda$. Prove that μ is minimal if and only if μ^T has distinct parts.
e) Suppose $\alpha = \alpha_0 \vdash\!\!> \alpha_1 \vdash\!\!> \cdots \vdash\!\!> \alpha_k$ and $\alpha = \alpha'_0 \vdash\!\!> \alpha'_1 \vdash\!\!> \cdots \vdash\!\!> \alpha'_{k'}$, where α_k and $\alpha'_{k'}$ are minimal partitions. Prove that $k = k'$ and $\alpha_k = \alpha'_{k'}$.
f) Explain how to compute the lexicographically smallest partition into distinct parts that majorizes a given partition α.
g) Describe λ_n, the lexicographically smallest partition of n into distinct parts. What is the length of all paths $n^1 = \alpha_0 \vdash\!\!> \alpha_1 \vdash\!\!> \cdots \vdash\!\!> \lambda_n^T$?

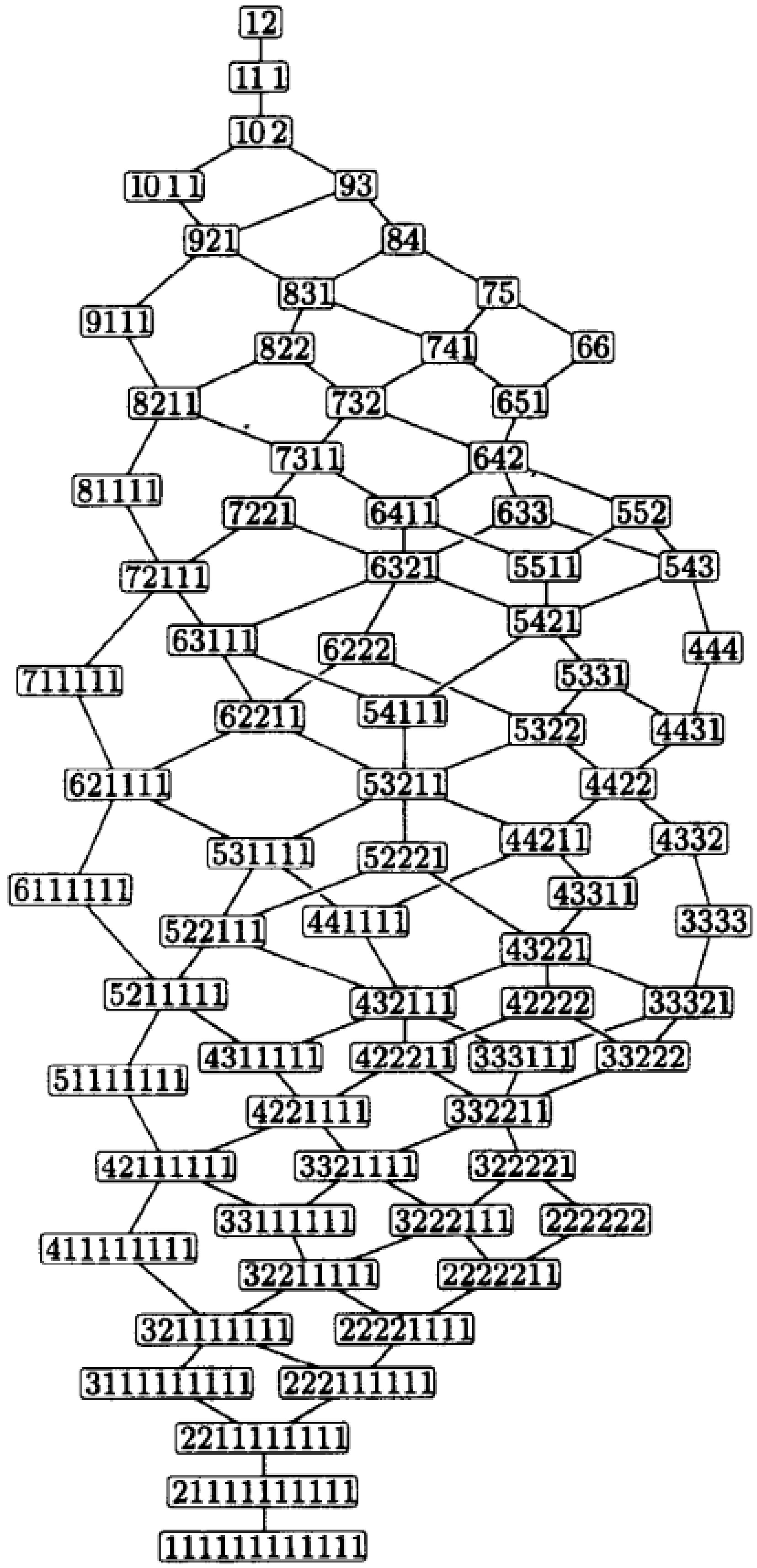


Fig. 32. The majorization lattice for partitions of 12. (See exercises 54–58.)

h) What are the lengths of the longest and shortest paths of the form $n^1 = \alpha_0$, α_1, $\ldots$, $\alpha_l = 1^n$, where α_j covers α_{j+1} for $0 \le j < l$?

▶ **56.** [*M27*] Design an algorithm to generate all partitions α such that $\lambda \preceq \alpha \preceq \mu$, given partitions λ and μ with $\lambda \preceq \mu$.

Note: Such an algorithm has numerous applications. For example, to generate all partitions that have m parts and no part exceeding l, we can let λ be the smallest such partition, namely $\lceil n/m \rceil \ldots \lfloor n/m \rfloor$ as in exercise 3, and let μ be the largest, namely $((n-m+1)1^{m-1}) \wedge (l^{\lfloor n/l \rfloor}(n \bmod l))$. Similarly, according to a well-known theorem of H. G. Landau [*Bull. Math. Biophysics* **15** (1953), 143–148], the partitions of $\binom{m}{2}$ such that

$$\left\lfloor \frac{m}{2} \right\rfloor^{\lfloor m/2 \rfloor} \left\lfloor \frac{m-1}{2} \right\rfloor^{\lceil m/2 \rceil} \preceq \alpha \preceq (m-1)(m-2)\ldots 21$$

are the possible "score vectors" of a round-robin tournament, namely the partitions $a_1 \ldots a_m$ such that the jth strongest player wins a_j games.

57. [*M22*] Suppose a matrix (a_{ij}) of 0s and 1s has row sums $r_i = \sum_j a_{ij}$ and column sums $c_j = \sum_i a_{ij}$. Then $\lambda = r_1 r_2 \ldots$ and $\mu = c_1 c_2 \ldots$ are partitions of $n = \sum_{i,j} a_{ij}$. Prove that such a matrix exists if and only if $\lambda \preceq \mu^T$.

58. [*M23*] (*Symmetrical means.*) Let $\alpha = a_1 \ldots a_m$ and $\beta = b_1 \ldots b_m$ be partitions of n. Prove that the inequality

$$\frac{1}{m!}\sum x_{p_1}^{a_1} \ldots x_{p_m}^{a_m} \geq \frac{1}{m!}\sum x_{p_1}^{b_1} \ldots x_{p_m}^{b_m}$$

holds for all nonnegative values of the variables $(x_1, \ldots, x_m)$, where the sums range over all $m!$ permutations of $\{1, \ldots, m\}$, if and only if $\alpha \succeq \beta$. (For example, this inequality reduces to $(y_1 + \cdots + y_n)/n \geq (y_1 \ldots y_n)^{1/n}$ in the special case $m = n$, $\alpha = n0\ldots0$, $\beta = 11\ldots1$, $x_j = y_j^{1/n}$.)

59. [*M22*] The Gray path (59) is symmetrical in the sense that the reversed sequence 6, 51, ..., 111111 is the same as conjugate sequence $(111111)^T$, $(21111)^T$, ..., $(6)^T$. Find all Gray paths $\alpha_1, \ldots, \alpha_{p(n)}$ that are symmetrical in this way.

60. [*23*] Complete the proof of Theorem S by modifying the definitions of $L(m,n)$ and $M(m,n)$ in all places where $L(4,6)$ is called in (62) and (63).

61. [*26*] Implement a partition-generation scheme based on Theorem S, always specifying the two parts that have changed between visits.

62. [*46*] Prove or disprove: For all sufficiently large integers n and $3 \leq m < n$ such that $n \bmod m \neq 0$, and for all partitions α of n with $a_1 \leq m$, there is a Gray path for all partitions with parts $\leq m$, beginning at 1^n and ending at α, unless $\alpha = 1^n$ or $\alpha = 21^{n-2}$.

63. [*47*] For which partitions λ and μ is there a Gray code through all partitions α such that $\lambda \preceq \alpha \preceq \mu$?

▶ **64.** [*32*] (*Binary partitions.*) Design a loopless algorithm that visits all partitions of n into powers of 2, where each step replaces $2^k + 2^k$ by 2^{k+1} or vice versa.

65. [*23*] It is well known that every commutative group of m elements can be represented as a discrete torus $T(m_1, \ldots, m_n)$ with the addition operation of 7.2.1.3–(66), where $m = m_1 \ldots m_n$ and m_j is a multiple of m_{j+1} for $1 \leq j < n$. For example, when $m = 360 = 2^3 \cdot 3^2 \cdot 5^1$ there are six such groups, corresponding to the factorizations $(m_1, m_2, m_3) = (30,6,2)$, $(60,6,1)$, $(90,2,2)$, $(120,3,1)$, $(180,2,1)$, and $(360,1,1)$.

Explain how to generate all such factorizations systematically with an algorithm that changes exactly two of the factors m_j at each step.

▶ **66.** [*M25*] (*P-partitions.*) Instead of insisting that $a_1 \geq a_2 \geq \cdots$, suppose we want to consider all nonnegative compositions of n that satisfy a given *partial* order. For example, P. A. MacMahon observed that all solutions to the "up-down" inequalities $a_4 \leq a_2 \geq a_3 \leq a_1$ can be divided into five nonoverlapping types:

$$a_1 \geq a_2 \geq a_3 \geq a_4; \quad a_1 \geq a_2 \geq a_4 > a_3;$$
$$a_2 > a_1 \geq a_3 \geq a_4; \quad a_2 > a_1 \geq a_4 > a_3; \quad a_2 \geq a_4 > a_1 \geq a_3.$$

Each of these types is easily enumerated since, for example, $a_2 > a_1 \geq a_4 > a_3$ is equivalent to $a_2 - 2 \geq a_1 - 1 \geq a_4 - 1 \geq a_3$; the number of solutions with $a_3 \geq 0$ and $a_1 + a_2 + a_3 + a_4 = n$ is the number of partitions of $n-1-2-0-1$ into at most four parts.

Explain how to solve a general problem of this kind: Given any partial order relation $\prec$ on m elements, consider all m-tuples $a_1 \ldots a_m$ with the property that $a_j \geq a_k$

when $j \prec k$. Assuming that the subscripts have been chosen so that $j \prec k$ implies $j \le k$, show that all of the desired m-tuples fall into exactly N classes, one for each of the outputs of the topological sorting algorithm 7.2.1.2V. What is the generating function for all such $a_1 \dots a_m$ that are nonnegative and sum to n? How could you generate them all?

67. [*M25*] (P. A. MacMahon, 1886.) A *perfect partition* of n is a multiset that has exactly $n+1$ submultisets, and these multisets are partitions of the integers $0, 1, \dots, n$. For example, the multisets $\{1,1,1,1,1\}$, $\{2,2,1\}$, and $\{3,1,1\}$ are perfect partitions of 5.

Explain how to construct the perfect partitions of n that have fewest elements.

68. [*M23*] What partition of n into m parts has the largest product $a_1 \dots a_m$, when (a) m is given; (b) m is arbitrary?

69. [*M30*] Find all $n < 10^9$ such that the equation $x_1 + x_2 + \cdots + x_n = x_1 x_2 \dots x_n$ has only one solution in positive integers $x_1 \ge x_2 \ge \cdots \ge x_n$. (There is, for example, only one solution when $n = 2$, 3, or 4; but $5+2+1+1+1 = 5\cdot 2\cdot 1\cdot 1\cdot 1$ and $3+3+1+1+1 = 3\cdot 3\cdot 1\cdot 1\cdot 1$ and $2+2+2+1+1 = 2\cdot 2\cdot 2\cdot 1\cdot 1$.)

70. [*M30*] ("Bulgarian solitaire.") Take n cards and divide them arbitrarily into one or more piles. Then repeatedly remove one card from each pile and form a new pile.

Show that if $n = 1+2+\cdots+m$, this process always reaches a self-repeating state with piles of sizes $\{m, m-1, \dots, 1\}$. For example, if $n = 10$ and if we start with piles whose sizes are $\{3,3,2,2\}$, we get the sequence of partitions

$$3322 \to 42211 \to 5311 \to 442 \to 3331 \to 4222 \to 43111 \to 532 \to 4321 \to 4321 \to \cdots.$$

What cycles of states are possible for other values of n?

71. [*M46*] Continuing the previous problem, what is the maximum number of steps that can occur before n-card Bulgarian solitaire reaches a cyclic state?

72. [*M25*] Suppose we write down all partitions of n, for example

$$6,\ 51,\ 42,\ 411,\ 33,\ 321,\ 3111,\ 222,\ 2211,\ 21111,\ 111111$$

when $n = 6$, and change each jth occurrence of k to j:

$$1,\ 11,\ 11,\ 112,\ 12,\ 111,\ 1123,\ 123,\ 1212,\ 11234,\ 123456.$$

a) Prove that this operation yields a permutation of the individual elements.
b) How many times does the element k appear altogether?

7.2.1.5. Generating all set partitions.

Now let's shift gears and concentrate on a rather different kind of partition. The *partitions of a set* are the ways to regard that set as a union of nonempty, disjoint subsets called *blocks*. For example, we listed the five essentially different partitions of $\{1,2,3\}$ at the beginning of the previous section, in 7.2.1.4–(2) and 7.2.1.4–(4). Those five partitions can also be written more compactly in the form

$$123,\quad 12|3,\quad 13|2,\quad 1|23,\quad 1|2|3, \tag{1}$$

using a vertical line to separate one block from another. In this list the elements of each block could have been written in any order, and so could the blocks themselves, because '13|2' and '31|2' and '2|13' and '2|31' all represent the same partition. But we can standardize the representation by agreeing, for example, to list the elements of each block in increasing order, and to arrange the blocks in increasing order of their smallest elements. With this convention the partitions of $\{1,2,3,4\}$ are

$$\begin{gathered} 1234,\ 123|4,\ 124|3,\ 12|34,\ 12|3|4,\ 134|2,\ 13|24,\ 13|2|4, \\ 14|23,\ 1|234,\ 1|23|4,\ 14|2|3,\ 1|24|3,\ 1|2|34,\ 1|2|3|4, \end{gathered} \tag{2}$$

obtained by placing 4 among the blocks of (1) in all possible ways.

Set partitions arise in many different contexts. Political scientists and economists, for example, often see them as "coalitions"; computer system designers may consider them to be "cache-hit patterns" for memory accesses; poets know them as "rhyme schemes" (see exercises 34–37). We saw in Section 2.3.3 that any *equivalence relation* between objects — namely any binary relation that is reflexive, symmetric, and transitive — defines a partition of those objects into so-called "equivalence classes." Conversely, every set partition defines an equivalence relation: If Π is a partition of $\{1,2,\ldots,n\}$ we can write

$$j \equiv k \quad (\text{modulo } \Pi) \tag{3}$$

whenever j and k belong to the same block of Π.

One of the most convenient ways to represent a set partition inside a computer is to encode it as a *restricted growth string*, namely as a string $a_1a_2\ldots a_n$ of nonnegative integers in which we have

$$a_1 = 0 \quad \text{and} \quad a_{j+1} \le 1 + \max(a_1,\ldots,a_j) \text{ for } 1 \le j < n. \tag{4}$$

The idea is to set $a_j = a_k$ if and only if $j \equiv k$, and to choose the smallest available number for a_j whenever j is smallest in its block. For example, the restricted growth strings for the fifteen partitions in (2) are respectively

$$\begin{gathered} 0000,\ 0001,\ 0010,\ 0011,\ 0012,\ 0100,\ 0101,\ 0102, \\ 0110,\ 0111,\ 0112,\ 0120,\ 0121,\ 0122,\ 0123. \end{gathered} \tag{5}$$

This convention suggests the following simple generation scheme, due to George Hutchinson [*CACM* **6** (1963), 613–614]:

Algorithm H (*Restricted growth strings in lexicographic order*). Given $n \ge 2$, this algorithm generates all partitions of $\{1,2,\ldots,n\}$ by visiting all strings $a_1a_2\ldots a_n$ that satisfy the restricted growth condition (4). We maintain an auxiliary array $b_1b_2\ldots b_n$, where $b_{j+1} = 1 + \max(a_1,\ldots,a_j)$; the value of b_n is actually kept in a separate variable, m, for efficiency.

H1. [Initialize.] Set $a_1\ldots a_n \leftarrow 0\ldots0$, $b_1\ldots b_{n-1} \leftarrow 1\ldots1$, and $m \leftarrow 1$.

H2. [Visit.] Visit the restricted growth string $a_1\ldots a_n$, which represents a partition into $m + [a_n = m]$ blocks. Then go to H4 if $a_n = m$.

H3. [Increase a_n.] Set $a_n \leftarrow a_n + 1$ and return to H2.

H4. [Find j.] Set $j \leftarrow n-1$; then, while $a_j = b_j$, set $j \leftarrow j-1$.

H5. [Increase a_j.] Terminate if $j = 1$. Otherwise set $a_j \leftarrow a_j + 1$.

H6. [Zero out $a_{j+1} \ldots a_n$.] Set $m \leftarrow b_j + [a_j = b_j]$ and $j \leftarrow j + 1$. Then, while $j < n$, set $a_j \leftarrow 0$, $b_j \leftarrow m$, and $j \leftarrow j + 1$. Finally set $a_n \leftarrow 0$ and go back to H2. ▮

Exercise 47 proves that steps H4–H6 are rarely necessary, and that the loops in H4 and H6 are almost always short. A linked-list variant of this algorithm appears in exercise 2.

Gray codes for set partitions. One way to pass quickly through all set partitions is to change just one digit of the restricted growth string $a_1 \ldots a_n$ at each step, because a change to a_j simply means that element j moves from one block to another. An elegant way to arrange such a list was proposed by Gideon Ehrlich [*JACM* **20** (1973), 507–508]: We can successively append the digits

$$0,\ m,\ m-1,\ \ldots,\ 1 \qquad \text{or} \qquad 1,\ \ldots,\ m-1,\ m,\ 0 \tag{6}$$

to each string $a_1 \ldots a_{n-1}$ in the list for partitions of $n - 1$ elements, where $m = 1 + \max(a_1, \ldots, a_{n-1})$, alternating between the two cases. Thus the list '00, 01' for $n = 2$ becomes '000, 001, 011, 012, 010' for $n = 3$; and that list becomes

$$\begin{gathered} 0000,\ 0001,\ 0011,\ 0012,\ 0010,\ 0110,\ 0112,\ 0111, \\ 0121,\ 0122,\ 0123,\ 0120,\ 0100,\ 0102,\ 0101 \end{gathered} \tag{7}$$

when we extend it to the case $n = 4$. Exercise 14 shows that Ehrlich's scheme leads to a simple algorithm that achieves this Gray-code order without doing much more work than Algorithm H.

Suppose, however, that we aren't interested in *all* of the partitions; we might want only the ones that have exactly m blocks. Can we run through this smaller collection of restricted growth strings, still changing only one digit at a time? Yes; a very pretty way to generate such a list has been discovered by Frank Ruskey [*Lecture Notes in Comp. Sci.* **762** (1993), 205–206]. He defined two such sequences, A_{mn} and A'_{mn}, both of which start with the lexicographically smallest m-block string $0^{n-m}01\ldots(m-1)$. The difference between them, if $n > m + 1$, is that A_{mn} ends with $01\ldots(m-1)0^{n-m}$ while A'_{mn} ends with $0^{n-m-1}01\ldots(m-1)0$. Here are Ruskey's recursive rules, when $1 < m < n$:

$$A_{m(n+1)} = \begin{cases} A_{(m-1)n}(m-1), A^R_{mn}(m-1), \ldots, A^R_{mn}1, A_{mn}0, & \text{if } m \text{ is even;} \\ A'_{(m-1)n}(m-1), A_{mn}(m-1), \ldots, A^R_{mn}1, A_{mn}0, & \text{if } m \text{ is odd;} \end{cases} \tag{8}$$

$$A'_{m(n+1)} = \begin{cases} A'_{(m-1)n}(m-1), A_{mn}(m-1), \ldots, A_{mn}1, A^R_{mn}0, & \text{if } m \text{ is even;} \\ A_{(m-1)n}(m-1), A^R_{mn}(m-1), \ldots, A_{mn}1, A^R_{mn}0, & \text{if } m \text{ is odd.} \end{cases} \tag{9}$$

Of course the base cases are simply one-element lists,

$$A_{1n} = A'_{1n} = \{0^n\} \qquad \text{and} \qquad A_{nn} = \{01\ldots(n-1)\}. \tag{10}$$

With these definitions the $\left\{{5 \atop 3}\right\} = 25$ partitions of $\{1,2,3,4,5\}$ into three blocks are

$$\begin{gathered}00012,\ 00112,\ 01112,\ 01012,\ 01002,\ 01102,\ 00102,\\ 00122,\ 01122,\ 01022,\ 01222,\ 01212,\ 01202,\\ 01201,\ 01211,\ 01221,\ 01021,\ 01121,\ 00121,\\ 00120,\ 01120,\ 01020,\ 01220,\ 01210,\ 01200.\end{gathered} \tag{11}$$

(See exercise 17 for an efficient implementation.)

In Ehrlich's scheme (7) the rightmost digits of $a_1 \ldots a_n$ vary most rapidly, but in Ruskey's scheme most of the changes occur near the left. In both cases, however, each step affects just one digit a_j, and the changes are quite simple: Either a_j changes by ±1, or it jumps between the two extreme values 0 and $1+\max(a_1,\ldots,a_{j-1})$. Under the same constraints, the sequence A'_{1n}, A'_{2n}, ..., A'_{nn} runs through *all* partitions, in increasing order of the number of blocks.

The number of set partitions. We've seen that there are 5 partitions of $\{1,2,3\}$ and 15 of $\{1,2,3,4\}$. A quick way to compute these counts was discovered by C. S. Peirce, who presented the following triangle of numbers in the *American Journal of Mathematics* **3** (1880), page 48:

$$\begin{matrix}1\\ 2 & 1\\ 5 & 3 & 2\\ 15 & 10 & 7 & 5\\ 52 & 37 & 27 & 20 & 15\\ 203 & 151 & 114 & 87 & 67 & 52\end{matrix} \tag{12}$$

Here the entries ϖ_{n1}, ϖ_{n2}, ..., ϖ_{nn} of the nth row obey the simple recurrence

$$\varpi_{nk} = \varpi_{(n-1)k} + \varpi_{n(k+1)} \text{ if } 1 \le k < n; \qquad \varpi_{nn} = \varpi_{(n-1)1} \text{ if } n > 1; \tag{13}$$

and $\varpi_{11} = 1$. Peirce's triangle has many remarkable properties, some of which are surveyed in exercises 26–31. For example, ϖ_{nk} is the number of partitions of $\{1,2,\ldots,n\}$ in which k is the smallest of its block.

The entries on the diagonal and in the first column of Peirce's triangle, which tell us the total number of set partitions, are commonly known as *Bell numbers*, because E. T. Bell wrote several influential papers about them [*AMM* **41** (1934), 411–419; *Annals of Math.* **35** (1934), 258–277; **39** (1938), 539–557]. We shall denote Bell numbers by ϖ_n, following the lead of Louis Comtet, in order to avoid confusion with the Bernoulli numbers B_n. The first few cases are

$n =$	0	1	2	3	4	5	6	7	8	9	10	11	12
$\varpi_n =$	1	1	2	5	15	52	203	877	4140	21147	115975	678570	4213597

Notice that this sequence grows rapidly, but not as fast as $n!$; we will prove below that $\varpi_n = \Theta(n/\log n)^n$.

The Bell numbers $\varpi_n = \varpi_{n1}$ for $n \ge 0$ must satisfy the recurrence formula

$$\varpi_{n+1} = \varpi_n + \binom{n}{1}\varpi_{n-1} + \binom{n}{2}\varpi_{n-2} + \cdots = \sum_k \binom{n}{k}\varpi_{n-k}, \tag{14}$$

because every partition of $\{1,\ldots,n+1\}$ is obtained by choosing k elements of $\{1,\ldots,n\}$ to put in the block containing $n+1$ and by partitioning the remaining elements in ϖ_{n-k} ways, for some k. This recurrence, found by Yoshisuke Matsunaga in the 18th century (see Section 7.2.1.7), leads to a nice generating function,

$$\Pi(z) \;=\; \sum_{n=0}^{\infty} \varpi_n \frac{z^n}{n!} \;=\; e^{e^z-1}, \tag{15}$$

discovered by W. A. Whitworth [*Choice and Chance*, 3rd edition (1878), 3.XXIV]. For if we multiply both sides of (14) by $z^n/n!$ and sum on n we get

$$\Pi'(z) = \sum_{n=0}^{\infty} \varpi_{n+1} \frac{z^n}{n!} = \Bigl(\sum_{k=0}^{\infty} \frac{z^k}{k!}\Bigr)\Bigl(\sum_{m=0}^{\infty} \varpi_m \frac{z^m}{m!}\Bigr) = e^z \Pi(z),$$

and (15) is the solution to this differential equation with $\Pi(0) = 1$.

The numbers ϖ_n had been studied for many years because of their curious properties related to this formula, long before Whitworth pointed out their combinatorial connection with set partitions. For example, we have

$$\varpi_n = \frac{n!}{e}\,[z^n]\, e^{e^z} = \frac{n!}{e}\,[z^n] \sum_{k=0}^{\infty} \frac{e^{kz}}{k!} = \frac{1}{e}\sum_{k=0}^{\infty} \frac{k^n}{k!} \tag{16}$$

[*Mat. Sbornik* **3** (1868), 62; **4** (1869), 39; G. Dobiński, *Archiv der Math. und Physik* **61** (1877), 333–336; **63** (1879), 108–110]. Christian Kramp discussed the expansion of e^{e^z} in *Der polynomische Lehrsatz*, ed. by C. F. Hindenburg (Leipzig: 1796), 112–113; he mentioned two ways to compute the coefficients, namely either to use (14) or to use a summation of $p(n)$ terms, one for each ordinary partition of n. (See Arbogast's formula, exercise 1.2.5–21. Kramp, who came close to discovering that formula, seemed to prefer his partition-based method, not realizing that it would require more than polynomial time as n got larger and larger; and he computed 116015, not 115975, for the coefficient of z^{10}.)

***Asymptotic estimates.** We can learn how fast ϖ_n grows by using one of the most basic principles of complex residue theory: If the power series $\sum_{k=0}^{\infty} a_k z^k$ converges whenever $|z| < r$, then

$$a_{n-1} \;=\; \frac{1}{2\pi i}\oint \frac{a_0 + a_1 z + a_2 z^2 + \cdots}{z^n}\, dz, \tag{17}$$

if the integral is taken along a simple closed path that goes counterclockwise around the origin and stays inside the circle $|z| = r$. Let $f(z) = \sum_{k=0}^{\infty} a_k z^{k-n}$ be the integrand. We're free to choose any such path, but special techniques often apply when the path goes through a point z_0 at which the derivative $f'(z_0)$ is zero, because we have

$$f(z_0 + \epsilon e^{i\theta}) \;=\; f(z_0) + \frac{f''(z_0)}{2}\epsilon^2 e^{2i\theta} + O(\epsilon^3) \tag{18}$$

in the vicinity of such a point. If, for example, $f(z_0)$ and $f''(z_0)$ are real and positive, say $f(z_0) = u$ and $f''(z_0) = 2v$, this formula says that the value of

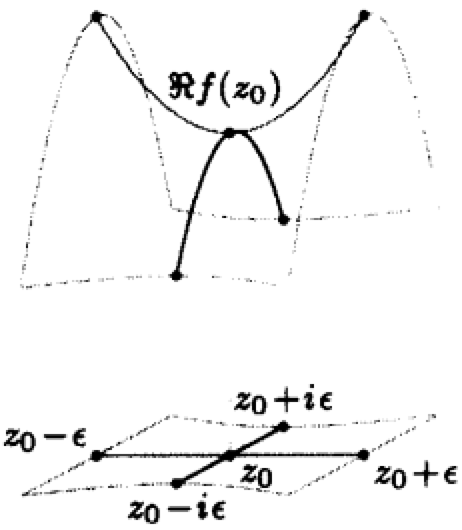


Fig. 33. The behavior of an analytic function near a saddle point.

$f(z_0 \pm \epsilon)$ is approximately $u + v\epsilon^2$ while $f(z_0 \pm i\epsilon)$ is approximately $u - v\epsilon^2$. If z moves from $z_0 - i\epsilon$ to $z_0 + i\epsilon$, the value of $f(z)$ rises to a maximum value u, then falls again; but the larger value $u + v\epsilon^2$ occurs both to the left and to the right of this path. In other words, a mountaineer who goes hiking on the complex plane, when the altitude at point z is $\Re f(z)$, encounters a "pass" at z_0; the terrain looks like a saddle at that point. The overall integral of $f(z)$ will be the same if taken around any path, but a path that doesn't go through the pass won't be as nice because it will have to cancel out some higher values of $f(z)$ that could have been avoided. Therefore we tend to get best results by choosing a path that goes through z_0, in the direction of increasing imaginary part. This important technique, due to P. Debye [*Math. Annalen* **67** (1909), 535–558], is called the "saddle point method."

Let's get familiar with the saddle point method by starting with an example for which we already know the answer:

$$\frac{1}{(n-1)!} = \frac{1}{2\pi i} \oint \frac{e^z}{z^n}\, dz. \tag{19}$$

Our goal is to find a good approximation for the value of the integral on the right when n is large. It will be convenient to deal with $f(z) = e^z/z^n$ by writing it as $e^{g(z)}$ where $g(z) = z - n \ln z$; then the saddle point occurs where $g'(z_0) = 1 - n/z_0$ is zero, namely at $z_0 = n$. If $z = n + it$ we have

$$\begin{aligned} g(z) &= g(n) + \sum_{k=2}^{\infty} \frac{g^{(k)}(n)}{k!}(it)^k \\ &= n - n\ln n - \frac{t^2}{2n} + \frac{it^3}{3n^2} + \frac{t^4}{4n^3} - \frac{it^5}{5n^4} + \cdots \end{aligned}$$

because $g^{(k)}(z) = (-1)^k (k-1)!\, n/z^k$ when $k \ge 2$. Let's integrate $f(z)$ on a rectangular path from $n - im$ to $n + im$ to $-n + im$ to $-n - im$ to $n - im$:

$$\begin{aligned} \frac{1}{2\pi i} \oint \frac{e^z}{z^n}\, dz &= \frac{1}{2\pi} \int_{-m}^{m} f(n+it)\, dt + \frac{1}{2\pi i} \int_{n}^{-n} f(t+im)\, dt \\ &\quad + \frac{1}{2\pi} \int_{m}^{-m} f(-n+it)\, dt + \frac{1}{2\pi i} \int_{-n}^{n} f(t-im)\, dt. \end{aligned}$$

Clearly $|f(z)| \le 2^{-n} f(n)$ on the last three sides of this path if we choose $m = 2n$, because $|e^z| = e^{\Re z}$ and $|z| \ge \max(\Re z, \Im z)$; so we're left with

$$\frac{1}{2\pi i}\oint \frac{e^z}{z^n}\,dz \;=\; \frac{1}{2\pi}\int_{-m}^{m} e^{g(n+it)}\,dt + O\Big(\frac{ne^n}{2^n n^n}\Big).$$

Now we fall back on a technique that we've used several times before— for example to derive Eq. 5.1.4–(53): If $\hat f(t)$ is a good approximation to $f(t)$ when $t \in A$, and if the sums $\sum_{t\in B\setminus A} f(t)$ and $\sum_{t\in C\setminus A}\hat f(t)$ are both small, then $\sum_{t\in C}\hat f(t)$ is a good approximation to $\sum_{t\in B} f(t)$. The same idea applies to integrals as well as sums. [This general method, introduced by Laplace in 1782, is often called "trading tails"; see *CMath* §9.4.] If $|t| \le n^{1/2+\epsilon}$ we have

$$\begin{aligned} e^{g(n+it)} &= \exp\Big(g(n) - \frac{t^2}{2n} + \frac{it^3}{3n^2} + \cdots\Big)\\ &= \frac{e^n}{n^n}\exp\Big(-\frac{t^2}{2n} + \frac{it^3}{3n^2} + \frac{t^4}{4n^3} + O(n^{5\epsilon-3/2})\Big)\\ &= \frac{e^n}{n^n}e^{-t^2/(2n)}\Big(1 + \frac{it^3}{3n^2} + \frac{t^4}{4n^3} - \frac{t^6}{18n^4} + O(n^{9\epsilon-3/2})\Big). \end{aligned}$$

And when $|t| > n^{1/2+\epsilon}$ we have

$$|e^{g(n+it)}| < |f(n+in^{1/2+\epsilon})| = \frac{e^n}{n^n}\exp\Big(-\frac{n}{2}\ln(1+n^{2\epsilon-1})\Big) = O\Big(\frac{e^{n-n^{2\epsilon}/2}}{n^n}\Big).$$

Furthermore the incomplete gamma function

$$\int_{n^{1/2+\epsilon}}^{\infty} e^{-t^2/(2n)}t^k\,dt = 2^{(k-1)/2}n^{(k+1)/2}\,\Gamma\Big(\frac{k+1}{2},\frac{n^{2\epsilon}}{2}\Big) = O(n^{O(1)}e^{-n^{2\epsilon}/2})$$

is negligible. Thus we can trade tails and obtain the approximation

$$\begin{aligned} \frac{1}{2\pi i}\oint\frac{e^z}{z^n}\,dz &= \frac{e^n}{2\pi n^n}\int_{-\infty}^{\infty} e^{-t^2/(2n)}\Big(1 + \frac{it^3}{3n^2} + \frac{t^4}{4n^3} - \frac{t^6}{18n^4} + O(n^{9\epsilon-3/2})\Big)\,dt\\ &= \frac{e^n}{2\pi n^n}\Big(I_0 + \frac{i}{3n^2}I_3 + \frac{1}{4n^3}I_4 - \frac{1}{18n^4}I_6 + O(n^{9\epsilon-3/2})\Big), \end{aligned}$$

where $I_k = \int_{-\infty}^{\infty} e^{-t^2/(2n)}t^k\,dt$. Of course $I_k = 0$ when k is odd. Otherwise we can evaluate I_k by using the well-known fact that

$$\int_{-\infty}^{\infty} e^{-at^2}t^{2l}\,dt = \frac{\Gamma\big((2l+1)/2\big)}{a^{(2l+1)/2}} = \frac{\sqrt{2\pi}}{(2a)^{(2l+1)/2}}\prod_{j=1}^{l}(2j-1) \qquad (20)$$

when $a > 0$; see exercise 39. Putting everything together gives us, for all $\epsilon > 0$, the asymptotic estimate

$$\frac{1}{(n-1)!} = \frac{e^n}{\sqrt{2\pi}\,n^{n-1/2}}\Big(1 + 0 + \frac{3}{4n} - \frac{15}{18n} + O(n^{9\epsilon-3/2})\Big); \qquad (21)$$

this result agrees perfectly with Stirling's approximation, which we derived by quite different methods in 1.2.11.2–(19). Further terms in the expansion of

$g(n+it)$ would allow us to prove that the true error in (21) is only $O(n^{-2})$, because the same procedure yields an asymptotic series of the general form $e^n/(\sqrt{2\pi}n^{n-1/2})(1+c_1/n+c_2/n^2+\cdots+c_m/n^m+O(n^{-m-1}))$ for all m.

Our derivation of this result has glossed over an important technicality: The function $\ln z$ is not single-valued along the path of integration, because it grows by $2\pi i$ when we loop around the origin. Indeed, this fact underlies the basic mechanism that makes the residue theorem work. But our reasoning was valid because the ambiguity of the logarithm does not affect the integrand $f(z) = e^z/z^n$ when n is an integer. Furthermore, if n were not an integer, we could have adapted the argument and kept it rigorous by choosing to carry out the integral (19) along a path that starts at $-\infty$, circles the origin counterclockwise and returns to $-\infty$. That would have given us Hankel's integral for the gamma function, Eq. 1.2.5–(17); we could thereby have derived the asymptotic formula

$$\frac{1}{\Gamma(x)} = \frac{1}{2\pi i}\oint \frac{e^z}{z^x}\,dz = \frac{e^x}{\sqrt{2\pi}x^{x-1/2}}\Bigl(1-\frac{1}{12x}+O(x^{-2})\Bigr), \tag{22}$$

valid for all real x as $x\to\infty$.

So the saddle point method seems to work—although it isn't the simplest way to get this particular result. Let's apply it now to deduce the approximate size of the Bell numbers:

$$\frac{\varpi_{n-1}}{(n-1)!} = \frac{1}{2\pi i e}\oint e^{g(z)}\,dz, \qquad g(z) = e^z - n\ln z. \tag{23}$$

A saddle point now occurs at the point $z_0 = \xi > 0$, where

$$\xi e^\xi = n. \tag{24}$$

(We should actually write $\xi(n)$ to indicate that ξ depends on n; but that would clutter up the formulas below.) Let's assume for the moment that a little bird has told us the value of ξ. Then we want to integrate on a path where $z=\xi+it$, and we have

$$g(\xi+it) = e^\xi - n\Bigl(\ln\xi - \frac{(it)^2}{2!}\frac{\xi+1}{\xi^2} - \frac{(it)^3}{3!}\frac{\xi^2-2!}{\xi^3} - \frac{(it)^4}{4!}\frac{\xi^3+3!}{\xi^4} + \cdots\Bigr).$$

By integrating on a suitable rectangular path, we can prove as above that the integral in (23) is well approximated by

$$\int_{-n^{\epsilon-1/2}}^{n^{\epsilon-1/2}} e^{g(\xi)-na_2t^2-nia_3t^3+na_4t^4+\cdots}\,dt, \qquad a_k = \frac{\xi^{k-1}+(-1)^k(k-1)!}{k!\,\xi^k}; \tag{25}$$

see exercise 43. Noting that a_kt^k is $O(n^{k\epsilon-k/2})$ inside this integral, we obtain an asymptotic expansion of the form

$$\varpi_{n-1} = \frac{e^{e^\xi-1}(n-1)!}{\xi^{n-1}\sqrt{2\pi n(\xi+1)}}\Bigl(1+\frac{b_1}{n}+\frac{b_2}{n^2}+\cdots+\frac{b_m}{n^m}+O\Bigl(\Bigl(\frac{\log n}{n}\Bigr)^{m+1}\Bigr)\Bigr), \tag{26}$$

where $(\xi+1)^{3k}b_k$ is a polynomial of degree $4k$ in ξ. (See exercise 44.) For example,

$$b_1 = -\frac{2\xi^4-3\xi^3-20\xi^2-18\xi+2}{24(\xi+1)^3}; \qquad (27)$$

$$b_2 = \frac{4\xi^8-156\xi^7-695\xi^6-696\xi^5+1092\xi^4+2916\xi^3+1972\xi^2-72\xi+4}{1152(\xi+1)^6}. \qquad (28)$$

Stirling's approximation (21) can be used in (26) to prove that

$$\varpi_{n-1} = \exp\Bigl(n\Bigl(\xi-1+\frac{1}{\xi}\Bigr)-\xi-\frac{1}{2}\ln(\xi+1)-1-\frac{\xi}{12n}+O\Bigl(\Bigl(\frac{\log n}{n}\Bigr)^2\Bigr)\Bigr); \qquad (29)$$

and exercise 45 proves the similar formula

$$\varpi_n = \exp\Bigl(n\Bigl(\xi-1+\frac{1}{\xi}\Bigr)-\frac{1}{2}\ln(\xi+1)-1-\frac{\xi}{12n}+O\Bigl(\Bigl(\frac{\log n}{n}\Bigr)^2\Bigr)\Bigr). \qquad (30)$$

Consequently we have $\varpi_n/\varpi_{n-1} \approx e^\xi = n/\xi$. More precisely,

$$\frac{\varpi_{n-1}}{\varpi_n} = \frac{\xi}{n}\Bigl(1+O\Bigl(\frac{1}{n}\Bigr)\Bigr). \qquad (31)$$

But what is the asymptotic value of ξ? The definition (24) implies that

$$\begin{aligned}\xi &= \ln n - \ln\xi = \ln n - \ln(\ln n - \ln\xi)\\ &= \ln n - \ln\ln n + O\Bigl(\frac{\log\log n}{\log n}\Bigr);\end{aligned} \qquad (32)$$

and we can go on in this vein, as shown in exercise 49. But the asymptotic series for ξ developed in this way never gives better accuracy than $O(1/(\log n)^m)$ for larger and larger m; so it is hugely inaccurate when multiplied by n in formula (29) for ϖ_{n-1} or formula (30) for ϖ_n.

Thus if we want to use (29) or (30) to calculate good numerical approximations to Bell numbers, our best strategy is to start by computing a good numerical value for ξ, without using a slowly convergent series. Newton's rootfinding method, discussed in the remarks preceding Algorithm 4.7N, yields the efficient iterative scheme

$$\xi_0 = \ln n, \qquad \xi_{k+1} = \frac{\xi_k}{\xi_k+1}(1+\xi_0-\ln\xi_k), \qquad (33)$$

which converges rapidly to the correct value. For example, when $n = 100$ the fifth iterate

$$\xi_5 = 3.38563\,01402\,90050\,18488\,82443\,64529\,72686\,74917- \qquad (34)$$

is already correct to 40 decimal places. Using this value in (29) gives us successive approximations

$$(1.6176088053\ldots, 1.6187421339\ldots, 1.6187065391\ldots, 1.6187060254\ldots)\times 10^{114}$$

when we take terms up to b_0, b_1, b_2, b_3 into account; the true value of ϖ_{99} is the 115-digit integer 16187060274460 . . . 20741.

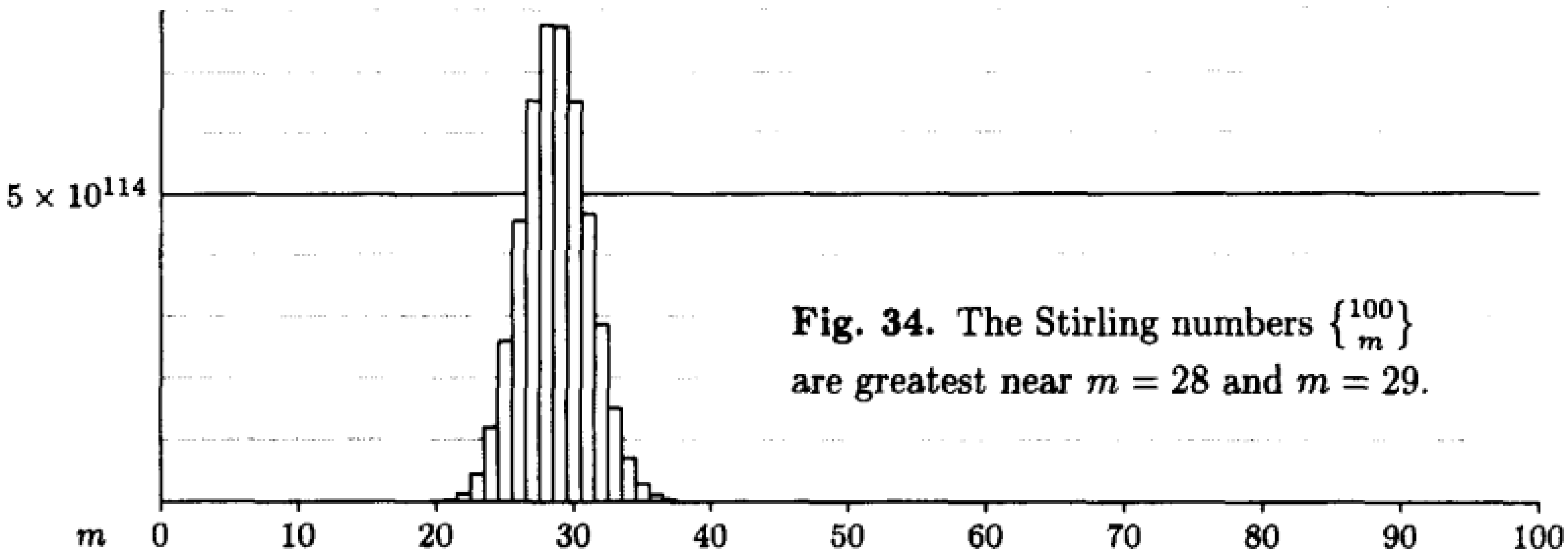


Fig. 34. The Stirling numbers $\left\{{100 \atop m}\right\}$ are greatest near $m = 28$ and $m = 29$.

Now that we know the number of set partitions ϖ_n, let's try to figure out how many of them have exactly m blocks. It turns out that nearly all partitions of $\{1, \ldots, n\}$ have roughly $n/\xi = e^\xi$ blocks, with about ξ elements per block. For example, Fig. 34 shows a histogram of the numbers $\left\{{n \atop m}\right\}$ when $n = 100$ and $e^\xi \approx 29.54$.

We can investigate the size of $\left\{{n \atop m}\right\}$ by applying the saddle point method to formula 1.2.9–(23), which states that

$$\left\{{n \atop m}\right\} = \frac{n!}{m!}[z^n]\,(e^z - 1)^m = \frac{n!}{m!}\frac{1}{2\pi i}\oint e^{m\ln(e^z-1)-(n+1)\ln z}\,dz. \tag{35}$$

Let $\alpha = (n+1)/m$. The function $g(z) = \alpha^{-1}\ln(e^z - 1) - \ln z$ has a saddle point at $\sigma > 0$ when

$$\frac{\sigma}{1 - e^{-\sigma}} = \alpha. \tag{36}$$

Notice that $\alpha > 1$ for $1 \le m \le n$. This special value σ is given by

$$\sigma = \alpha - \beta, \qquad \beta = T(\alpha e^{-\alpha}), \tag{37}$$

where T is the tree function of Eq. 2.3.4.4–(30). Indeed, β is the value between 0 and 1 for which we have

$$\beta e^{-\beta} = \alpha e^{-\alpha}; \tag{38}$$

the function xe^{-x} increases from 0 to e^{-1} when x increases from 0 to 1, then it decreases to 0 again. Therefore β is uniquely defined, and we have

$$e^\sigma = \frac{\alpha}{\beta}. \tag{39}$$

All such pairs α and β are obtainable by using the inverse formulas

$$\alpha = \frac{\sigma e^\sigma}{e^\sigma - 1}, \qquad \beta = \frac{\sigma}{e^\sigma - 1}; \tag{40}$$

for example, the values $\alpha = \ln 4$ and $\beta = \ln 2$ correspond to $\sigma = \ln 2$.

We can show as above that the integral in (35) is asymptotically equivalent to an integral of $e^{(n+1)g(z)}\,dz$ over the path $z = \sigma + it$. (See exercise 58.) Exercise 56 proves that the Taylor series about $z = \sigma$,

$$g(\sigma + it) = g(\sigma) - \frac{t^2(1-\beta)}{2\sigma^2} - \sum_{k=3}^{\infty} \frac{(it)^k}{k!} g^{(k)}(\sigma), \tag{41}$$

has the property that

$$|g^{(k)}(\sigma)| < 2(k-1)!\,(1-\beta)/\sigma^k \qquad \text{for all } k > 0. \tag{42}$$

Therefore we can conveniently remove a factor of $N = (n+1)(1-\beta)$ from the power series $(n+1)g(z)$, and the saddle point method leads to the formula

$$\left\{ {n \atop m} \right\} = \frac{n!}{m!}\frac{1}{(\alpha-\beta)^{n-m}\beta^m\sqrt{2\pi N}}\left(1 + \frac{b_1}{N} + \frac{b_2}{N^2} + \cdots + \frac{b_l}{N^l} + O\Big(\frac{1}{N^{l+1}}\Big)\right) \tag{43}$$

as $N \to \infty$, where $(1-\beta)^{2k} b_k$ is a polynomial in α and β. (The quantity $(\alpha-\beta)^{n-m}\beta^m$ in the denominator comes from the fact that $(e^\sigma - 1)^m/\sigma^n = (\alpha/\beta - 1)^m/(\alpha-\beta)^n$, by (37) and (39).) For example,

$$b_1 = \frac{6 - \beta^3 - 4\alpha\beta^2 - \alpha^2\beta}{8(1-\beta)} - \frac{5(2-\beta^2-\alpha\beta)^2}{24(1-\beta)^2}. \tag{44}$$

Exercise 57 proves that $N \to \infty$ if and only if $n - m \to \infty$. An asymptotic expansion for $\left\{ {n \atop m} \right\}$ similar to (43), but somewhat more complicated, was first obtained by Leo Moser and Max Wyman, *Duke Math. J.* **25** (1957), 29–43.

Formula (43) looks a bit scary because it is designed to apply over the entire range of block counts m. Significant simplifications are possible when m is relatively small or relatively large (see exercises 60 and 61); but the simplified formulas don't give accurate results in the important cases when $\left\{ {n \atop m} \right\}$ is largest. Let's look at those crucial cases more closely now, so that we can account for the sharp peak illustrated in Fig. 34.

Let $\xi e^\xi = n$ as in (24), and suppose $m = \exp(\xi + r/\sqrt{n}) = ne^{r/\sqrt{n}}/\xi$; we will assume that $|r| \le n^\epsilon$, so that m is near e^ξ. The leading term of (43) can be rewritten

$$\frac{n!}{m!}\frac{1}{(\alpha-\beta)^{n-m}\beta^m\sqrt{2\pi(n+1)(1-\beta)}} = \frac{m^n}{m!}\frac{(n+1)!}{(n+1)^{n+1}}\frac{e^{n+1}}{\sqrt{2\pi(n+1)}}\left(1-\frac{\beta}{\alpha}\right)^{m-n}\frac{e^{-\beta m}}{\sqrt{1-\beta}}, \tag{45}$$

and Stirling's approximation for $(n+1)!$ is evidently ripe for cancellation in the midst of this expression. With the help of computer algebra we find

$$\frac{m^n}{m!} = \frac{1}{\sqrt{2\pi}}\exp\Big(n\Big(\xi - 1 + \frac{1}{\xi}\Big) - \frac{1}{2}\Big(\xi + r^2 + \frac{r^2}{\xi}\Big) - \Big(\frac{r}{2} + \frac{r^3}{6} + \frac{r^3}{3\xi}\Big)\frac{1}{\sqrt{n}} + O(n^{4\epsilon-1})\Big);$$

and the relevant quantities related to α and β are

$$\frac{\beta}{\alpha} = \frac{\xi}{n} + \frac{r\xi^2}{n\sqrt{n}} + O(\xi^3 n^{2\epsilon-2});$$

$$e^{-\beta m} = \exp\Bigl(-\xi - \frac{r\xi^2}{\sqrt{n}} + O(\xi^3 n^{2\epsilon-1})\Bigr);$$

$$\Bigl(1-\frac{\beta}{\alpha}\Bigr)^{m-n} = \exp\Bigl(\xi - 1 + \frac{r(\xi^2-\xi-1)}{\sqrt{n}} + O(\xi^3 n^{2\epsilon-1})\Bigr).$$

Therefore the overall result is

$$\left\{ {n \atop e^{\xi+r/\sqrt{n}}} \right\} = \frac{1}{\sqrt{2\pi}} \exp\Bigl(n\Bigl(\xi - 1 + \frac{1}{\xi}\Bigr) - \frac{\xi}{2} - 1 - \frac{\xi+1}{2\xi}\Bigl(r + \frac{3\xi(2\xi+3)+(\xi+2)r^2}{6(\xi+1)\sqrt{n}}\Bigr)^2 + O(\xi^3 n^{4\epsilon-1})\Bigr). \quad (46)$$

The squared expression on the last line is zero when

$$r = -\frac{\xi(2\xi+3)}{2(\xi+1)\sqrt{n}} + O(\xi^2 n^{-3/2});$$

thus the maximum occurs when the number of blocks is

$$m = \frac{n}{\xi} - \frac{3+2\xi}{2+2\xi} + O\Bigl(\frac{\xi}{n}\Bigr). \quad (47)$$

By comparing (47) to (30) we see that the largest Stirling number $\left\{ {n \atop m} \right\}$ for a given value of n is approximately equal to $\xi\varpi_n/\sqrt{2\pi n}$.

The saddle point method applies to problems that are considerably more difficult than the ones we have considered here. Excellent expositions of advanced techniques can be found in several books: N. G. de Bruijn, *Asymptotic Methods in Analysis* (1958), Chapters 5 and 6; F. W. J. Olver, *Asymptotics and Special Functions* (1974), Chapter 4; R. Wong, *Asymptotic Approximations of Integrals* (2001), Chapters 2 and 7.

***Random set partitions.** The sizes of blocks in a partition of $\{1,\ldots,n\}$ constitute by themselves an ordinary partition of the number n. Therefore we might wonder what sort of partition they are likely to be. Figure 30 in Section 7.2.1.4 showed the result of superimposing the Ferrers diagrams of all $p(25) = 1958$ partitions of 25; those partitions tended to follow the symmetrical curve of Eq. 7.2.1.4–(49). By contrast, Fig. 35 shows what happens when we superimpose the corresponding diagrams of all $\varpi_{25} \approx 4.6386 \times 10^{18}$ partitions of the set $\{1,\ldots,25\}$. Evidently the "shape" of a random set partition is quite different from the shape of a random integer partition.

This change is due to the fact that some integer partitions occur only a few times as block sizes of set partitions, while others are extremely common. For example, the partition $n = 1+1+\cdots+1$ arises in only one way, but if n is

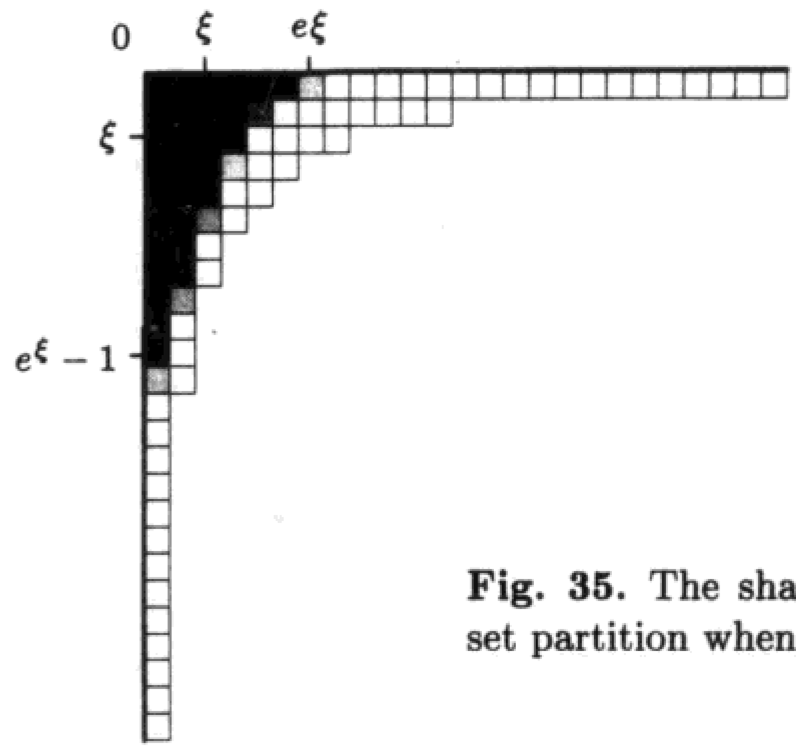


Fig. 35. The shape of a random set partition when $n = 25$.

even the partition $n = 2 + 2 + \cdots + 2$ arises in $(n-1)(n-3)\ldots(1)$ ways. When $n = 25$, the integer partition

$$25 \;=\; 4+4+3+3+3+2+2+2+1+1$$

actually occurs in more than 2% of all possible set partitions. (This particular partition turns out to be most common in the case $n = 25$. The answer to exercise 1.2.5–21 explains that exactly

$$\frac{n!}{c_1!\,1!^{c_1}\,c_2!\,2!^{c_2}\ldots c_n!\,n!^{c_n}} \tag{48}$$

set partitions correspond to the integer partition $n = c_1 \cdot 1 + c_2 \cdot 2 + \cdots + c_n \cdot n$.)

We can easily determine the average number of k-blocks in a random partition of $\{1,\ldots,n\}$: If we write out all ϖ_n of the possibilities, every particular k-element block occurs exactly ϖ_{n-k} times. Therefore the average number is

$$\binom{n}{k}\frac{\varpi_{n-k}}{\varpi_n}. \tag{49}$$

An extension of Eq. (31) above, proved in exercise 64, shows moreover that

$$\frac{\varpi_{n-k}}{\varpi_n} \;=\; \Bigl(\frac{\xi}{n}\Bigr)^k\Bigl(1+\frac{k\xi(k\xi+k+1)}{2(\xi+1)^2 n}+O\Bigl(\frac{k^3}{n^2}\Bigr)\Bigr) \qquad \text{if } k \le n^{2/3}, \tag{50}$$

where ξ is defined in (24). Therefore if, say, $k \le n^{\epsilon}$, formula (49) simplifies to

$$\frac{n^{\underline{k}}}{k!}\Bigl(\frac{\xi}{n}\Bigr)^k\Bigl(1+O\Bigl(\frac{1}{n}\Bigr)\Bigr) \;=\; \frac{\xi^k}{k!}\bigl(1+O(n^{2\epsilon-1})\bigr). \tag{51}$$

There are, on average, about ξ blocks of size 1, and $\xi^2/2!$ blocks of size 2, etc.

The variance of these quantities is small (see exercise 65), and it turns out that a random partition behaves essentially as if the number of k-blocks were a Poisson deviate with mean $\xi^k/k!$. The smooth curve shown in Fig. 35 runs through the points $\bigl(f(k),k\bigr)$ in Ferrers-like coordinates, where

$$f(k) \;=\; \xi^{k+1}/(k+1)! + \xi^{k+2}/(k+2)! + \xi^{k+3}/(k+3)! + \cdots \tag{52}$$

is the approximate distance from the top line corresponding to block size $k \geq 0$. (This curve becomes more nearly vertical when n is larger.)

The largest block tends to contain approximately $e\xi$ elements. Furthermore, the probability that the block containing element 1 has size less than $\xi + a\sqrt{\xi}$ approaches the probability that a normal deviate is less than a. [See John Haigh, *J. Combinatorial Theory* **A13** (1972), 287–295; V. N. Sachkov, *Probabilistic Methods in Combinatorial Analysis* (1997), Chapter 4, translated from a Russian book published in 1978; Yu. Yakubovich, *J. Mathematical Sciences* **87** (1997), 4124–4137, translated from a Russian paper published in 1995; B. Pittel, *J. Combinatorial Theory* **A79** (1997), 326–359.]

A nice way to generate random partitions of $\{1, 2, \ldots, n\}$ was introduced by A. J. Stam in the *Journal of Combinatorial Theory* **A35** (1983), 231–240: Let M be a random integer that takes the value m with probability

$$p_m = \frac{m^n}{e\,m!\,\varpi_n}; \tag{53}$$

these probabilities sum to 1 because of (16). Once M has been chosen, generate a random n-tuple $X_1X_2\ldots X_n$, where each X_j is uniformly and independently distributed between 0 and $M-1$. Then let $i \equiv j$ in the partition if and only if $X_i = X_j$. This procedure works because each k-block partition is obtained with probability $\sum_{m\geq 0}(m^{\underline{k}}/m^n)p_m = 1/\varpi_n$.

For example, if $n = 25$ we have

$p_4 \approx .00000372$	$p_9 \approx .15689865$	$p_{14} \approx .04093663$	$p_{19} \approx .00006068$
$p_5 \approx .00019696$	$p_{10} \approx .21855285$	$p_{15} \approx .01531445$	$p_{20} \approx .00001094$
$p_6 \approx .00313161$	$p_{11} \approx .21526871$	$p_{16} \approx .00480507$	$p_{21} \approx .00000176$
$p_7 \approx .02110279$	$p_{12} \approx .15794784$	$p_{17} \approx .00128669$	$p_{22} \approx .00000026$
$p_8 \approx .07431024$	$p_{13} \approx .08987171$	$p_{18} \approx .00029839$	$p_{23} \approx .00000003$

and the other probabilities are negligible. So we can usually get a random partition of 25 elements by looking at a random 25-digit integer in radix 9, 10, 11, or 12. The number M can be generated using 3.4.1–(3); it tends to be approximately $n/\xi = e^{\xi}$ (see exercise 67).

***Partitions of a multiset.** The partitions of an integer and the partitions of a set are just the extreme cases of a far more general problem, the partitions of a multiset. Indeed, the partitions of n are essentially the same as the partitions of $\{1, 1, \ldots, 1\}$, where there are n 1s.

From this standpoint there are essentially $p(n)$ different multisets with n elements. For example, five different cases of multiset partitions arise when $n = 4$:

1234, 123|4, 124|3, 12|34, 12|3|4, 134|2, 13|24, 13|2|4,
14|23, 14|2|3, 1|234, 1|23|4, 1|24|3, 1|2|34, 1|2|3|4;
1123, 112|3, 113|2, 11|23, 11|2|3, 123|1, 12|13, 12|1|3, 13|1|2, 1|1|23, 1|1|2|3;
1122, 112|2, 11|22, 11|2|2, 122|1, 12|12, 12|1|2, 1|1|22, 1|1|2|2;
1112, 111|2, 112|1, 11|12, 11|1|2, 12|1|1, 1|1|1|2;
1111, 111|1, 11|11, 11|1|1, 1|1|1|1. (54)

When the multiset contains m distinct elements, with n_1 of one kind, n_2 of another, ..., and n_m of the last, we write $p(n_1, n_2, \ldots, n_m)$ for the total number of partitions. Thus the examples in (54) show that

$$p(1,1,1,1) = 15, \quad p(2,1,1) = 11, \quad p(2,2) = 9, \quad p(3,1) = 7, \quad p(4) = 5. \qquad (55)$$

Partitions with $m = 2$ are often called "bipartitions"; those with $m = 3$ are "tripartitions"; and in general these combinatorial objects are known as *multipartitions*. The study of multipartitions was inaugurated long ago by P. A. MacMahon [*Philosophical Transactions* **181** (1890), 481–636; **217** (1917), 81–113; *Proc. Cambridge Philos. Soc.* **22** (1925), 951–963]; but the subject is so vast that many unsolved problems remain. In the remainder of this section and in the exercises below we shall take a glimpse at some of the most interesting and instructive aspects of the theory that have been discovered so far.

In the first place it is important to notice that multipartitions are essentially the partitions of *vectors* with nonnegative integer components, namely the ways to decompose such a vector as a sum of such vectors. For example, the nine partitions of $\{1,1,2,2\}$ listed in (54) are the same as the nine partitions of the bipartite column vector $\substack{2\\2}$, namely

$$\substack{2\\2},\quad \substack{20\\11},\quad \substack{20\\02},\quad \substack{200\\011},\quad \substack{11\\20},\quad \substack{11\\11},\quad \substack{110\\101},\quad \substack{110\\002},\quad \substack{1100\\0011}. \qquad (56)$$

(We drop the + signs for brevity, as in the case of one-dimensional integer partitions.) Each partition can be written in canonical form if we list its parts in nonincreasing lexicographic order.

A simple algorithm suffices to generate the partitions of any given multiset. In the following procedure we represent partitions on a stack that contains triples of elements (c, u, v), where c denotes a component number, $u > 0$ denotes the yet-unpartitioned amount remaining in component c, and $v \le u$ denotes the c component of the current part. Triples are actually kept in three arrays $(c_0, c_1, \ldots)$, $(u_0, u_1, \ldots)$, and $(v_0, v_1, \ldots)$ for convenience, and a "stack frame" array $(f_0, f_1, \ldots)$ is also maintained so that the $(l+1)$st vector of the partition consists of elements f_l through $f_{l+1} - 1$ in the c, u, and v arrays. For example, the following arrays would represent the bipartition $\substack{3221100\\1201131}$:

j	0	1	2	3	4	5	6	7	8	9	10	
c_j	1	2	1	2	1	1	2	1	2	2	2	
u_j	9	9	6	8	4	2	6	1	5	4	1	
v_j	3	1	2	2	2	1	1	1	1	3	1	
	$f_0 = 0$		$f_1 = 2$		$f_2 = 4$	$f_3 = 5$		$f_4 = 7$		$f_5 = 9$	$f_6 = 10$	$f_7 = 11$

(57)

Algorithm M (*Multipartitions in decreasing lexicographic order*). Given a multiset $\{n_1 \cdot 1, \ldots, n_m \cdot m\}$, this algorithm visits all of its partitions using arrays $f_0 f_1 \ldots f_n$, $c_0 c_1 \ldots c_n$, $u_0 u_1 \ldots u_n$, and $v_0 v_1 \ldots v_n$ as described above, where $n = n_1 + \cdots + n_m$. We assume that $m > 0$ and $n_1, \ldots, n_m > 0$.

M1. [Initialize.] Set $c_j \leftarrow j+1$ and $u_j \leftarrow v_j \leftarrow n_{j+1}$ for $0 \le j < m$; also set $f_0 \leftarrow a \leftarrow l \leftarrow 0$ and $f_1 \leftarrow b \leftarrow m$. (In the following steps, the current stack frame runs from a to $b-1$, inclusive.)

M2. [Subtract v from u.] (At this point we want to find all partitions of the vector u in the current frame, into parts that are lexicographically $\le v$. First we will use v itself.) Set $j \leftarrow a$ and $k \leftarrow b$. Then while $j < b$ do the following: Set $u_k \leftarrow u_j - v_j$, and if $u_k \ge v_j$ set $c_k \leftarrow c_j$, $v_k \leftarrow v_j$, $k \leftarrow k+1$, $j \leftarrow j+1$. But if u_k is less than v_j after it has been decreased, the action changes: First set $c_k \leftarrow c_j$, $v_k \leftarrow u_k$, and $k \leftarrow k+1$ if u_k was nonzero; then set $j \leftarrow j+1$. While $j < b$, set $u_k \leftarrow u_j - v_j$, $c_k \leftarrow c_j$, $v_k \leftarrow u_k$, and $k \leftarrow k+1$ if $u_j \ne v_j$; then again $j \leftarrow j+1$, until $j = b$.

M3. [Push if nonzero.] If $k > b$, set $a \leftarrow b$, $b \leftarrow k$, $l \leftarrow l+1$, $f_{l+1} \leftarrow b$, and return to M2.

M4. [Visit a partition.] Visit the partition represented by the $l+1$ vectors currently in the stack. (For $0 \le k \le l$, the vector has v_j in component c_j, for $f_k \le j < f_{k+1}$.)

M5. [Decrease v.] Set $j \leftarrow b-1$, and if $v_j = 0$ set $j \leftarrow j-1$ until $v_j > 0$. Then if $j = a$ and $v_j = 1$, go to M6. Otherwise set $v_j \leftarrow v_j - 1$, and $v_k \leftarrow u_k$ for $j < k < b$. Return to M2.

M6. [Backtrack.] Terminate if $l = 0$. Otherwise set $l \leftarrow l-1$, $b \leftarrow a$, $a \leftarrow f_l$, and return to M5. ▌

The key to this algorithm is step M2, which decreases the current residual vector, u, by the largest permissible part, v; that step also decreases v, if necessary, to the lexicographically largest vector $\le v$ that is less than or equal to the new residual amount in every component.

Let us conclude this section by discussing an amusing connection between multipartitions and the least-significant-digit-first procedure for radix sorting (Algorithm 5.2.5R). The idea is best understood by considering an example. See Table 1, where Step (0) shows nine 4-partite column vectors in lexicographic order. Serial numbers ①–⑨ have been attached at the bottom for identification. Step (1) performs a stable sort of the vectors, bringing their fourth (least significant) entries into decreasing order; similarly, Steps (2), (3), and (4) do a stable sort on the third, second, and top rows. The theory of radix sorting tells us that the original lexicographic order is thereby restored.

Suppose the serial number sequences after these stable sorting operations are respectively α_4, $\alpha_3\alpha_4$, $\alpha_2\alpha_3\alpha_4$, and $\alpha_1\alpha_2\alpha_3\alpha_4$, where the α's are permutations; Table 1 shows the values of α_4, α_3, α_2, and α_1 in parentheses. And now comes the point: Wherever the permutation α_j has a descent, the numbers in row j after sorting must also have a descent, because the sorting is stable. (These descents are indicated by caret marks in the table.) For example, where α_3 has 8 followed by 7, we have 5 followed by 3 in row 3. Therefore the entries $a_1 \ldots a_9$ in row 3 after Step (2) are not an arbitrary partition of their sum; they must satisfy

$$a_1 \ge a_2 \ge a_3 \ge a_4 > a_5 \ge a_6 > a_7 \ge a_8 \ge a_9. \qquad (58)$$

Table 1

RADIX SORTING AND MULTIPARTITIONS

Step (0): Original partition	Step (1): Sort row 4	Step (2): Sort row 3	Step (3): Sort row 2	Step (4): Sort row 1
6 5 5 4 3 2 1 0 0	0 6 4 3 5 0 5 2 1	0 6 5 2 5 1 4 3 0	1 2 3 0 6 0 5 5 4	$6\ 5\ 5\ 4_\wedge 3_\wedge 2_\wedge 1\ 0\ 0$
3 2 1 0 4 5 6 4 2	2 3 0 4 2 4 1 5 6	2 3 2 5 1 6 0 4 4	$6_\wedge 5\ 4\ 4_\wedge 3_\wedge 2\ 2\ 1\ 0$	3 2 1 0 4 5 6 4 2
6 6 3 1 1 5 2 0 7	7 6 1 1 6 0 3 5 2	$7\ 6\ 6\ 5_\wedge 3\ 2_\wedge 1\ 1\ 0$	2 5 1 0 6 7 6 3 1	6 6 3 1 1 5 2 0 7
4 2 1 3 3 1 1 2 5	$5_\wedge 4\ 3\ 3_\wedge 2\ 2_\wedge 1\ 1\ 1$	5 4 2 1 1 1 3 3 2	1 1 3 2 4 5 2 1 3	4 2 1 3 3 1 1 2 5
①②③④⑤⑥⑦⑧⑨	⑨①④⑤②⑧③⑥⑦	⑨①②⑥③⑦④⑤⑧	⑦⑥⑤⑧①⑨②③④	①②③④⑤⑥⑦⑧⑨
	$\alpha_4 = (9_\wedge 1\ 4\ 5_\wedge 2\ 8_\wedge 3\ 6\ 7)$	$\alpha_3 = (1\ 2\ 5\ 8_\wedge 7\ 9_\wedge 3\ 4\ 6)$	$\alpha_2 = (6_\wedge 4\ 8\ 9_\wedge 2_\wedge 1\ 3\ 5\ 7)$	$\alpha_1 = (5\ 7\ 8\ 9_\wedge 3_\wedge 2_\wedge 1\ 4\ 6)$

But the numbers $(a_1-2, a_2-2, a_3-2, a_4-2, a_5-1, a_6-1, a_7, a_8, a_9)$ do form an essentially arbitrary partition of the original sum, minus $(4+6)$. The amount of decrease, $4+6$, is the sum of the indices where descents occur; this number is what we called $\operatorname{ind}\alpha_3$, the "index" of α_3, in Section 5.1.1.

Thus we see that any given partition of an m-partite number into at most r parts, with extra zeros added so that the number of columns is exactly r, can be encoded as a sequence of permutations $\alpha_1, \ldots, \alpha_m$ of $\{1,\ldots,r\}$ such that the product $\alpha_1 \ldots \alpha_m$ is the identity, together with a sequence of ordinary one-dimensional partitions of the numbers $(n_1 - \operatorname{ind}\alpha_1, \ldots, n_m - \operatorname{ind}\alpha_m)$ into at most r parts. For example, the vectors in Table 1 represent a partition of $(26, 27, 31, 22)$ into 9 parts; the permutations $\alpha_1, \ldots, \alpha_4$ appear in the table, and we have $(\operatorname{ind}\alpha_1, \ldots, \operatorname{ind}\alpha_4) = (15, 10, 10, 11)$; the partitions are respectively

$$26-15 = (322111100), \qquad 27-10 = (332222210),$$
$$31-10 = (544321110), \qquad 22-11 = (221111111).$$

Conversely, any such permutations and partitions will yield a multipartition of $(n_1, \ldots, n_m)$. If r and m are small, it can be helpful to consider these $r!^{m-1}$ sequences of one-dimensional partitions when listing or reasoning about multipartitions, especially in the bipartite case. [This construction is due to Basil Gordon, *J. London Math. Soc.* **38** (1963), 459–464.]

A good summary of early work on multipartitions, including studies of partitions into distinct parts and/or strictly positive parts, appears in a paper by M. S. Cheema and T. S. Motzkin, *Proc. Symp. Pure Math.* **19** (Amer. Math. Soc., 1971), 39–70.

EXERCISES

1. [*20*] (G. Hutchinson.) Show that a simple modification to Algorithm H will generate all partitions of $\{1, \ldots, n\}$ into *at most* r blocks, given n and $r \geq 2$.

▶ **2.** [*22*] When set partitions are used in practice, we often want to link the elements of each block together. Thus it is convenient to have an array of links $l_1 \ldots l_n$ and an array of headers $h_1 \ldots h_t$ so that the elements of the jth block of a t-block partition are $i_1 > \cdots > i_k$, where

$$i_1 = h_j, \quad i_2 = l_{i_1}, \quad \ldots, \quad i_k = l_{i_{k-1}}, \quad \text{and} \quad l_{i_k} = 0.$$

For example, the representation of 137|25|489|6 would have $t = 4$, $l_1 \ldots l_9 = 001020348$, and $h_1 \ldots h_4 = 7596$.

Design a variant of Algorithm H that generates partitions using this representation.

3. [*M23*] What is the millionth partition of $\{1, \ldots, 12\}$ generated by Algorithm H?

▶ **4.** [*21*] If $x_1 \ldots x_n$ is any string, let $\rho(x_1 \ldots x_n)$ be the restricted growth string that corresponds to the equivalence relation $j \equiv k \iff x_j = x_k$. Classify each of the five-letter English words in the Stanford GraphBase by applying this ρ function; for example, $\rho(\texttt{tooth}) = 01102$. How many of the 52 set partitions of five elements are representable by English words in this way? What's the most common word of each type?

5. [*22*] Guess the next elements of the following two sequences: (a) 0, 1, 1, 1, 12, 12, 12, 12, 12, 12, 100, 121, 122, 123, 123, ...; (b) 0, 1, 12, 100, 112, 121, 122, 123, ...

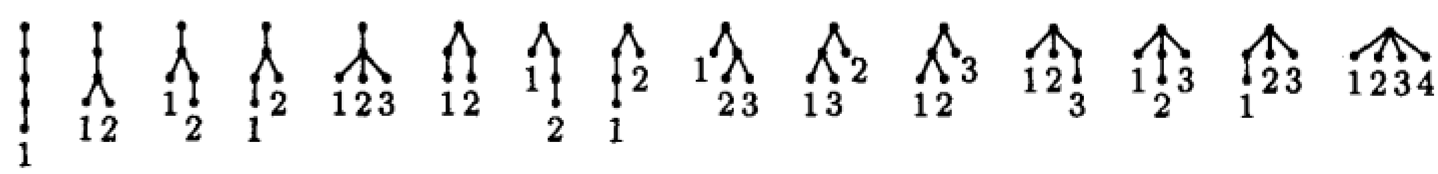
▶ **6.** [*25*] Suggest an algorithm to generate all partitions of $\{1, \ldots, n\}$ in which there are exactly c_1 blocks of size 1, c_2 blocks of size 2, etc.

7. [*M20*] How many permutations $a_1 \ldots a_n$ of $\{1, \ldots, n\}$ have the property that $a_{k-1} > a_k > a_j$ implies $j > k$?

8. [*20*] Suggest a way to generate all permutations of $\{1, \ldots, n\}$ that have exactly m left-to-right minima.

9. [*M20*] How many restricted growth strings $a_1 \ldots a_n$ contain exactly k_j occurrences of j, given the integers $k_0, k_1, \ldots, k_{n-1}$?

10. [*25*] A *semilabeled tree* is an oriented tree in which the leaves are labeled with the integers $\{1, \ldots, k\}$, but the other nodes are unlabeled. Thus there are 15 semilabeled trees with 5 vertices:

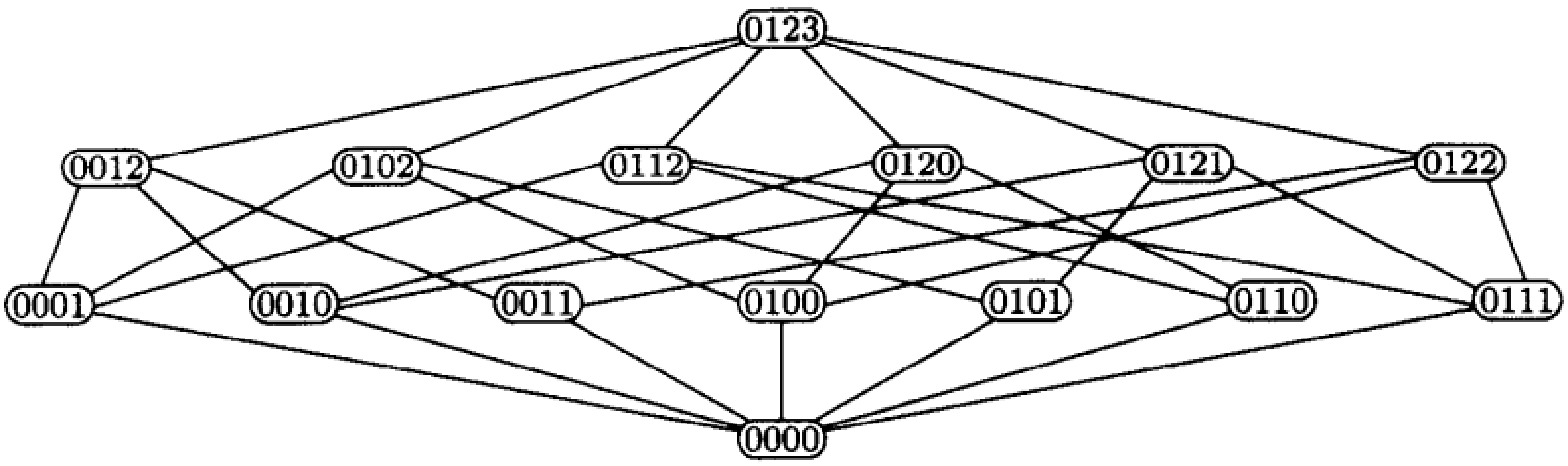


Find a one-to-one correspondence between partitions of $\{1, \ldots, n\}$ and semilabeled trees with $n + 1$ vertices.

▶ **11.** [*28*] We observed in Section 7.2.1.2 that Dudeney's famous problem **send**+**more** = **money** is a "pure" alphametic, namely an alphametic with a unique solution. His puzzle corresponds to a set partition on 13 digit positions, for which the restricted growth string $\rho(\textbf{sendmoremoney})$ is 0123456145217; and we might wonder how lucky he had to be in order to come up with such a construction. How many restricted growth strings of length 13 define pure alphametics of the form $a_1a_2a_3a_4 + a_5a_6a_7a_8 = a_9a_{10}a_{11}a_{12}a_{13}$?

12. [*M31*] (*The partition lattice.*) If Π and Π' are partitions of the same set, we write $\Pi \preceq \Pi'$ if $x \equiv y$ (modulo Π) whenever $x \equiv y$ (modulo Π'). In other words, $\Pi \preceq \Pi'$ means that Π' is a "refinement" of Π, obtained by partitioning zero or more of the latter's blocks; and Π is a "crudification" or *coalescence* of Π', obtained by merging zero or more blocks together. This partial ordering is easily seen to be a lattice, with

$\Pi \vee \Pi'$ the greatest common refinement of Π and Π', and with $\Pi \wedge \Pi'$ their least common coalescence. For example, the lattice of partitions of $\{1, 2, 3, 4\}$ is



if we represent partitions by restricted growth strings $a_1a_2a_3a_4$; upward paths in this diagram take each partition into its refinements. Partitions with t blocks appear on level t from the bottom, and their descendants form the partition lattice of $\{1, \ldots, t\}$.

a) Explain how to compute $\Pi \vee \Pi'$, given $a_1 \ldots a_n$ and $a'_1 \ldots a'_n$.
b) Explain how to compute $\Pi \wedge \Pi'$, given $a_1 \ldots a_n$ and $a'_1 \ldots a'_n$.
c) When does Π' cover Π in this lattice? (See exercise 7.2.1.4–55.)
d) If Π has t blocks of sizes $s_1, \ldots, s_t$, how many partitions does it cover?
e) If Π has t blocks of sizes $s_1, \ldots, s_t$, how many partitions cover it?
f) True or false: If $\Pi \vee \Pi'$ covers Π, then Π' covers $\Pi \wedge \Pi'$.
g) True or false: If Π' covers $\Pi \wedge \Pi'$, then $\Pi \vee \Pi'$ covers Π.
h) Let $b(\Pi)$ denote the number of blocks of Π. Prove that

$$b(\Pi) + b(\Pi') \le b(\Pi \vee \Pi') + b(\Pi \wedge \Pi').$$

13. [*M28*] (Stephen C. Milne, 1977.) If A is a set of partitions of $\{1, \ldots, n\}$, its *shadow* ∂A is the set of all partitions Π' such that Π covers Π' for some $\Pi \in A$. (We considered the analogous concept for the subset lattice in 7.2.1.3–(54).)

Let $\Pi_1, \Pi_2, \ldots$ be the partitions of $\{1, \ldots, n\}$ into t blocks, in lexicographic order of their restricted growth strings; and let $\Pi'_1, \Pi'_2, \ldots$ be the $(t-1)$-block partitions, also in lexicographic order. Prove that there is a function $f_{nt}(N)$ such that

$$\partial\{\Pi_1, \ldots, \Pi_N\} = \{\Pi'_1, \ldots, \Pi'_{f_{nt}(N)}\} \qquad \text{for } 0 \le N \le \left\{ {n \atop t} \right\}.$$

Hint: The diagram in exercise 12 shows that $(f_{43}(0), \ldots, f_{43}(6)) = (0, 3, 5, 7, 7, 7, 7)$.

14. [*23*] Design an algorithm to generate set partitions in Gray-code order like (7).

15. [*M21*] What is the final partition generated by the algorithm of exercise 14?

16. [*16*] The list (11) is Ruskey's A_{35}; what is A'_{35}?

17. [*26*] Implement Ruskey's Gray code (8) for all m-block partitions of $\{1, \ldots, n\}$.

18. [*M46*] For which n is it possible to generate all restricted growth strings $a_1 \ldots a_n$ in such a way that some a_j changes by ± 1 at each step?

19. [*28*] Prove that there's a Gray code for restricted growth strings in which, at each step, some a_j changes by either ± 1 or ± 2, when (a) we want to generate all ϖ_n strings $a_1 \ldots a_n$; or (b) we want to generate only the $\left\{ {n \atop m} \right\}$ cases with $\max(a_1, \ldots, a_n) = m - 1$.

20. [*17*] If Π is a partition of $\{1,\ldots,n\}$, its conjugate Π^T is defined by the rule

$$j \equiv k \pmod{\Pi^T} \quad\iff\quad n+1-j \equiv n+1-k \pmod{\Pi}.$$

Suppose Π has the restricted growth string 001010202013; what is the restricted growth string of Π^T?

21. [*M27*] How many partitions of $\{1,\ldots,n\}$ are self-conjugate?

22. [*M23*] If X is a random variable with a given distribution, the expected value of X^n is called the nth *moment* of that distribution. What is the nth moment when X is (a) a Poisson deviate with mean 1 (Eq. 3.4.1–(40))? (b) the number of fixed points of a random permutation of $\{1,\ldots,m\}$, when $m \ge n$ (Eq. 1.3.3–(27))?

23. [*HM30*] If $f(x) = \sum a_k x^k$ is a polynomial, let $f(\varpi)$ stand for $\sum a_k \varpi_k$.

a) Prove the symbolic formula $f(\varpi+1) = \varpi f(\varpi)$. (For example, if $f(x)$ is the polynomial x^2, this formula states that $\varpi_2 + 2\varpi_1 + \varpi_0 = \varpi_3$.)

b) Similarly, prove that $f(\varpi+k) = \varpi^{\underline{k}} f(\varpi)$ for all positive integers k.

c) If p is prime, prove that $\varpi_{n+p} \equiv \varpi_n + \varpi_{n+1} \pmod{p}$. *Hint:* Show first that $x^{\underline{p}} \equiv x^p - x$.

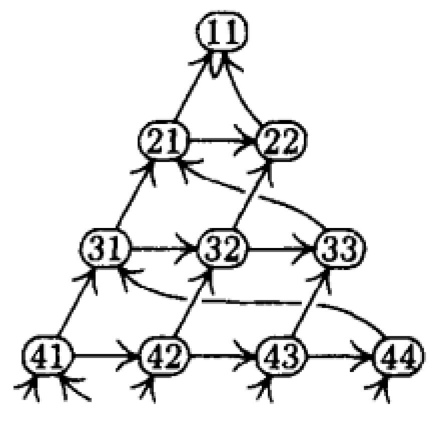
d) Consequently $\varpi_{n+N} \equiv \varpi_n \pmod{p}$ when $N = p^{p-1} + p^{p-2} + \cdots + p + 1$.

24. [*HM35*] Continuing the previous exercise, prove that the Bell numbers satisfy the periodic law $\varpi_{n+p^{e-1}N} \equiv \varpi_n$ (modulo p^e), if p is an odd prime. *Hint:* Show that

$x^{\underline{p^e}} \equiv g_e(x)+1$ (modulo p^e, $p^{e-1}g_1(x)$, ..., and $pg_{e-1}(x)$), where $g_j(x) = (x^p - x - 1)^{p^j}$.

25. [*M27*] Prove that $\varpi_n/\varpi_{n-1} \le \varpi_{n+1}/\varpi_n \le \varpi_n/\varpi_{n-1} + 1$.

▶ **26.** [*M22*] According to the recurrence equations (13), the numbers ϖ_{nk} in Peirce's triangle count the paths from ⓝⓚ to ⑪ in the infinite directed graph



Explain why each path from ⓝ① to ⑪ corresponds to a partition of $\{1,\ldots,n\}$.

▶ **27.** [*M35*] A "vacillating tableau loop" of order n is a sequence of integer partitions $\lambda_k = a_{k1}a_{k2}a_{k3}\ldots$ with $a_{k1} \ge a_{k2} \ge a_{k3} \ge \cdots$ for $0 \le k \le 2n$, such that $\lambda_0 = \lambda_{2n} = e_0$ and $\lambda_k = \lambda_{k-1} + (-1)^k e_{t_k}$ for $1 \le k \le 2n$ and for some t_k with $0 \le t_k \le n$; here e_t denotes the unit vector $0^{t-1}10^{n-t}$ when $0 < t \le n$, and e_0 is all zeros.

a) List all the vacillating tableau loops of order 4. [*Hint:* There are 15 altogether.]

b) Prove that exactly ϖ_{nk} vacillating tableau loops of order n have $t_{2k-1} = 0$.

▶ **28.** [*M25*] (*Generalized rook polynomials.*) Consider an arrangement of $a_1 + \cdots + a_m$ square cells in rows and columns, where row k contains cells in columns $1, \ldots, a_k$. Place zero or more "rooks" into the cells, with at most one rook in each row and at most one in each column. An empty cell is called "free" if there is no rook to its right and no rook below. For example, Fig. 35 shows two such placements, one with four rooks in rows of lengths (3,1,4,1,5,9,2,6,5), and another with nine on a 9×9 square board. Rooks are indicated by solid circles; hollow circles have been placed above and to the left of each rook, thereby leaving the free cells blank.

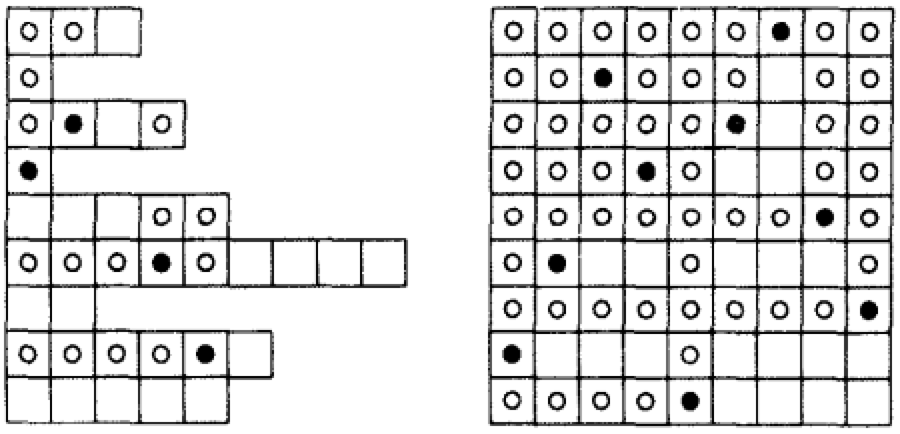


Fig. 35. Rook placements and free cells.

Let $R(a_1, \ldots, a_m)$ be the polynomial in x and y obtained by summing $x^r y^f$ over all legal rook placements, where r is the number of rooks and f is the number of free cells; for example, the left-hand placement in Fig. 35 contributes x^4y^{17} to the polynomial $R(3,1,4,1,5,9,2,6,5)$.

a) Prove that we have $R(a_1,\ldots,a_m) = R(a_1,\ldots,a_{j-1},a_{j+1},a_j,a_{j+2},\ldots,a_m)$; in other words, the order of the row lengths is irrelevant, and we can assume that $a_1 \ge \cdots \ge a_m$ as in a Ferrers diagram like 7.2.1.4–(13).
b) If $a_1 \ge \cdots \ge a_m$ and if $b_1 \ldots b_n = (a_1 \ldots a_m)^T$ is the conjugate partition, prove that $R(a_1,\ldots,a_m) = R(b_1,\ldots,b_n)$.
c) Find a recurrence for evaluating $R(a_1,\ldots,a_m)$ and use it to compute $R(3,2,1)$.
d) Generalize Peirce's triangle (12) by changing the addition rule (13) to

$$\varpi_{nk}(x,y) \;=\; x\varpi_{(n-1)k}(x,y) + y\varpi_{n(k+1)}(x,y), \qquad 1 \le k < n.$$

Thus $\varpi_{21}(x,y) = x+y$, $\varpi_{32}(x,y) = x+xy+y^2$, $\varpi_{31}(x,y) = x^2+2xy+xy^2+y^3$, etc. Prove that the resulting quantity $\varpi_{nk}(x,y)$ is the rook polynomial $R(a_1,\ldots,a_{n-1})$ where $a_j = n-j-[j<k]$.
e) The polynomial $\varpi_{n1}(x,y)$ in part (d) can be regarded as a generalized Bell number $\varpi_n(x,y)$, representing paths from ⓝ① to ①① in the digraph of exercise 26 that have a given number of "x steps" to the northeast and a given number of "y steps" to the east. Prove that

$$\varpi_n(x,y) \;=\; \sum_{a_1\ldots a_n} x^{n-1-\max(a_1,\ldots,a_n)}\,y^{a_1+\cdots+a_n}$$

summed over all restricted growth strings $a_1 \ldots a_n$ of length n.

29. [*M26*] Continuing the previous exercise, let $R_r(a_1,\ldots,a_m) = [x^r]\,R(a_1,\ldots,a_m)$ be the polynomial in y that enumerates free cells when r rooks are placed.

a) Show that the number of ways to place n rooks on an $n\times n$ board, leaving f cells free, is the number of permutations of $\{1,\ldots,n\}$ that have f inversions. Thus, by Eq. 5.1.1–(8) and exercise 5.1.2–16, we have

$$R_n(\overbrace{n,\ldots,n}^{n}) \;=\; n!_y \;=\; \prod_{k=1}^{n}(1+y+\cdots+y^{k-1}).$$

b) What is $R_r(\overbrace{n,\ldots,n}^{m})$, the generating function for r rooks on an $m\times n$ board?
c) If $a_1 \ge \cdots \ge a_m$ and t is a nonnegative integer, prove the general formula

$$\prod_{j=1}^{m} \frac{1-y^{a_j+m-j+t}}{1-y} \;=\; \sum_{k=0}^{m} \frac{t!_y}{(t-k)!_y}\,R_{m-k}(a_1,\ldots,a_m).$$

[*Note:* The quantity $t!_y/(t-k)!_y = \prod_{j=0}^{k-1}((1-y^{t-j})/(1-y))$ is zero when $k > t \geq 0$. Thus, for example, when $t = 0$ the right-hand side reduces to $R_m(a_1, \ldots, a_m)$. We can compute R_m, R_{m-1}, ..., R_0 by successively setting $t = 0, 1, \ldots, m$.]

d) If $a_1 \geq a_2 \geq \cdots \geq a_m \geq 0$ and $a'_1 \geq a'_2 \geq \cdots \geq a'_m \geq 0$, show that we have $R(a_1, a_2 \ldots, a_m) = R(a'_1, a'_2, \ldots, a'_m)$ if and only if the associated multisets $\{a_1{+}m, a_2{+}m{-}1, \ldots, a_m{+}1\}$ and $\{a'_1{+}m, a'_2{+}m{-}1, \ldots, a'_m{+}1\}$ are the same.

30. [*HM30*] The generalized Stirling number $\left\{{n \atop m}\right\}_q$ is defined by the recurrence

$$\left\{{n+1 \atop m}\right\}_q = (1+q+\cdots+q^{m-1})\left\{{n \atop m}\right\}_q + \left\{{n \atop m-1}\right\}_q; \qquad \left\{{0 \atop m}\right\}_q = \delta_{m0}.$$

Thus $\left\{{n \atop m}\right\}_q$ is a polynomial in q; and $\left\{{n \atop m}\right\}_1$ is the ordinary Stirling number $\left\{{n \atop m}\right\}$, because it satisfies the recurrence relation in Eq. 1.2.6–(46).

a) Prove that the generalized Bell number $\varpi_n(x, y) = R(n-1, \ldots, 1)$ of exercise 28(e) has the explicit form

$$\varpi_n(x, y) = \sum_{m=0}^{n} x^{n-m} y^{\binom{m}{2}} \left\{{n \atop m}\right\}_y.$$

b) Show that generalized Stirling numbers also obey the recurrence

$$q^m\left\{{n+1 \atop m+1}\right\}_q = q^n\left\{{n \atop m}\right\}_q + \binom{n}{1} q^{n-1}\left\{{n-1 \atop m}\right\}_q + \cdots = \sum_k \binom{n}{k} q^k \left\{{k \atop m}\right\}_q.$$

c) Find generating functions for $\left\{{n \atop m}\right\}_q$, generalizing 1.2.9–(23) and 1.2.9–(28).

31. [*HM23*] Generalizing (15), show that the elements of Peirce's triangle have a simple generating function, if we compute the sum

$$\sum_{n,k} \varpi_{nk} \frac{w^{n-k}}{(n-k)!} \frac{z^{k-1}}{(k-1)!}.$$

32. [*M22*] Let δ_n be the number of restricted growth strings $a_1 \ldots a_n$ for which the sum $a_1 + \cdots + a_n$ is even minus the number for which $a_1 + \cdots + a_n$ is odd. Prove that

$$\delta_n = (1, 0, -1, -1, 0, 1) \qquad \text{when} \qquad n \bmod 6 = (1, 2, 3, 4, 5, 0).$$

Hint: See exercise 28(e).

33. [*M21*] How many partitions of $\{1, 2, \ldots, n\}$ have $1 \not\equiv 2$, $2 \not\equiv 3$, ..., $k-1 \not\equiv k$?

34. [*14*] Many poetic forms involve *rhyme schemes*, which are partitions of the lines of a stanza with the property that $j \equiv k$ if and only if line j rhymes with line k. For example, a "limerick" is generally a 5-line poem with certain rhythmic constraints and with a rhyme scheme described by the restricted growth string 00110.

What rhyme schemes were used in the classical *sonnets* by (a) Guittone d'Arezzo (c. 1270)? (b) Petrarch (c. 1350)? (c) Spenser (1595)? (d) Shakespeare (1609)? (e) Elizabeth Barrett Browning (1850)?

35. [*M21*] Let ϖ'_n be the number of schemes for n-line poems that are "completely rhymed," in the sense that every line rhymes with at least one other. Thus we have $\langle \varpi'_0, \varpi'_1, \varpi'_2, \ldots \rangle = \langle 1, 0, 1, 1, 4, 11, 41, \ldots \rangle$. Give a combinatorial proof of the fact that $\varpi'_n + \varpi'_{n+1} = \varpi_n$.

36. [*M22*] Continuing exercise 35, what is the generating function $\sum_n \varpi'_n z^n/n!$?

37. [*M18*] Alexander Pushkin adopted an elaborate structure in his poetic novel *Eugene Onegin* (1833), based not only on "masculine" rhymes in which the sounds of accented final syllables agree with each other (pain–gain, form–warm, pun–fun, bucks–crux), but also on "feminine" rhymes in which one or two unstressed syllables also participate (humor–tumor, tetrameter–pentameter, lecture–conjecture, iguana–piranha). Every stanza of *Eugene Onegin* is a sonnet with the strict scheme 01012233455477, where the rhyme is feminine or masculine according as the digit is even or odd. Several modern translators of Pushkin's novel have also succeeded in retaining the same form in English and German.

How do I justify this stanza? / These feminine rhymes? My wrinkled muse?
This whole passé extravaganza? / How can I (careless of time) use
The dusty bread molds of Onegin / In the brave bakery of Reagan?
The loaves will surely fail to rise / Or else go stale before my eyes.
The truth is, I can't justify it. / But as no shroud of critical terms
Can save my corpse from boring worms, / I may as well have fun and try it.
If it works, good; and if not, well, / A theory won't postpone its knell.
— VIKRAM SETH, *The Golden Gate* (1986)

A 14-line poem might have any of $\varpi'_{14} = 24{,}011{,}157$ complete rhyme schemes, according to exercise 35. But how many schemes are possible if we are allowed to specify, for each block, whether its rhyme is to be feminine or masculine?

▸ **38.** [*M30*] Let σ_k be the cyclic permutation $(1, 2, \ldots, k)$. The object of this exercise is to study the sequences $k_1k_2\ldots k_n$, called *σ-cycles*, for which $\sigma_{k_1}\sigma_{k_2}\ldots\sigma_{k_n}$ is the identity permutation. For example, when $n = 4$ there are exactly 15 σ-cycles, namely

1111, 1122, 1212, 1221, 1333, 2112, 2121, 2211, 2222, 2323, 3133, 3232, 3313, 3331, 4444.

a) Find a one-to-one correspondence between partitions of $\{1, 2, \ldots, n\}$ and σ-cycles of length n.
b) How many σ-cycles of length n have $1 \le k_1, \ldots, k_n \le m$, given m and n?
c) How many σ-cycles of length n have $k_i = j$, given i, j, and n?
d) How many σ-cycles of length n have $k_1, \ldots, k_n \ge 2$?
e) How many partitions of $\{1, \ldots, n\}$ have $1 \not\equiv 2$, $2 \not\equiv 3$, …, $n-1 \not\equiv n$, and $n \not\equiv 1$?

39. [*HM16*] Evaluate $\int_0^\infty e^{-t^{p+1}} t^q\,dt$ when p and q are nonnegative integers. *Hint:* See exercise 1.2.5–20.

40. [*HM20*] Suppose the saddle point method is used to estimate $[z^{n-1}]\,e^{cz}$. The text's derivation of (21) from (19) deals with the case $c = 1$; how should that derivation change if c is an arbitrary positive constant?

41. [*HM21*] Solve the previous exercise when $c = -1$.

42. [*HM23*] Use the saddle point method to estimate $[z^{n-1}]\,e^{z^2}$ with relative error $O(1/n^2)$.

43. [*HM22*] Justify replacing the integral in (23) by (25).

44. [*HM22*] Explain how to compute $b_1, b_2, \ldots$ in (26) from $a_2, a_3, \ldots$ in (25).

▸ **45.** [*HM23*] Show that, in addition to (26), we also have the expansion

$$\varpi_n = \frac{e^{e^\xi - 1}\, n!}{\xi^n\sqrt{2\pi n(\xi+1)}}\left(1 + \frac{b'_1}{n} + \frac{b'_2}{n^2} + \cdots + \frac{b'_m}{n^m} + O\Big(\frac{1}{n^{m+1}}\Big)\right),$$

where $b'_1 = -(2\xi^4 + 9\xi^3 + 16\xi^2 + 6\xi + 2)/(24(\xi+1)^3)$.

46. [*HM25*] Estimate the value of ϖ_{nk} in Peirce's triangle when $n \to \infty$.

47. [*M21*] Analyze the running time of Algorithm H.

48. [*HM25*] If n is not an integer, the integral in (23) can be taken over a Hankel contour to define a generalized Bell number ϖ_x for all real $x > 0$. Show that, as in (16),

$$\varpi_x = \frac{1}{e}\sum_{k=0}^{\infty}\frac{k^x}{k!}.$$

▸ **49.** [*HM35*] Prove that, for large n, the number ξ defined in Eq. (24) is equal to

$$\ln n - \ln\ln n + \sum_{j,k\ge 0}\left[{j+k \atop j+1}\right]\alpha^j\frac{\beta^k}{k!}, \qquad \alpha = -\frac{1}{\ln n}, \quad \beta = \frac{\ln\ln n}{\ln n}.$$

▸ **50.** [*HM21*] If $\xi(n)e^{\xi(n)} = n$ and $\xi(n) > 0$, how does $\xi(n+k)$ relate to $\xi(n)$?

51. [*HM27*] Use the saddle point method to estimate $t_n = n!\,[z^n]\,e^{z+z^2/2}$, the number of *involutions* on n elements (aka partitions of $\{1,\ldots,n\}$ into blocks of sizes ≤ 2).

52. [*HM22*] The *cumulants* of a probability distribution are defined in Eq. 1.2.10–(23). What are the cumulants, when the probability that a random integer equals k is (a) $e^{1-e^{\xi}}\varpi_k\xi^k/k!$? (b) $\sum_j {k \brace j}\, e^{e^{-1}-1-j}/k!$?

▸ **53.** [*HM30*] Let $G(z) = \sum_{k=0}^{\infty} p_k z^k$ be the generating function for a discrete probability distribution, converging for $|z| < 1+\delta$; thus the coefficients p_k are nonnegative, $G(1) = 1$, and the mean and variance are respectively $\mu = G'(1)$ and $\sigma^2 = G''(1) + G'(1) - G'(1)^2$. If $X_1, \ldots, X_n$ are independent random variables having this distribution, the probability that $X_1 + \cdots + X_n = m$ is $[z^m]\,G(z)^n$, and we often want to estimate this probability when m is near the mean value μn.

Assume that $p_0 \ne 0$ and that no integer $d > 1$ is a common divisor of all subscripts k with $p_k \ne 0$; this assumption means that m does not have to satisfy any special congruence conditions mod d when n is large. Prove that

$$[z^{\mu n+r}]\,G(z)^n = \frac{e^{-r^2/(2\sigma^2 n)}}{\sigma\sqrt{2\pi n}} + O\Big(\frac{1}{n}\Big) \qquad \text{as } n \to \infty,$$

when $\mu n + r$ is an integer. *Hint:* Integrate $G(z)^n/z^{\mu n+r}$ on the circle $|z| = 1$.

54. [*HM20*] If α and β are defined by (40), show that their arithmetic and geometric means are respectively $\frac{\alpha+\beta}{2} = s\coth s$ and $\sqrt{\alpha\beta} = s\,\mathrm{csch}\, s$, where $s = \sigma/2$.

55. [*HM20*] Suggest a good way to compute the number β needed in (43).

▸ **56.** [*HM26*] Let $g(z) = \alpha^{-1}\ln(e^z - 1) - \ln z$ and $\sigma = \alpha - \beta$ as in (37).

a) Prove that $(-\sigma)^{n+1}g^{(n+1)}(\sigma) = n! - \sum_{k=0}^{n}\left\langle {n \atop k}\right\rangle\alpha^k\beta^{n-k}$, where the Eulerian numbers $\left\langle {n \atop k}\right\rangle$ are defined in Section 5.1.3.

b) Prove that $\frac{\beta}{\alpha}n! < \sum_{k=0}^{n}\left\langle {n \atop k}\right\rangle\alpha^k\beta^{n-k} < n!$ for all $\sigma > 0$. *Hint:* See exercise 5.1.3–25.

c) Now verify the inequality (42).

57. [*HM22*] In the notation of (43), prove that (a) $n+1-m < 2N$; (b) $N < 2(n+1-m)$.

58. [*HM31*] Complete the proof of (43) as follows.

a) Show that for all $\sigma > 0$ there is a number $\tau \ge 2\sigma$ such that τ is a multiple of 2π and $|e^{\sigma+it} - 1|/|\sigma + it|$ is monotone decreasing for $0 \le t \le \tau$.

b) Prove that $\int_{-\tau}^{\tau}\exp\big((n+1)g(\sigma+it)\big)\,dt$ leads to (43).

c) Show that the corresponding integrals over the straight-line paths $z = t \pm i\tau$ for $-n \le t \le \sigma$ and $z = -n \pm it$ for $-\tau \le t \le \tau$ are negligible.

▶ **59.** [*HM23*] What does (43) predict for the approximate value of $\left\{{n \atop n}\right\}$?

60. [*HM25*] (a) Show that the partial sums in the identity

$$\left\{{n \atop m}\right\} = \frac{m^n}{m!} - \frac{(m-1)^n}{1!\,(m-1)!} + \frac{(m-2)^n}{2!\,(m-2)!} - \cdots + (-1)^m \frac{0^n}{m!\,0!}$$

alternately overestimate and underestimate the final value. (b) Conclude that

$$\left\{{n \atop m}\right\} = \frac{m^n}{m!}(1 - O(ne^{-n^\epsilon})) \qquad \text{when } m \le n^{1-\epsilon}.$$

(c) Derive a similar result from (43).

61. [*HM26*] Prove that if $m = n - r$ where $r \le n^\epsilon$ and $\epsilon \le n^{1/2}$, Eq. (43) yields

$$\left\{{n \atop n-r}\right\} = \frac{n^{2r}}{2^r r!}\Bigl(1 + O(n^{2\epsilon-1}) + O\Bigl(\frac{1}{r}\Bigr)\Bigr).$$

62. [*HM40*] Prove rigorously that if $\xi e^\xi = n$, the maximum $\left\{{n \atop m}\right\}$ occurs either when $m = \lfloor e^\xi - 1\rfloor$ or when $m = \lceil e^\xi - 1\rceil$.

▶ **63.** [*M35*] (J. Pitman.) Prove that there is an elementary way to locate the maximum Stirling numbers, and many similar quantities, as follows: Suppose $0 \le p_j \le 1$.

a) Let $f(z) = (1+p_1(z-1))\ldots(1+p_n(1-z))$ and $a_k = [z^k]\,f(z)$; thus a_k is the probability that k heads turn up after n independent coin flips with the respective probabilities $p_1, \ldots, p_n$. Prove that $a_{k-1} < a_k$ whenever $k \le \mu = p_1 + \cdots + p_n$, $a_k \ne 0$.

b) Similarly, prove that $a_{k+1} < a_k$ whenever $k \ge \mu$ and $a_k \ne 0$.

c) If $f(x) = a_0 + a_1x + \cdots + a_nx^n$ is any nonzero polynomial with nonnegative coefficients and with n real roots, prove that $a_{k-1} < a_k$ when $k \le \mu$ and $a_{k+1} < a_k$ when $k \ge \mu$, where $\mu = f'(1)/f(1)$. Therefore if $a_m = \max(a_0, \ldots, a_n)$ we must have either $m = \lfloor\mu\rfloor$ or $m = \lceil\mu\rceil$.

d) Under the hypotheses of (c), and with $a_j = 0$ when $j < 0$ or $j > n$, show that there are indices $s \le t$, such that $a_{k+1} - a_k < a_k - a_{k-1}$ if and only if $s \le k \le t$. (Thus, a histogram of the sequence $(a_0, a_1, \ldots, a_n)$ is always "bell-shaped.")

e) What do these results tell us about Stirling numbers?

64. [*HM21*] Prove the approximate ratio (50), using (30) and exercise 50.

▶ **65.** [*HM22*] What is the variance of the number of blocks of size k in a random partition of $\{1, \ldots, n\}$?

66. [*M46*] What partition of n leads to the most partitions of $\{1, \ldots, n\}$?

67. [*HM20*] What are the mean and variance of M in Stam's method (53)?

68. [*20*] How large can the stack get in Algorithm M, when it is generating all $p(n_1, \ldots, n_m)$ partitions of $\{n_1 \cdot 1, \ldots, n_m \cdot m\}$?

▶ **69.** [*21*] Modify Algorithm M so that it produces only partitions into at most r parts.

▶ **70.** [*M22*] Analyze the number of r-block partitions possible in the n-element multisets (a) $\{0, \ldots, 0, 1\}$; (b) $\{1, 2, \ldots, n-1, n-1\}$. What is the total, summed over r?

71. [*M20*] How many partitions of $\{n_1 \cdot 1, \ldots, n_m \cdot m\}$ have exactly 2 parts?

72. [*M26*] Can $p(n, n)$ be evaluated in polynomial time?

▶ **73.** [*M32*] Can $p(2, \ldots, 2)$ be evaluated in polynomial time when there are n 2s?

74. [*M46*] Can $p(n, \ldots, n)$ be evaluated in polynomial time when there are n ns?

75. [*HM41*] Find the asymptotic value of $p(n, n)$.

76. [*HM36*] Find the asymptotic value of $p(2, \ldots, 2)$ when there are n 2s.

77. [*HM46*] Find the asymptotic value of $p(n, \ldots, n)$ when there are n ns.

78. [*20*] What partition of $(15,10,10,11)$ leads to the permutations α_1, α_2, α_3, and α_4 shown in Table 1?

79. [*22*] A sequence u_1, u_2, u_3, ... is called *universal* for partitions of $\{1,\ldots,n\}$ if its subsequences $(u_{m+1},u_{m+2},\ldots,u_{m+n})$ for $0 \le m < \varpi_n$ represent all possible set partitions under the convention "$j \equiv k$ if and only if $u_{m+j} = u_{m+k}$." For example, $(0,0,0,1,0,2,2)$ is a universal sequence for partitions of $\{1,2,3\}$.

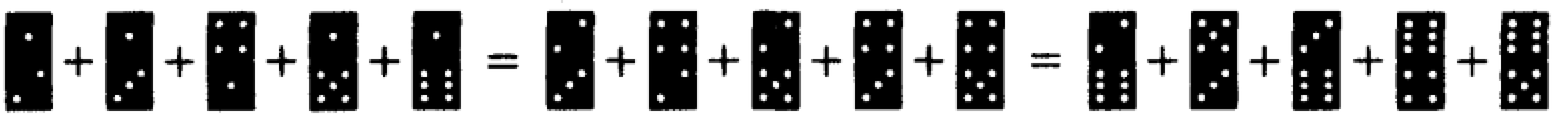
Write a program to find all universal sequences for partitions of $\{1,2,3,4\}$ with the properties that (i) $u_1 = u_2 = u_3 = u_4 = 0$; (ii) the sequence has restricted growth; (iii) $0 \le u_j \le 3$; and (iv) $u_{16} = u_{17} = u_{18} = 0$ (hence the sequence is essentially *cyclic*).

80. [*M28*] Prove that universal cycles for partitions of $\{1,2,\ldots,n\}$ exist in the sense of the previous exercise whenever $n \ge 4$.

81. [*29*] Find a way to arrange an ordinary deck of 52 playing cards so that the following trick is possible: Five players each cut the deck (applying a cyclic permutation) as often as they like. Then each player takes a card from the top. A magician tells them to look at their cards and to form affinity groups, joining with others who hold the same suit: Everybody with clubs gets together, everybody with diamonds forms another group, and so on. (The Jack of Spades is, however, considered to be a "joker"; its holder, if any, should remain aloof.)

Observing the affinity groups, but not being told any of the suits, the magician can name all five cards, if the cards were suitably arranged in the first place.

82. [*22*] In how many ways can the following 15 dominoes, optionally rotated, be partitioned into three sets of five having the same sum when regarded as fractions?



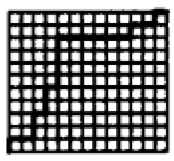
SECTION 7.2.1.3

1. Given a multiset, form the sequence $e_t \ldots e_2 e_1$ from right to left by listing the distinct elements first, then those that appear twice, then those that appear thrice, etc. Let us set $e_{-j} \leftarrow s - j$ for $0 \le j \le s = n - t$, so that every element e_j for $1 \le j \le t$ is equal to some element to its right in the sequence $e_t \ldots e_1 e_0 \ldots e_{-s}$. If the first such element is $e_{c_j - s}$, we obtain a solution to (3). Conversely, every solution to (3) yields a unique multiset $\{e_1, \ldots, e_t\}$, because $c_j < s + j$ for $1 \le j \le t$.

[A similar correspondence was proposed by E. Catalan: If $0 \le e_1 \le \cdots \le e_t \le s$, let

$$\{c_1, \ldots, c_t\} \;=\; \{e_1, \ldots, e_t\} \cup \{s + j \mid 1 \le j < t \text{ and } e_j = e_{j+1}\}.$$

See *Mémoires de la Soc. roy. des Sciences de Liège* (2) **12** (1885), *Mélanges Math.*, 3.]

2. Start at the bottom left corner; then go up for each 0, go right for each 1. The result is .

3. In this algorithm, variable r is the least positive index such that $q_r > 0$.

F1. [Initialize.] Set $q_j \leftarrow 0$ for $1 \le j \le t$, and $q_0 \leftarrow s$. (We assume that $st > 0$.)

F2. [Visit.] Visit the composition $q_t \ldots q_0$. Go to F4 if $q_0 = 0$.

F3. [Easy case.] Set $q_0 \leftarrow q_0 - 1$, $r \leftarrow 1$, and go to F5.

F4. [Tricky case.] Terminate if $r = t$. Otherwise set $q_0 \leftarrow q_r - 1$, $q_r \leftarrow 0$, $r \leftarrow r + 1$.

F5. [Increase q_r.] Set $q_r \leftarrow q_r + 1$ and return to F2. ▮

[See *CACM* **11** (1968), 430; **12** (1969), 187. The task of generating such compositions in *decreasing* lexicographic order is more difficult.]

4. We can reverse the roles of 0 and 1 in (14), so that $0^{q_t}10^{q_{t-1}}1\ldots10^{q_1}10^{q_0} = 1^{r_s}01^{r_{s-1}}0\ldots01^{r_1}01^{r_0}$. This gives $0^1 1 0^0 1 0^2 1 0^2 1 0^4 1 0^0 1 0^0 1 0^0 1 0^0 1 0^0 1 0^1 1 0^0 1 0^1 1 0^0 = 1^0 0 1^2 0 1^0 0 1^1 0 1^0 0 1^1 0 1^0 0 1^0 0 1^0 0 1^6 0 1^2 0 1^1$. Lexicographic order of $a_{n-1} \ldots a_1 a_0$ corresponds to lexicographic order of $r_s \ldots r_1 r_0$.

Incidentally, there's also a multiset connection: $\{d_t, \ldots, d_1\} = \{r_s \cdot s, \ldots, r_0 \cdot 0\}$. For example, $\{10, 10, 8, 6, 2, 2, 2, 2, 2, 2, 1, 1, 0\} = \{0 \cdot 11, 2 \cdot 10, 0 \cdot 9, 1 \cdot 8, 0 \cdot 7, 1 \cdot 6, 0 \cdot 5, 0 \cdot 4, 0 \cdot 3, 6 \cdot 2, 2 \cdot 1, 1 \cdot 0\}$.

5. (a) Set $x_j = c_j - \lfloor (j-1)/2 \rfloor$ in each t-combination of $n + \lfloor t/2 \rfloor$. (b) Set $x_j = c_j + j + 1$ in each t-combination of $n - t - 2$.

(A similar approach finds all solutions $(x_t, \ldots, x_1)$ to the inequalities $x_{j+1} \ge x_j + \delta_j$ for $0 \le j \le t$, given the values of x_{t+1}, $(\delta_t, \ldots, \delta_1)$, and x_0.)

6. Assume that $t > 0$. We get to T3 when $c_1 > 0$; to T5 when $c_2 = c_1 + 1 > 1$; to T4 for $2 \le j \le t+1$ when $c_j = c_1 + j - 1 \ge j$. So the counts are: T1, 1; T2, $\binom{n}{t}$; T3, $\binom{n-1}{t}$; T4, $\binom{n-2}{t-1} + \binom{n-2}{t-2} + \cdots + \binom{n-t-1}{0} = \binom{n-1}{t-1}$; T5, $\binom{n-2}{t-1}$; T6, $\binom{n-1}{t-1} + \binom{n-2}{t-1} - 1$.

7. A procedure slightly simpler than Algorithm T suffices: Assume that $s < n$.

S1. [Initialize.] Set $b_j \leftarrow j + n - s - 1$ for $1 \le j \le s$; then set $j \leftarrow 1$.

S2. [Visit.] Visit the combination $b_s \ldots b_2 b_1$. Terminate if $j > s$.

S3. [Decrease b_j.] Set $b_j \leftarrow b_j - 1$. If $b_j < j$, set $j \leftarrow j + 1$ and return to S2.

S4. [Reset $b_{j-1} \ldots b_1$.] While $j > 1$, set $b_{j-1} \leftarrow b_j - 1$, $j \leftarrow j - 1$, and repeat until $j = 1$. Go to S2. ▮

(See S. Dvořák, *Comp. J.* **33** (1990), 188. Notice that if $x_k = n - b_k$ for $1 \le k \le s$, this algorithm runs through all combinations $x_s \ldots x_2 x_1$ of $\{1, 2, \ldots, n\}$ with $1 \le x_s < \cdots < x_2 < x_1 \le n$, in *increasing* lexicographic order.)

8. **A1.** [Initialize.] Set $a_n \ldots a_0 \leftarrow 0^{s+1}1^t$, $q \leftarrow t$, $r \leftarrow 0$. (We assume that $0 < t < n$.)

A2. [Visit.] Visit the combination $a_{n-1} \ldots a_1 a_0$. Go to A4 if $q = 0$.

A3. [Replace $\ldots 01^q$ by $\ldots 101^{q-1}$.] Set $a_q \leftarrow 1$, $a_{q-1} \leftarrow 0$, $q \leftarrow q - 1$; then if $q = 0$, set $r \leftarrow 1$. Return to A2.

A4. [Shift block of 1s.] Set $a_r \leftarrow 0$ and $r \leftarrow r + 1$. Then if $a_r = 1$, set $a_q \leftarrow 1$, $q \leftarrow q + 1$, and repeat step A4.

A5. [Carry to left.] Terminate if $r = n$; otherwise set $a_r \leftarrow 1$.

A6. [Odd?] If $q > 0$, set $r \leftarrow 0$. Return to A2. ▌

In step A2, q and r point respectively to the rightmost 0 and 1 in $a_{n-1} \ldots a_0$. Steps A1, …, A6 are executed with frequency 1, $\binom{n}{t}$, $\binom{n-1}{t-1}$, $\binom{n}{t} - 1$, $\binom{n-1}{t}$, $\binom{n-1}{t} - 1$.

9. (a) The first $\binom{n-1}{t}$ strings begin with 0 and have $2A_{(s-1)t}$ bit changes; the other $\binom{n-1}{t-1}$ begin with 1 and have $2A_{s(t-1)}$. And $\nu(01^t0^{s-1} \oplus 10^s1^{t-1}) = 2\min(s, t)$.

(b) Solution 1 (direct): Let $B_{st} = A_{st} + \min(s, t) + 1$. Then

$$B_{st} = B_{(s-1)t} + B_{s(t-1)} + [s = t] \quad \text{when } st > 0; \qquad B_{st} = 1 \quad \text{when } st = 0.$$

Consequently $B_{st} = \sum_{k=0}^{\min(s,t)} \binom{s+t-2k}{s-k}$. If $s \le t$ this is $\le \sum_{k=0}^{s} \binom{s+t-k}{s-k} = \binom{s+t+1}{s} = \binom{s+t}{s} \frac{s+t+1}{t+1} < 2\binom{s+t}{t}$.

Solution 2 (indirect): The algorithm in answer 8 makes $2(x + y)$ bit changes when steps (A3, A4) are executed (x, y) times. Thus $A_{st} \le \binom{n-1}{t-1} + \binom{n}{t} - 1 < 2\binom{n}{t}$.

[The comment in answer 7.2.1.1–3 therefore applies to combinations as well.]

10. Each scenario corresponds to a (4, 4)-combination $b_4b_3b_2b_1$ or $c_4c_3c_2c_1$ in which A wins games $\{8-b_4, 8-b_3, 8-b_2, 8-b_1\}$ and N wins games $\{8-c_4, 8-c_3, 8-c_2, 8-c_1\}$, because we can assume that the losing team wins the remaining games in a series of 8. (Equivalently, we can generate all permutations of {A, A, A, A, N, N, N, N} and omit the trailing run of As or Ns.) The American League wins if and only if $b_1 \neq 0$, if and only if $c_1 = 0$. The formula $\binom{c_4}{4} + \binom{c_3}{3} + \binom{c_2}{2} + \binom{c_1}{1}$ assigns a unique integer between 0 and 69 to each scenario.

For example, ANANAA $\Longleftrightarrow a_7 \ldots a_1 a_0 = 01010011 \Longleftrightarrow b_4b_3b_2b_1 = 7532 \Longleftrightarrow c_4c_3c_2c_1 = 6410$, and this is the scenario of rank $\binom{6}{4} + \binom{4}{3} + \binom{1}{2} + \binom{0}{1} = 19$ in lexicographic order. (Notice that the term $\binom{c_j}{j}$ will be zero if and only if it corresponds to a trailing N.)

11. AAAA (9 times), NNNN (8), and ANAAA (7) were most common. Exactly 27 of the 70 failed to occur, including all four beginning with NNNA. (We disregard the games that were tied because of darkness, in 1907, 1912, and 1922. The case ANNAAA should perhaps be excluded too, because it occurred only in 1920 as part of ANNAAAA in a best-of-nine series. The scenario NNAAANN occurred for the first time in 2001.)

12. (a) Let V_j be the subspace $\{a_{n-1} \ldots a_0 \in V \mid a_k = 0 \text{ for } 0 \le k < j\}$, so that $\{0 \ldots 0\} = V_n \subseteq V_{n-1} \subseteq \cdots \subseteq V_0 = V$. Then $\{c_1, \ldots, c_t\} = \{c \mid V_c \neq V_{c+1}\}$, and α_k is the unique element $a_{n-1} \ldots a_0$ of V with $a_{c_j} = [j = k]$ for $1 \le j \le t$.

Incidentally, the $t \times n$ matrix corresponding to a canonical basis is said to be in *reduced row-echelon form*. It can be found by a standard "triangulation" algorithm (see exercise 4.6.1–19 and Algorithm 4.6.2N).

(b) The 2-nomial coefficient $\binom{n}{t}_2 = 2^t\binom{n-1}{t}_2 + \binom{n-1}{t-1}_2$ of exercise 1.2.6–58 has the right properties, because $2^t\binom{n-1}{t}_2$ binary vector spaces have $c_t < n-1$ and $\binom{n-1}{t-1}_2$ have $c_t = n-1$. [In general the number of canonical bases with r asterisks is the number of partitions of r into at most t parts, with no part exceeding $n-t$, and this is $[z^r]\binom{n}{t}_z$ by Eq. 7.2.1.4–(51). See D. E. Knuth, *J. Combinatorial Theory* **10** (1971), 178–180.]

(c) The following algorithm assumes that $n > t > 0$ and that $a_{(t+1)j} = 0$ for $t \le j \le n$.

V1. [Initialize.] Set $a_{kj} \leftarrow [j=k-1]$ for $1 \le k \le t$ and $0 \le j < n$. Also set $q \leftarrow t$, $r \leftarrow 0$.

V2. [Visit.] (At this point we have $a_{k(k-1)} = 1$ for $1 \le k \le q$, $a_{(q+1)q} = 0$, and $a_{1r} = 1$.) Visit the canonical basis $(a_{1(n-1)} \ldots a_{11}a_{10}, \ldots, a_{t(n-1)} \ldots a_{t1}a_{t0})$. Go to V4 if $q > 0$.

V3. [Find block of 1s.] Set $q \leftarrow 1, 2, \ldots,$ until $a_{(q+1)(q+r)} = 0$. Terminate if $q + r = n$.

V4. [Add 1 to column $q+r$.] Set $k \leftarrow 1$. If $a_{k(q+r)} = 1$, set $a_{k(q+r)} \leftarrow 0$, $k \leftarrow k+1$, and repeat until $a_{k(q+r)} = 0$. Then if $k \le q$, set $a_{k(q+r)} \leftarrow 1$; otherwise set $a_{q(q+r)} \leftarrow 1$, $a_{q(q+r-1)} \leftarrow 0$, $q \leftarrow q-1$.

V5. [Shift block right.] If $q = 0$, set $r \leftarrow r+1$. Otherwise, if $r > 0$, set $a_{k(k-1)} \leftarrow 1$ and $a_{k(r+k-1)} \leftarrow 0$ for $1 \le k \le q$, then set $r \leftarrow 0$. Go to V2. ▮

Step V2 finds $q > 0$ with probability $1 - (2^{n-t} - 1)/(2^n - 1) \approx 1 - 2^{-t}$, so we could save time by treating this case separately.

(d) Since $999999 = 4\binom{8}{4}_2 + 16\binom{7}{4}_2 + 5\binom{6}{3}_2 + 5\binom{5}{3}_2 + 8\binom{4}{3}_2 + 0\binom{3}{2}_2 + 4\binom{2}{2}_2 + 1\binom{1}{1}_2 + 2\binom{0}{1}_2$, the millionth output has binary columns 4, 16/2, 5, 5, 8/2, 0, 4/2, 1, 2/2, namely

$$\begin{aligned}\alpha_1 &= 0\,0\,1\,1\,0\,0\,0\,1\,1,\\ \alpha_2 &= 0\,0\,0\,0\,0\,0\,1\,0\,0,\\ \alpha_3 &= 1\,0\,1\,1\,1\,0\,0\,0\,0,\\ \alpha_4 &= 0\,1\,0\,0\,0\,0\,0\,0\,0.\end{aligned}$$

[*Reference:* E. Calabi and H. S. Wilf, *J. Combinatorial Theory* **A22** (1977), 107–109.]

13. Let $n = s + t$. There are $\binom{s-1}{\lceil (r-1)/2\rceil}\binom{t-1}{\lfloor (r-1)/2\rfloor}$ configurations beginning with 0 and $\binom{s-1}{\lfloor (r-1)/2\rfloor}\binom{t-1}{\lceil (r-1)/2\rceil}$ beginning with 1, because an Ising configuration that begins with 0 corresponds to a composition of s 0s into $\lceil (r+1)/2\rceil$ parts and a composition of t 1s into $\lfloor (r+1)/2\rfloor$ parts. We can generate all such pairs of compositions and weave them into configurations. [See E. Ising, *Zeitschrift für Physik* **31** (1925), 253–258; J. M. S. Simões Pereira, *CACM* **12** (1969), 562.]

14. Start with $l[j] \leftarrow j-1$ and $r[j-1] \leftarrow j$ for $1 \le j \le n$; $l[0] \leftarrow n$, $r[n] \leftarrow 0$. To get the next combination, assuming that $t > 0$, set $p \leftarrow s$ if $l[0] > s$, otherwise $p \leftarrow r[n]-1$. Terminate if $p \le 0$; otherwise set $q \leftarrow r[p]$, $l[q] \leftarrow l[p]$, and $r[l[p]] \leftarrow q$. Then if $r[q] > s$ and $p < s$, set $r[p] \leftarrow r[n]$, $l[r[n]] \leftarrow p$, $r[s] \leftarrow r[q]$, $l[r[q]] \leftarrow s$, $r[n] \leftarrow 0$, $l[0] \leftarrow n$; otherwise set $r[p] \leftarrow r[q]$, $l[r[q]] \leftarrow p$. Finally set $r[q] \leftarrow p$ and $l[p] \leftarrow q$.

[See Korsh and Lipschutz, *J. Algorithms* **25** (1997), 321–335, where the idea is extended to a loopless algorithm for multiset permutations. *Caution:* This exercise, like exercise 7.2.1.1–16, is more academic than practical, because the routine that visits the linked list might need a loop that nullifies any advantage of loopless generation.]

15. (The stated fact is true because lexicographic order of $c_t \ldots c_1$ corresponds to lexicographic order of $a_{n-1} \ldots a_0$, which is reverse lexicographic order of the complementary sequence $1 \ldots 1 \oplus a_{n-1} \ldots a_0$.) By Theorem L, the combination $c_t \ldots c_1$ is visited *before* exactly $\binom{b_s}{s} + \cdots + \binom{b_2}{2} + \binom{b_1}{1}$ others have been visited, and we must have

$$\binom{b_s}{s} + \cdots + \binom{b_1}{1} + \binom{c_t}{t} + \cdots + \binom{c_1}{1} = \binom{s+t}{t} - 1.$$

This general identity can be written

$$\sum_{j=0}^{n-1} x_j \binom{j}{x_0 + \cdots + x_j} + \sum_{j=0}^{n-1} \bar{x}_j \binom{j}{\bar{x}_0 + \cdots + \bar{x}_j} = \binom{n}{x_0 + \cdots + x_{n-1}} - 1$$

when each x_j is 0 or 1, and $\bar{x}_j = 1 - x_j$; it follows also from the equation

$$x_n \binom{n}{x_0 + \cdots + x_n} + \bar{x}_n \binom{n}{\bar{x}_0 + \cdots + \bar{x}_n} = \binom{n+1}{x_0 + \cdots + x_n} - \binom{n}{x_0 + \cdots + x_{n-1}}.$$

16. Since $999999 = \binom{1414}{2} + \binom{1008}{1} = \binom{182}{3} + \binom{153}{2} + \binom{111}{1} = \binom{71}{4} + \binom{56}{3} + \binom{36}{2} + \binom{14}{1} = \binom{43}{5} + \binom{32}{4} + \binom{21}{3} + \binom{15}{2} + \binom{6}{1}$, the answers are (a) 1414 1008; (b) 182 153 111; (c) 71 56 36 14; (d) 43 32 21 15 6; (e) 1000000 999999 … 2 0.

17. By Theorem L, n_t is the largest integer such that $N \geq \binom{n_t}{t}$; the remaining terms are the degree-$(t-1)$ representation of $N - \binom{n_t}{t}$.

A simple sequential method for $t > 1$ starts with $x = 1$, $c = t$, and sets $c \leftarrow c+1$, $x \leftarrow xc/(c-t)$ zero or more times until $x > N$; then we complete the first phase by setting $x \leftarrow x(c-t)/c$, $c \leftarrow c-1$, at which point we have $x = \binom{c}{t} \leq N < \binom{c+1}{t}$. Set $n_t \leftarrow c$, $N \leftarrow N - x$; terminate with $n_1 \leftarrow N$ if $t = 2$; otherwise set $x \leftarrow xt/c$, $t \leftarrow t-1$, $c \leftarrow c-1$; while $x > N$ set $x \leftarrow x(c-t)/c$, $c \leftarrow c-1$; repeat. This method requires $O(n)$ arithmetic operations if $N < \binom{n}{t}$, so it is suitable unless t is small and N is large.

When $t = 2$, exercise 1.2.4–41 tells us that $n_2 = \lfloor \sqrt{2N+2} + \frac{1}{2} \rfloor$. In general, n_t is $\lfloor x \rfloor$ where x is the largest root of $x^{\underline{t}} = t!\,N$; this root can be approximated by reverting the series $y = (x^{\underline{t}})^{1/t} = x - \frac{1}{2}(t-1) + \frac{1}{24}(t^2-1)x^{-1} + \cdots$ to get $x = y + \frac{1}{2}(t-1) + \frac{1}{24}(t^2-1)/y + O(y^{-3})$. Setting $y = (t!\,N)^{1/t}$ in this formula gives a good approximation, after which we can check that $\binom{\lfloor x \rfloor}{t} \leq N < \binom{\lfloor x \rfloor + 1}{t}$ or make a final adjustment. [See A. S. Fraenkel and M. Mor, *Comp. J.* **26** (1983), 336–343.]

18. A complete binary tree of $2^n - 1$ nodes is obtained, with an extra node at the top, like the "tree of losers" in replacement selection sorting (Fig. 63 in Section 5.4.1). Therefore explicit links aren't necessary; the right child of node k is node $2k+1$, and the left sibling is node $2k$, for $1 \leq k < 2^{n-1}$.

This representation of a binomial tree has the curious property that node $k = (0^a 1\alpha)_2$ corresponds to the combination whose binary string is $0^a 1 \alpha^R$.

19. It is post(1000000), where $\text{post}(n) = 2^k + \text{post}(n - 2^k + 1)$ if $2^k \leq n < 2^{k+1}$, and $\text{post}(0) = 0$. So it is 11110100001001000100.

20. $f(z) = (1 + z^{w_{n-1}}) \ldots (1 + z^{w_1})/(1-z)$, $g(z) = (1 + z^{w_0}) f(z)$, $h(z) = z^{w_0} f(z)$.

21. The rank of $c_t \ldots c_2 c_1$ is $\binom{c_t+1}{t} - 1$ minus the rank of $c_{t-1} \ldots c_2 c_1$. [See H. Lüneburg, *Abh. Math. Sem. Hamburg* **52** (1982), 208–227.]

22. Since $999999 = \binom{1415}{2} - \binom{406}{1} = \binom{183}{3} - \binom{98}{2} + \binom{21}{1} = \binom{72}{4} - \binom{57}{3} + \binom{32}{2} - \binom{27}{1} = \binom{44}{5} - \binom{40}{4} + \binom{33}{3} - \binom{13}{2} + \binom{3}{1}$, the answers are (a) 1414 405; (b) 182 97 21; (c) 71 56 31 26; (d) 43 39 32 12 3; (e) 1000000 999999 999998 999996 … 0.

23. There are $\binom{n-r}{t-r}$ combinations with $j > r$, for $r = 1, 2, \ldots, t$. (If $r = 1$ we have $c_2 = c_1 + 1$; if $r = 2$ we have $c_1 = 0$, $c_2 = 1$; if $r = 3$ we have $c_1 = 0$, $c_2 = 1$, $c_4 = c_3 + 1$; etc.) Thus the mean is $(\binom{n}{t} + \binom{n-1}{t-1} + \cdots + \binom{n-t}{0})/\binom{n}{t} = \binom{n+1}{t}/\binom{n}{t} = (n+1)/(n+1-t)$. The average running time per step is approximately proportional to this quantity; thus the algorithm is quite fast when t is small, but slow if t is near n.

24. In fact $j_k - 2 \le j_{k+1} \le j_k + 1$ when $j_k \equiv t$ (modulo 2) and $j_k - 1 \le j_{k+1} \le j_k + 2$ when $j_k \not\equiv t$, because R5 is performed only when $c_i = i - 1$ for $1 \le i < j$.

Thus we could say, "If $j \ge 4$, set $j \leftarrow j - 1 - [j \text{ odd}]$ and go to R5" at the end of R2, if t is odd; "If $j \ge 3$, set $j \leftarrow j - 1 - [j \text{ even}]$ and go to R5" if t is even. The algorithm will then be loopless, since R4 and R5 will be performed at most twice per visit.

25. Assume that $N > N'$ and $N - N'$ is minimum; furthermore let t and c_t be minimum, subject to those assumptions. Then $c_t > c'_t$.

If there is an element $x \notin C \cup C'$ with $0 \le x < c_t$, map each t-combination of $C \cup C'$ by changing $j \mapsto j - 1$ for $j > x$; or, if there is an element $x \in C \cap C'$, map each t-combination that contains x into a $(t-1)$-combination by omitting x and changing $j \mapsto x - j$ for $j < x$. In either case the mapping preserves alternating lexicographic order; hence $N - N'$ must exceed the number of combinations between the images of C and C'. But c_t is minimum, so no such x can exist. Consequently $t = m$ and $c_t = 2m - 1$.

Now if $c'_m < c_m - 1$, we could decrease $N - N'$ by increasing c'_m. Therefore $c'_m = 2m-2$, and the problem has been reduced to finding the *maximum* of $\mathrm{rank}(c_{m-1} \ldots c_1) - \mathrm{rank}(c'_{m-1} \ldots c'_1)$, where rank is calculated as in (30).

Let $f(s,t) = \max(\mathrm{rank}(b_s \ldots b_1) - \mathrm{rank}(c_t \ldots c_1))$ over all $\{b_s, \ldots, b_1, c_t \ldots, c_1\} = \{0, \ldots, s+t-1\}$. Then $f(s,t)$ satisfies the curious recurrence

$$f(s,0) = f(0,t) = 0; \qquad f(1,t) = t;$$
$$f(s,t) = \binom{s+t-1}{s} + \max(f(t-1,s-1), f(s-2,t)) \quad \text{if } st > 0 \text{ and } s > 1.$$

When $s + t = 2u + 2$ the solution turns out to be

$$f(s,t) = \binom{2u+1}{t-1} + \sum_{j=1}^{u-r}\binom{2u+1-2j}{r} + \sum_{j=0}^{r-1}\binom{2j+1}{j}, \qquad r = \min(s-2, t-1),$$

with the maximum occurring at $f(t-1, s-1)$ when $s \le t$ and at $f(s-2,t)$ when $s \ge t+2$.

Therefore the minimum $N - N'$ occurs for

$$C = \{2m-1\} \cup \{2m-2-x \mid 1 \le x \le 2m-2,\ \ x \bmod 4 \le 1\},$$
$$C' = \{2m-2\} \cup \{2m-2-x \mid 1 \le x \le 2m-2,\ \ x \bmod 4 \ge 2\};$$

and it equals $\binom{2m-1}{m-1} - \sum_{k=0}^{m-2}\binom{2k+1}{k} = 1 + \sum_{k=1}^{m-1}\binom{2k}{k-1}$. [See A. J. van Zanten, *IEEE Trans.* **IT-37** (1991), 1229–1233.]

26. (a) Yes: The first is $0^{n-\lceil t/2\rceil}1^{t \bmod 2}2^{\lfloor t/2\rfloor}$ and the last is $2^{\lfloor t/2\rfloor}1^{t \bmod 2}0^{n-\lceil t/2\rceil}$; transitions are substrings of the forms $02^a1 \leftrightarrow 12^a0$, $02^a2 \leftrightarrow 12^a1$, $10^a1 \leftrightarrow 20^a0$, $10^a2 \leftrightarrow 20^a1$.

(b) No: If $s = 0$ there is a big jump from 02^t0^{r-1} to 20^r2^{t-1}.

27. The following procedure extracts all combinations $c_1 \ldots c_k$ of Γ_n that have weight $\le t$: Begin with $k \leftarrow 0$ and $c_0 \leftarrow n$. Visit $c_1 \ldots c_k$. If k is even and $c_k = 0$, set $k \leftarrow k - 1$; if k is even and $c_k > 0$, set $c_k \leftarrow c_k - 1$ if $k = t$, otherwise $k \leftarrow k + 1$ and $c_k \leftarrow 0$. On the other hand if k is odd and $c_k + 1 = c_{k-1}$, set $k \leftarrow k - 1$ and

$c_k \leftarrow c_{k+1}$ (but terminate if $k = 0$); if k is odd and $c_k + 1 < c_{k-1}$, set $c_k \leftarrow c_k + 1$ if $k = t$, otherwise $k \leftarrow k+1$, $c_k \leftarrow c_{k-1}$, $c_{k-1} \leftarrow c_k + 1$. Repeat.

(This loopless algorithm reduces to that of exercise 7.2.1.1–12(b) when $t = n$, with slight changes of notation.)

28. True. Bit strings $a_{n-1} \ldots a_0 = \alpha\beta$ and $a'_{n-1} \ldots a'_0 = \alpha\beta'$ correspond to index lists $(b_s \ldots b_1 = \theta\chi,\ c_t \ldots c_1 = \phi\psi)$ and $(b'_s \ldots b'_1 = \theta\chi',\ c'_t \ldots c'_1 = \phi\psi')$ such that everything between $\alpha\beta$ and $\alpha\beta'$ begins with α if and only if everything between $\theta\chi$ and $\theta\chi'$ begins with θ and everything between $\phi\psi$ and $\phi\psi'$ begins with ϕ. For example, if $n = 10$, the prefix $\alpha = 01101$ corresponds to prefixes $\theta = 96$ and $\phi = 875$.

(But just having $c_t \ldots c_1$ in genlex order is a much weaker condition. For example, *every* such sequence is genlex when $t = 1$.)

29. (a) $\texttt{-}^k\texttt{0}^{l+1}$ or $\texttt{-}^k\texttt{0}^{l+1}\texttt{+}\pm^m$ or $\pm^k$, for $k, l, m \ge 0$.

(b) No; the successor is always smaller in balanced ternary notation.

(c) For all α and all $k, l, m \ge 0$ we have $\alpha\texttt{0-}^{k+1}\texttt{0}^l\texttt{+}\pm^m \to \alpha\texttt{-+}^k\texttt{0}^{l+1}\texttt{-}\pm^m$ and $\alpha\texttt{+-}^k\texttt{0}^{l+1}\texttt{+}\pm^m \to \alpha\texttt{0+}^{k+1}\texttt{0}^l\texttt{-}\pm^m$; also $\alpha\texttt{0-}^{k+1}\texttt{0}^l \to \alpha\texttt{-+}^k\texttt{0}^{l+1}$ and $\alpha\texttt{+-}^k\texttt{0}^{l+1} \to \alpha\texttt{0+}^{k+1}\texttt{0}^l$.

(d) Let the jth sign of α_i be $(-1)^{a_{ij}}$, and let it be in position b_{ij}. Then we have $(-1)^{a_{ij}+b_{i(j-1)}} = (-1)^{a_{(i+1)j}+b_{(i+1)(j-1)}}$ for $0 \le i < k$ and $1 \le j \le t$, if we let $b_{i0} = 0$.

(e) By parts (a), (b), and (c), α belongs to some chain $\alpha_0 \to \cdots \to \alpha_k$, where α_k is final (has no successor) and α_0 is initial (has no predecessor). By part (d), every such chain has at most $\binom{s+t}{t}$ elements. But there are 2^s final strings, by (a), and there are $2^s\binom{s+t}{t}$ strings with s signs and t zeros; so k must be $\binom{s+t}{t} - 1$.

Reference: SICOMP **2** (1973), 128–133.

30. Assume that $t > 0$. Initial strings are the negatives of final strings. Let σ_j be the initial string $\texttt{0}^t\texttt{-}\tau_j$ for $0 \le j < 2^{s-1}$, where the kth character of τ_j for $1 \le k < s$ is the sign of $(-1)^{a_k}$ when j is the binary number $(a_{s-1} \ldots a_1)_2$; thus $\sigma_0 = \texttt{0}^t\texttt{-++}\ldots\texttt{+}$, $\sigma_1 = \texttt{0}^t\texttt{--+}\ldots\texttt{+}$, $\ldots$, $\sigma_{2^{s-1}-1} = \texttt{0}^t\texttt{---}\ldots\texttt{-}$. Let ρ_j be the final string obtained by inserting $\texttt{-0}^t$ after the first (possibly empty) run of minus signs in τ_j; thus $\rho_0 = \texttt{-0}^t\texttt{++}\ldots\texttt{+}$, $\rho_1 = \texttt{--0}^t\texttt{+}\ldots\texttt{+}$, $\ldots$, $\rho_{2^{s-1}-1} = \texttt{--}\ldots\texttt{-0}^t$. We also let $\sigma_{2^{s-1}} = \sigma_0$ and $\rho_{2^{s-1}} = \rho_0$. Then we can prove by induction that the chain beginning with σ_j ends with ρ_j when t is even, with ρ_{j-1} when t is odd, for $1 \le j \le 2^{s-1}$. Therefore the chain beginning with $-\rho_j$ ends with $-\sigma_j$ or $-\sigma_{j+1}$.

Let $A_j(s,t)$ be the sequence of (s,t)-combinations derived by mapping the chain that starts with σ_j, and let $B_j(s,t)$ be the analogous sequence derived from $-\rho_j$. Then, for $1 \le j \le 2^{s-1}$, the reverse sequence $A_j(s,t)^R$ is $B_j(s,t)$ when t is even, $B_{j-1}(s,t)$ when t is odd. The corresponding recurrences when $st > 0$ are

$$A_j(s,t) = \begin{cases} 1A_j(s,t-1),\ 0A_{\lfloor(2^{s-1}-1-j)/2\rfloor}(s-1,t)^R, & \text{if } j+t \text{ is even;}\\ 1A_j(s,t-1),\ 0A_{\lfloor j/2\rfloor}(s-1,t), & \text{if } j+t \text{ is odd;}\end{cases}$$

and when $st > 0$ all 2^{s-1} of these sequences are distinct.

Chase's sequence C_{st} is $A_{\lfloor 2^s/3\rfloor}(s,t)$, and $\widehat{C}_{st}$ is $A_{\lfloor 2^{s-1}/3\rfloor}(s,t)$. Incidentally, the homogeneous sequence K_{st} of (31) is $A_{2^{s-1}-[t\text{ even}]}(s,t)^R$.

31. (a) $2^{\binom{s+t}{t}-1}$ solves the recurrence $f(s,t) = 2f(s-1,t)f(s,t-1)$ when $f(s,0) = f(0,t) = 1$. (b) Now $f(s,t) = (s+1)!\,f(s,t-1)\ldots f(0,t-1)$ has the solution

$$(s+1)!^t\, s!^{\binom{t}{2}}\,(s-1)!^{\binom{t+1}{3}} \ldots 2!^{\binom{s+t-2}{s}} = \prod_{r=1}^{s}(r+1)!^{\binom{s+t-1-r}{t-2}+[r=s]}.$$

32. (a) No simple formula seems to exist, but the listings can be counted for small s and t by systematically computing the number of genlex paths that run through all weight-t strings from a given starting point to a given ending point via revolving-door moves. The totals for $s+t \le 6$ are

$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 4 & & 4 & & 1 & & & \\
 & & 1 & & 8 & & 20 & & 8 & & 1 & & \\
 & 1 & & 16 & & 160 & & 160 & & 16 & & 1 & \\
1 & & 32 & & 2264 & & 17152 & & 2264 & & 32 & & 1
\end{array}$$

and $f(4,4) = 95{,}304{,}112{,}865{,}280$; $f(5,5) \approx 5.92646 \times 10^{48}$. [This class of combination generators was first studied by G. Ehrlich, *JACM* **20** (1973), 500–513, but he did not attempt to enumerate them.]

(b) By extending the proof of Theorem N, one can show that all such listings or their reversals must run from 1^t0^s to $0^a1^t0^{s-a}$ for some a, $1 \le a \le s$. Moreover, the number n_{sta} of possibilities, given s, t, and a with $st > 0$, satisfies $n_{1t1} = 1$ and

$$n_{sta} = \begin{cases} n_{s(t-1)1}\,n_{(s-1)t(a-1)}, & \text{if } a > 1; \\ n_{s(t-1)2}\,n_{(s-1)t1} + \cdots + n_{s(t-1)s}\,n_{(s-1)t(s-1)}, & \text{if } a = 1 < s. \end{cases}$$

This recurrence has the remarkable solution $n_{sta} = 2^{m(s,t,a)}$, where

$$m(s,t,a) = \begin{cases} \binom{s+t-3}{t} + \binom{s+t-5}{t-2} + \cdots + \binom{s-1}{2}, & \text{if } t \text{ is even;} \\ \binom{s+t-3}{t} + \binom{s+t-5}{t-2} + \cdots + \binom{s}{3} + s - a - [a<s], & \text{if } t \text{ is odd.} \end{cases}$$

33. Consider first the case $t = 1$: The number of near-perfect paths from i to $j > i$ is $f(j-i-[i>0]-[j<n-1])$, where $\sum_j f(j)z^j = 1/(1-z-z^3)$. (By coincidence, the same sequence $f(j)$ arises in Caron's polyphase merge on 6 tapes, Table 5.4.2–2.) The sum over $0 \le i < j < n$ is $3f(n) + f(n-1) + f(n-2) + 2 - n$; and we must double this, to cover cases with $j > i$.

When $t > 1$ we can construct $\binom{n}{t} \times \binom{n}{t}$ matrices that tell how many genlex listings begin and end with particular combinations. The entries of these matrices are sums of products of matrices for the case $t-1$, summed over all paths of the type considered for $t = 1$. The totals for $s+t \le 6$ turn out to be

$$\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 1 & & 1 & & \\
 & & 1 & 2 & 1 & & \\
 & 1 & 6 & 2 & 1 & & \\
1 & 12 & 10 & 2 & 1 & & \\
1 & 20 & 44 & 10 & 2 & 1 & \\
1 & 34 & 238 & 68 & 10 & 2 & 1
\end{array}
\qquad
\begin{array}{l}
1 \\ 1\ 1 \\ 1\ 2\ 1 \\ 1\ 2\ 0\ 1 \\ 1\ 2\ 2\ 0\ 1 \\ 1\ 2\ 0\ 0\ 0\ 1 \\ 1\ 2\ 6\ 0\ 0\ 0\ 1
\end{array}$$

where the right-hand triangle shows the number of *cycles*, $g(s,t)$. Further values include $f(4,4) = 17736$; $f(5,5) = 9{,}900{,}888{,}879{,}984$; $g(4,4) = 96$; $g(5,5) = 30{,}961{,}456{,}320$.

There are exactly 10 such schemes when $s = 2$ and $n \ge 4$. For example, when $n = 7$ they run from 43210 to 65431 or 65432, or from 54321 to 65420 or 65430 or 65432, or the reverse.

34. The minimum can be computed as in the previous answer, but using min-plus matrix multiplication $c_{ij} = \min_k(a_{ik} + b_{kj})$ instead of ordinary matrix multiplication $c_{ij} = \sum_k a_{ik}b_{kj}$. (When $s = t = 5$, the genlex path in Fig. 26(e) with only 49 imperfect transitions is essentially unique. There is a genlex cycle for $s = t = 5$ that has only 55 imperfections.)

35. From the recurrences (35) we have $a_{st} = b_{s(t-1)} + [s>1][t>0] + a_{(s-1)t}$, $b_{st} = a_{s(t-1)} + a_{(s-1)t}$; consequently $a_{st} = b_{st} + [s>1][t\,\text{odd}]$ and $a_{st} = a_{s(t-1)} + a_{(s-1)t} + [s>1][t\,\text{odd}]$. The solution is

$$a_{st} = \sum_{k=0}^{t/2} \binom{s+t-2-2k}{s-2} - [s>1][t\,\text{even}];$$

this sum is approximately $s/(s+2t)$ times $\binom{s+t}{t}$.

36. Consider the binary tree with root node (s,t) and with recursively defined subtrees rooted at $(s-1,t)$ and $(s,t-1)$ whenever $st>0$; the node (s,t) is a leaf if $st=0$. Then the subtree rooted at (s,t) has $\binom{s+t}{t}$ leaves, corresponding to all (s,t)-combinations $a_{n-1}\ldots a_1a_0$. Nodes on level l correspond to prefixes $a_{n-1}\ldots a_{n-l}$, and leaves on level l are combinations with $r = n-l$.

Any genlex algorithm for combinations $a_{n-1}\ldots a_1a_0$ corresponds to preorder traversal of such a tree, after the children of the $\binom{s+t}{t}-1$ branch nodes have been ordered in any desired way; that, in fact, is why there are $2^{\binom{s+t}{t}-1}$ such genlex schemes (exercise 31(a)). And the operation $j \leftarrow j+1$ is performed exactly once per branch node, namely after both children have been processed.

Incidentally, exercise 7.2.1.2–6(a) implies that the average value of r is $s/(t+1) + t/(s+1)$, which can be $\Omega(n)$; thus the extra time needed to keep track of r is worthwhile.

37. (a) In the lexicographic case we needn't maintain the w_j table, since a_j is active for $j \ge r$ if and only if $a_j = 0$. After setting $a_j \leftarrow 1$ and $a_{j-1} \leftarrow 0$ there are two cases to consider if $j>1$: If $r=j$, set $r \leftarrow j-1$; otherwise set $a_{j-2}\ldots a_0 \leftarrow 0^r1^{j-1-r}$ and $r \leftarrow j-1-r$ (or $r \leftarrow j$ if r was $j-1$).

(b) Now the transitions to be handled when $j>1$ are to change $a_j\ldots a_0$ as follows: $01^r \to 1101^{r-2}$, $010^r \to 10^{r+1}$, $010^a1^r \to 110^{a+1}1^{r-1}$, $10^r \to 010^{r-1}$, $110^r \to 010^{r-1}1$, $10^a1^r \to 0^a1^{r+1}$; these six cases are easily distinguished. The value of r should change appropriately.

(c) Again the case $j=1$ is trivial. Otherwise $01^a0^r \to 101^{a-1}0^r$; $0^a1^r \to 10^a1^{r-1}$; $101^a0^r \to 01^{a+1}0^r$; $10^a1^r \to 0^a1^{r+1}$; and there is also an ambiguous case, which can occur only if $a_{n-1}\ldots a_{j+1}$ contains at least one 0: Let $k>j$ be minimal with $a_k = 0$. Then $10^r \to 010^{r-1}$ if k is odd, $10^r \to 0^r1$ if k is even.

38. The same algorithm works, except that (i) step C1 sets $a_{n-1}\ldots a_0 \leftarrow 01^t0^{s-1}$ if n is odd or $s=1$, $a_{n-1}\ldots a_0 \leftarrow 001^t0^{s-2}$ if n is even and $s>1$, with an appropriate value of r; (ii) step C3 interchanges the roles of even and odd; (iii) step C5 goes to C4 also if $j=1$.

39. In general, start with $r \leftarrow 0$, $j \leftarrow s+t-1$, and repeat the following steps until $st=0$:

$$r \leftarrow r + [w_j = 0]\binom{j}{s-a_j}, \quad s \leftarrow s - [a_j = 0], \quad t \leftarrow t - [a_j = 1], \quad j \leftarrow j-1.$$

Then r is the rank of $a_{n-1}\ldots a_1a_0$. So the rank of 110010010000111111101101010 is $\binom{23}{12}+\binom{22}{11}+\binom{21}{9}+\binom{17}{8}+\binom{16}{7}+\binom{14}{5}+\binom{13}{3}+\binom{12}{3}+\binom{11}{3}+\binom{10}{3}+\binom{9}{3}+\binom{8}{3}+\binom{4}{3}+\binom{3}{1}+\binom{1}{0} = 2390131$.

40. We start with $N \leftarrow 999999$, $v \leftarrow 0$, and repeat the following steps until $st = 0$: If $v = 0$, set $t \leftarrow t - 1$ and $a_{s+t} \leftarrow 1$ if $N < \binom{s+t-1}{s}$, otherwise set $N \leftarrow N - \binom{s+t-1}{s}$, $v \leftarrow (s + t) \bmod 2$, $s \leftarrow s - 1$, $a_{s+t} \leftarrow 0$. If $v = 1$, set $v \leftarrow (s + t) \bmod 2$, $s \leftarrow s - 1$, and $a_{s+t} \leftarrow 0$ if $N < \binom{s+t-1}{t}$, otherwise set $N \leftarrow N - \binom{s+t-1}{t}$, $t \leftarrow t - 1$, $a_{s+t} \leftarrow 1$. Finally if $s = 0$, set $a_{t-1} \ldots a_0 \leftarrow 1^t$; if $t = 0$, set $a_{s-1} \ldots a_0 \leftarrow 0^s$. The answer is $a_{25} \ldots a_0 = 11101001111110101001000001$.

41. Let $c(0), \ldots, c(2^n - 1) = C_n$ where $C_{2n} = 0C_{2n-1},\ 1C_{2n-1};\ \ C_{2n+1} = 0C_{2n},\ 1\widehat{C}_{2n};\ \ \widehat{C}_{2n} = 1C_{2n-1},\ 0\widehat{C}_{2n-1};\ \ \widehat{C}_{2n+1} = 1\widehat{C}_{2n},\ 0\widehat{C}_{2n};\ C_0 = \widehat{C}_0 = \epsilon$. Then $a_j \oplus b_j = b_{j+1} \mathbin{\&} (b_{j+2} \mid (b_{j+3} \mathbin{\&} (b_{j+4} \mid \cdots)))$ if j is even, $b_{j+1} \mid (b_{j+2} \mathbin{\&} (b_{j+3} \mid (b_{j+4} \mathbin{\&} \cdots)))$ if j is odd. Curiously we also have the inverse relation $c((\ldots a_4\bar{a}_3a_2\bar{a}_1a_0)_2) = (\ldots b_4\bar{b}_3b_2\bar{b}_1b_0)_2$.

42. Equation (40) shows that the left context $a_{n-1} \ldots a_{l+1}$ does not affect the behavior of the algorithm on $a_{l-1} \ldots a_0$ if $a_l = 0$ and $l > r$. Therefore we can analyze Algorithm C by counting combinations that end with certain bit patterns, and it follows that the number of times each operation is performed can be represented as $[w^s z^t]\, p(w, z)/(1 - w^2)^2(1 - z^2)^2(1 - w - z)$ for an appropriate polynomial $p(w, z)$.

For example, the algorithm goes from C5 to C4 once for each combination that ends with $01^{2a+1}01^{2b+1}$ or has the form $1^{a+1}01^{2b+1}$, for integers $a, b \ge 0$; the corresponding generating functions are $w^2z^2/(1 - z^2)^2(1 - w - z)$ and $w(z^2 + z^3)/(1 - z^2)^2$.

Here are the polynomials $p(w, z)$ for key operations. Let $W = 1 - w^2$, $Z = 1 - z^2$.

C3 → C4:	$wzW(1+wz)(1-w-z^2)$;	$\mathrm{C5}(r \leftarrow 1)$:	$w^2zW^2Z(1-wz-z^2)$;
C3 → C5:	$wzW(w+z)(1-wz-z^2)$;	$\mathrm{C5}(r \leftarrow j-1)$:	$w^2z^3W^2(1-wz-z^2)$;
C3 → C6:	$w^2z^2W(w+z)$;	$\mathrm{C6}(j = 1)$:	w^2zW^2Z;
C3 → C7:	$w^2zW(1+wz)$;	$\mathrm{C6}(r \leftarrow j-1)$:	$w^2z^3W^2$;
$\mathrm{C4}(j = 1)$:	$wzW^2Z(1-w-z^2)$;	$\mathrm{C6}(r \leftarrow j)$:	w^3z^2WZ;
$\mathrm{C4}(r \leftarrow j-1)$:	$w^3zWZ(1-w-z^2)$;	C7 → C6:	w^2zW^2;
$\mathrm{C4}(r \leftarrow j)$:	$wz^2W^2(1+z-2wz-z^2-z^3)$;	$\mathrm{C7}(r \leftarrow j)$:	w^4zWZ;
C5 → C4:	$wz^2W^2(1-wz-z^2)$;	$\mathrm{C7}(r \leftarrow j-2)$:	$w^3z^2W^2$.
$\mathrm{C5}(r \leftarrow j-2)$:	$w^4zWZ(1-wz-z^2)$;		

The asymptotic value is $\binom{s+t}{t}\bigl(p(1 - x, x)/(2x - x^2)^2(1 - x^2)^2 + O(n^{-1})\bigr)$, for fixed $0 < x < 1$, if $t = xn + O(1)$ as $n \to \infty$. Thus we find, for example, that the four-way branching in step C3 takes place with relative frequencies $x + x^2 - x^3 : 1 : x : 1 + x - x^2$.

Incidentally, the number of cases with j odd exceeds the number of cases with j even by

$$\sum_{k,l \ge 1} \binom{s + t - 2k - 2l}{s - 2k} [2k + 2l \le s + t] + [s\,\mathrm{odd}][t\,\mathrm{odd}],$$

in *any* genlex scheme that uses (39). This quantity has the interesting generating function $wz/(1 + w)(1 + z)(1 - w - z)$.

43. The identity is true for all nonnegative integers x, except when $x = 1$.

44. In fact, $C_t(n) - 1 = \widehat{C}_t(n - 1)^R$, and $\widehat{C}_t(n) - 1 = C_t(n - 1)^R$. (Hence $C_t(n) - 2 = C_t(n - 2)$, etc.)

45. In the following algorithm, r is the least subscript with $c_r \ge r$.

CC1. [Initialize.] Set $c_j \leftarrow n - t - 1 + j$ and $z_j \leftarrow 0$ for $1 \le j \le t + 1$. Also set $r \leftarrow 1$. (We assume that $0 < t < n$.)

CC2. [Visit.] Visit the combination $c_t \ldots c_2c_1$. Then set $j \leftarrow r$.

CC3. [Branch.] Go to CC5 if $z_j \ne 0$.

CC4. [Try to decrease c_j.] Set $x \leftarrow c_j + (c_j \bmod 2) - 2$. If $x \geq j$, set $c_j \leftarrow x$, $r \leftarrow 1$; otherwise if $c_j = j$, set $c_j \leftarrow j - 1$, $z_j \leftarrow c_{j+1} - ((c_{j+1} + 1) \bmod 2)$, $r \leftarrow j$; otherwise if $c_j < j$, set $c_j \leftarrow j$, $z_j \leftarrow c_{j+1} - ((c_{j+1} + 1) \bmod 2)$, $r \leftarrow \max(1, j - 1)$; otherwise set $c_j \leftarrow x$, $r \leftarrow j$. Return to CC2.

CC5. [Try to increase c_j.] Set $x \leftarrow c_j + 2$. If $x < z_j$, set $c_j \leftarrow x$; otherwise if $x = z_j$ and $z_{j+1} \neq 0$, set $c_j \leftarrow x - (c_{j+1} \bmod 2)$; otherwise set $z_j \leftarrow 0$, $j \leftarrow j + 1$, and go to CC3 (but terminate if $j > t$). If $c_1 > 0$, set $r \leftarrow 1$; otherwise set $r \leftarrow j - 1$. Return to CC2. ▮

46. Equation (40) implies that $u_k = (b_j + k + 1) \bmod 2$ when j is minimal with $b_j > k$. Then (37) and (38) yield the following algorithm, where we assume for convenience that $3 \leq s < n$.

CB1. [Initialize.] Set $b_j \leftarrow j - 1$ for $1 \leq j \leq s$; also set $z \leftarrow s + 1$, $b_z \leftarrow 1$. (When subsequent steps examine the value of z, it is the smallest index such that $b_z \neq z - 1$.)

CB2. [Visit.] Visit the dual combination $b_s \ldots b_2 b_1$.

CB3. [Branch.] If b_2 is odd: Go to CB4 if $b_2 \neq b_1 + 1$, otherwise to CB5 if $b_1 > 0$, otherwise to CB6 if b_z is odd. Go to CB9 if b_2 is even and $b_1 > 0$. Otherwise go to CB8 if $b_{z+1} = b_z + 1$, otherwise to CB7.

CB4. [Increase b_1.] Set $b_1 \leftarrow b_1 + 1$ and return to CB2.

CB5. [Slide b_1 and b_2.] If b_3 is odd, set $b_1 \leftarrow b_1 + 1$ and $b_2 \leftarrow b_2 + 1$; otherwise set $b_1 \leftarrow b_1 - 1$, $b_2 \leftarrow b_2 - 1$, $z \leftarrow 3$. Go to CB2.

CB6. [Slide left.] If z is odd, set $z \leftarrow z - 2$, $b_{z+1} \leftarrow z + 1$, $b_z \leftarrow z$; otherwise set $z \leftarrow z - 1$, $b_z \leftarrow z$. Go to CB2.

CB7. [Slide b_z.] If b_{z+1} is odd, set $b_z \leftarrow b_z + 1$ and terminate if $b_z \geq n$; otherwise set $b_z \leftarrow b_z - 1$, then if $b_z < z$ set $z \leftarrow z + 1$. To CB2.

CB8. [Slide b_z and b_{z+1}.] If b_{z+2} is odd, set $b_z \leftarrow b_{z+1}$, $b_{z+1} \leftarrow b_z + 1$, and terminate if $b_{z+1} \geq n$. Otherwise set $b_{z+1} \leftarrow b_z$, $b_z \leftarrow b_z - 1$, then if $b_z < z$ set $z \leftarrow z + 2$. To CB2.

CB9. [Decrease b_1.] Set $b_1 \leftarrow b_1 - 1$, $z \leftarrow 2$, and return to CB2. ▮

Notice that this algorithm is *loopless*. Chase gave a similar procedure for the sequence $\widehat{C}^R_{st}$ in *Cong. Num.* **69** (1989), 233–237. It is truly amazing that this algorithm defines precisely the complements of the indices $c_t \ldots c_1$ produced by the algorithm in the previous exercise.

47. We can, for example, use Algorithm C and its reverse (exercise 38), with w_j replaced by a d-bit number whose bits represent activity at different levels of the recursion. Separate pointers $r_0, r_1, \ldots, r_{d-1}$ are needed to keep track of the r-values on each level. (Many other solutions are possible.)

48. There are permutations $\pi_1, \ldots, \pi_M$ such that the kth element of Λ_j is $\pi_k\alpha_j \uparrow \beta_{k-1}$. And $\pi_k\alpha_j$ runs through all permutations of $\{s_1 \cdot 1, \ldots, s_d \cdot d\}$ as j varies from 0 to $N-1$.

Historical note: The first publication of a homogeneous revolving-door scheme for (s,t)-combinations was by Éva Török, *Matematikai Lapok* **19** (1968), 143–146, who was motivated by the generation of multiset permutations. Many authors have subsequently relied on the homogeneity condition for similar constructions, but this exercise shows that homogeneity is not necessary.

49. We have $\lim_{z\to q}(z^{km+r}-1)/(z^{lm+r}-1) = 1$ when $0 < r < m$, and the limit is $\lim_{z\to q}(kmz^{km-1})/(lmz^{lm-1}) = k/l$ when $r = 0$. So we can pair up factors of the numerator $\prod_{n-k<a\le n}(z^a - 1)$ with factors of the denominator $\prod_{0<b\le k}(z^b - 1)$ when $a \equiv b$ (modulo m).

Notes: This formula was discovered by G. Olive, *AMM* **72** (1965), 619. In the special case $m = 2$, $q = -1$, the second factor vanishes only when n is even and k is odd. The formula $\binom{n}{k}_q = \binom{n}{n-k}_q$ holds for all $n \ge 0$, but $\binom{\lfloor n/m\rfloor}{\lfloor k/m\rfloor}$ is *not* always equal to $\binom{\lfloor n/m\rfloor}{\lfloor (n-k)/m\rfloor}$. We do, however, have $\lfloor k/m\rfloor + \lfloor (n-k)/m\rfloor = \lfloor n/m\rfloor$ in the case when $n \bmod m \ge k \bmod m$; otherwise the second factor is zero.

50. The stated coefficient is zero when $n_1 \bmod m + \cdots + n_t \bmod m \ge m$. Otherwise it equals

$$\binom{\lfloor (n_1+\cdots+n_t)/m\rfloor}{\lfloor n_1/m\rfloor,\ldots,\lfloor n_t/m\rfloor}\binom{(n_1+\cdots+n_t) \bmod m}{n_1 \bmod m,\ldots,n_t \bmod m}_q,$$

by Eq. 1.2.6–(43); here each upper index is the sum of the lower indices.

51. All paths clearly run between 000111 and 111000, since those vertices have degree 1. Fourteen total paths reduce to four under the stated equivalences. The path in (50), which is equivalent to itself under reflection-and-reversal, can be described by the delta sequence A = 3452132523414354123; the other three classes are B = 3452541453414512543, C = 3452541453252154123, D = 3452134145341432543. D. H. Lehmer found path C [*AMM* **72** (1965), Part II, 36–46]; D is essentially the path constructed by Eades, Hickey, and Read.

(Incidentally, perfect schemes aren't really rare, although they seem to be difficult to construct systematically. The case $(s,t) = (3,5)$ has 4,050,046 of them.)

52. We may assume that each s_j is nonzero and that $d > 1$. Then the difference between permutations with an even and odd number of inversions is $\binom{\lfloor (s_0+\cdots+s_d)/2\rfloor}{\lfloor s_0/2\rfloor,\ldots,\lfloor s_d/2\rfloor} \ge 2$, by exercise 50, unless at least two of the multiplicities s_j are odd.

Conversely, if at least two multiplicities are odd, a general construction by G. Stachowiak [*SIAM J. Discrete Math.* **5** (1992), 199–206] shows that a perfect scheme exists. Indeed, his construction applies to a variety of topological sorting problems; in the special case of multisets it gives a Hamiltonian cycle in all cases with $d > 1$ and s_0s_1 odd, except when $d = 2$, $s_0 = s_1 = 1$, and s_2 is even.

53. See *AMM* **72** (1965), Part II, 36–46.

54. Assuming that $st \ne 0$, a Hamiltonian path exists if and only if s and t are not both even; a Hamiltonian cycle exists if and only if, in addition, ($s \ne 2$ and $t \ne 2$) or $n = 5$. [T. C. Enns, *Discrete Math.* **122** (1993), 153–165.]

55. [Solution by Aaron Williams.] The sequence 0^s1^t, W_{st} has the correct properties if

$$W_{st} = 0W_{(s-1)t},\ 1W_{s(t-1)},\ 10^s1^{t-1}, \quad \text{for } st > 0; \qquad W_{0t} = W_{s0} = \emptyset.$$

And there is an amazingly efficient, *loopless* implementation: Assume that $t > 0$.

W1. [Initialize.] Set $n \leftarrow s+t$, $a_j \leftarrow 1$ for $0 \le j < t$, and $a_j \leftarrow 0$ for $t \le j \le n$. Also set $j \leftarrow k \leftarrow t-1$. (This is tricky, but it works.)

W2. [Visit.] Visit the (s,t)-combination $a_{n-1}\ldots a_1a_0$.

W3. [Zero out a_j.] Set $a_j \leftarrow 0$ and $j \leftarrow j+1$.

W4. [Easy case?] If $a_j = 1$, set $a_k \leftarrow 1$, $k \leftarrow k+1$, and return to W2.

W5. [Wrap around.] Terminate if $j = n$. Otherwise set $a_j \leftarrow 1$. Then if $k > 0$, set $a_k \leftarrow 1$, $a_0 \leftarrow 0$, $j \leftarrow 1$, and $k \leftarrow 0$. Return to W2. ▌

After the second visit, j is the smallest index with $a_j a_{j-1} = 10$, and k is smallest with $a_k = 0$. The easy case occurs exactly $\binom{s+t-1}{s} - 1$ times; the condition $k = 0$ occurs in step W5 exactly $\binom{s+t-2}{t} + \delta_{t1}$ times. Curiously, if N has the combinatorial representation (57), the combination of rank N in Algorithm L has rank $N - t + \binom{n_v}{v-1} + v - 1$ in Algorithm W. [*Lecture Notes in Comp. Sci.* **3595** (2005), 570–576.]

(b) (This program assumes that $s > 0$ and $t > 0$.)

```
   SET bits,(1<<t)-1
1H PUSHJ $0,Visit                       Visit bits = (a_{s+t-1}...a_1a_0)_2.
   ADDU $0,bits,1; AND $0,$0,bits       Set $0 ← bits & (bits + 1).
   SUBU $1,$0,1; XOR $1,$0,$1           Set $1 ← $0 ⊕ ($0 − 1).
   ADDU $0,$1,1; AND $1,$1,bits         Set $0 ← $1 + 1, $1 ← $1 & bits.
   AND $0,$0,bits; ODIF $0,$0,1         Set $0 ← ($0 & bits) ∸ 1.
   SUBU $1,$1,$0; ADDU bits,bits,$1     Set bits ← bits + $1 − $0.
   SRU $0,bits,s+t; PBZ $0,1B           Repeat unless a_{s+t} = 1. ▌
```

56. [*Discrete Math.* **48** (1984), 163–171.] This problem is equivalent to the "middle levels conjecture," which states that there is a Gray path through all binary strings of length $2t-1$ and weights $\{t-1, t\}$. In fact, such strings can almost certainly be generated by a delta sequence of the special form $\alpha_0\alpha_1\ldots\alpha_{2t-2}$ where the elements of α_k are those of α_0 shifted by k, modulo $2t-1$. For example, when $t = 3$ we can start with $a_5a_4a_3a_2a_1a_0 = 000111$ and repeatedly swap $a_0 \leftrightarrow a_\delta$, where δ runs through the cycle (4134 5245 1351 2412 3523). The middle levels conjecture is known to be true for $t \le 15$ [see I. Shields and C. D. Savage, *Cong. Num.* **140** (1999), 161–178].

57. Yes; there is a near-perfect genlex solution for all m, n, and t when $n \ge m > t$. One such scheme, in bitstring notation, is $1A_{(m-t)(t-1)}0^{n-m}$, $01A_{(m-t)(t-1)}0^{n-m-1}$, $\ldots$, $0^{n-m}1A_{(m-t)(t-1)}$, $0^{n-m+1}1A_{(m-1-t)(t-1)}$, $\ldots$, $0^{n-t}1A_{0(t-1)}$, using the sequences A_{st} of (35).

58. Solve the previous problem with m and n reduced by $t-1$, then add $j-1$ to each c_j. (Case (a), which is particularly simple, was probably known to Czerny.)

59. The generating function $G_{mnt}(z) = \sum g_{mntk}z^k$ for the number g_{mntk} of chords reachable in k steps from $0^{n-t}1^t$ satisfies $G_{mmt}(z) = \binom{m}{t}_z$ and $G_{m(n+1)t}(z) = G_{mnt}(z) + z^{tn-(t-1)m}\binom{m-1}{t-1}_z$, because the latter term accounts for cases with $c_t = n$ and $c_1 > n - m$. A perfect scheme is possible only if $|G_{mnt}(-1)| \le 1$. But if $n \ge m > t \ge 2$, this condition holds only when $m = t+1$ or $(n-t)t$ is odd, by (49). So there is no perfect solution when $t = 4$ and $m > 5$. (Many chords have only two neighbors when $n = t+2$, so one can easily rule out that case. All cases with $n \ge m > 5$ and $t = 3$ apparently do have perfect paths when n is even.)

60. The following solution uses lexicographic order, taking care to ensure that the average amount of computation per visit is bounded. We may assume that $stm_s \ldots m_0 \ne 0$ and $t \le m_s + \cdots + m_1 + m_0$.

Q1. [Initialize.] Set $q_j \leftarrow 0$ for $s \ge j \ge 1$, and $x = t$.

Q2. [Distribute.] Set $j \leftarrow 0$. Then while $x > m_j$, set $q_j \leftarrow m_j$, $x \leftarrow x - m_j$, $j \leftarrow j+1$, and repeat until $x \le m_j$. Finally set $q_j \leftarrow x$.

Q3. [Visit.] Visit the bounded composition $q_s + \cdots + q_1 + q_0$.

Q4. [Pick up the rightmost units.] If $j = 0$, set $x \leftarrow q_0 - 1$, $j \leftarrow 1$. Otherwise if $q_0 = 0$, set $x \leftarrow q_j - 1$, $q_j \leftarrow 0$, and $j \leftarrow j+1$. Otherwise go to Q7.

Q5. [Full?] Terminate if $j > s$. Otherwise if $q_j = m_j$, set $x \leftarrow x + m_j$, $q_j \leftarrow 0$, $j \leftarrow j+1$, and repeat this step.

Q6. [Increase q_j.] Set $q_j \leftarrow q_j + 1$. Then if $x = 0$, set $q_0 \leftarrow 0$ and return to Q3. (In that case $q_{j-1} = \cdots = q_0 = 0$.) Otherwise go to Q2.

Q7. [Increase and decrease.] (Now $q_i = m_i$ for $j > i \geq 0$.) While $q_j = m_j$, set $j \leftarrow j + 1$ and repeat until $q_j < m_j$ (but terminate if $j > s$). Then set $q_j \leftarrow q_j + 1$, $j \leftarrow j - 1$, $q_j \leftarrow q_j - 1$. If $q_0 = 0$, set $j \leftarrow 1$. Return to Q3. ▌

For example, if $m_s = \cdots = m_0 = 9$, the successors of the composition 3+9+9+7+0+0 are 4+0+0+6+9+9, 4+0+0+7+8+9, 4+0+0+7+9+8, 4+0+0+8+7+9, ...

61. Let $F_s(t) = \emptyset$ if $t < 0$ or $t > m_s + \cdots + m_0$; otherwise let $F_0(t) = t$, and

$$F_s(t) = 0{+}F_{s-1}(t),\ 1{+}F_{s-1}(t-1)^R,\ 2{+}F_{s-1}(t-2),\ \ldots,\ m_s{+}F_{s-1}(t-m_s)^{R^{m_s}}$$

when $s > 0$. This sequence can be shown to have the required properties; it is, in fact, equivalent to the compositions defined by the homogeneous sequence K_{st} of (31) under the correspondence of exercise 4, when restricted to the subsequence defined by the bounds m_s, ..., m_0. [See T. Walsh, *J. Combinatorial Math. and Combinatorial Computing* **33** (2000), 323–345, who has implemented it looplessly.]

62. (a) A $2 \times n$ contingency table with row sums r and $c_1 + \cdots + c_n - r$ is equivalent to solving $r = a_1 + \cdots + a_n$ with $0 \leq a_1 \leq c_1$, ..., $0 \leq a_n \leq c_n$.

(b) We can compute it sequentially by setting $a_{ij} \leftarrow \min(r_i - a_{i1} - \cdots - a_{i(j-1)}, c_j - a_{1j} - \cdots - a_{(i-1)j})$ for $j = 1, \ldots, n$, for $i = 1, \ldots, m$. Alternatively, if $r_1 \leq c_1$, set $a_{11} \leftarrow r_1$, $a_{12} \leftarrow \cdots \leftarrow a_{1n} \leftarrow 0$, and do the remaining rows with c_1 decreased by r_1; if $r_1 > c_1$, set $a_{11} \leftarrow c_1$, $a_{21} \leftarrow \cdots \leftarrow a_{m1} \leftarrow 0$, and do the remaining columns with r_1 decreased by c_1. The second approach shows that at most $m + n - 1$ of the entries are nonzero. We can also write down the explicit formula

$$a_{ij} = \max(0, \min(r_i, c_j, r_1 + \cdots + r_i - c_1 - \cdots - c_{j-1}, c_1 + \cdots + c_j - r_1 - \cdots - r_{i-1})).$$

(c) The same matrix is obtained as in (b).

(d) Reverse left and right in (b) and (c); in both cases the answer is

$$a_{ij} = \max(0, \min(r_i, c_j, r_{i+1} + \cdots + r_m - c_1 - \cdots - c_{j-1}, c_1 + \cdots + c_j - r_i - \cdots - r_m)).$$

(e) Here we choose, say, row-wise order: Generate the first row just as for bounded compositions of r_1, with bounds $(c_1, \ldots, c_n)$; and for each row $(a_{11}, \ldots, a_{1n})$, generate the remaining rows recursively in the same way, but with the column sums $(c_1 - a_{11}, \ldots, c_n - a_{1n})$. Most of the action takes place on the bottom two rows, but when a change is made to an earlier row the later rows must be re-initialized.

63. If a_{ij} and a_{kl} are positive, we obtain another contingency table by setting $a_{ij} \leftarrow a_{ij} - 1$, $a_{il} \leftarrow a_{il} + 1$, $a_{kj} \leftarrow a_{kj} + 1$, $a_{kl} \leftarrow a_{kl} - 1$. We want to show that the graph G whose vertices are the contingency tables for $(r_1, \ldots, r_m; c_1, \ldots, c_n)$, adjacent if they can be obtained from each other by such a transformation, has a Hamiltonian path.

When $m = n = 2$, G is a simple path. When $m = 2$ and $n = 3$, G has a two-dimensional structure from which we can see that every vertex is the starting point of at least two Hamiltonian paths, having distinct endpoints. When $m = 2$ and $n \geq 4$ we can show, inductively, that G actually has Hamiltonian paths from any vertex to any other.

When $m \geq 3$ and $n \geq 3$, we can reduce the problem from m to $m - 1$ as in answer 62(e), if we are careful not to "paint ourselves into a corner." Namely, we must avoid reaching a state where the nonzero entries of the bottom two rows have the form $\binom{1\ a\ 0}{0\ b\ c}$ for some a, b, $c > 0$ and a change to row $m - 2$ forces this to become $\binom{0\ a\ 1}{0\ b\ c}$. The previous round of changes to rows $m-1$ and m can avoid such a trap unless $c=1$ and it begins with $\binom{0\ a+1\ 0}{1\ b-1\ 1}$ or $\binom{1\ a-1\ 1}{0\ b+1\ 0}$. But that situation can be avoided too.

(A genlex method based on exercise 61 would be considerably simpler, and it almost always would make only four changes per step. But it would occasionally need to update $2\min(m,n)$ entries at a time.)

64. When $x_1 \ldots x_s$ is a binary string and A is a list of subcubes, let $A \oplus x_1 \ldots x_s$ denote replacing the digits $(a_1, \ldots, a_s)$ in each subcube of A by $(a_1 \oplus x_1, \ldots, a_s \oplus x_s)$, from left to right. For example, $0{*}1{*}{*}10 \oplus 1010 = 1{*}1{*}{*}00$. Then the following mutual recursions define a Gray cycle, because A_{st} gives a Gray path from $0^s{*}^t$ to $10^{s-1}{*}^t$ and B_{st} gives a Gray path from $0^s{*}^t$ to ${*}01^{s-1}{*}^{t-1}$, when $st>0$:

$$A_{st} = 0B_{(s-1)t},\ {*}A_{s(t-1)} \oplus 001^{s-2},\ 1B^R_{(s-1)t};$$
$$B_{st} = 0A_{(s-1)t},\ 1B_{(s-1)t} \oplus 010^{s-2},\ {*}A_{s(t-1)} \oplus 1^s.$$

The strings 001^{s-2} and 010^{s-2} are simply 0^s when $s<2$; A_{s0} is Gray binary code; $A_{0t}=B_{0t}={*}^t$. (Incidentally, the somewhat simpler construction

$$G_{st} = {*}G_{s(t-1)},\ a_t G_{(s-1)t},\ a_{t-1}G^R_{(s-1)t}, \qquad a_t = t \bmod 2,$$

defines a pleasant Gray *path* from ${*}^t0^s$ to $a_{t-1}{*}^t0^{s-1}$.)

65. If a path P is considered equivalent to P^R and to $P \oplus x_1 \ldots x_s$, the total number can be computed systematically as in exercise 33, with the following results for $s+t \le 6$:

paths	cycles
1	1
1 1	1 1
1 2 1	1 1 1
1 3 3 1	1 1 1 1
1 5 10 4 1	1 2 1 1 1
1 6 36 35 5 1	1 2 3 1 1 1
1 9 310 4630 218 6 1	1 3 46 4 1 1 1

In general there are $t+1$ paths when $s=1$ and $\binom{\lceil s/2\rceil+2}{2}-(s \bmod 2)$ when $t=1$. The cycles for $s \le 2$ are unique. When $s=t=5$ there are approximately 6.869×10^{170} paths and 2.495×10^{70} cycles.

66. Let $G(n,0)=\epsilon$; $G(n,t)=\emptyset$ when $n<t$; and for $1 \le t \le n$, let $G(n,t)$ be

$$\hat{g}(0)G(n-1,t),\ \hat{g}(1)G(n-1,t)^R,\ \ldots,\ \hat{g}(2^t-1)G(n-1,t)^R,\ \hat{g}(2^t-1)G(n-1,t-1),$$

where $\hat{g}(k)$ is a t-bit column containing the Gray binary number $g(k)$ with its least significant bit at the top. In this general formula we implicitly add a row of zeros below the bases of $G(n-1,t-1)$.

This remarkable rule gives ordinary Gray binary code when $t=1$, omitting $0\ldots00$. A cyclic Gray code is impossible because $\binom{n}{t}_2$ is odd.

67. A Gray path for compositions corresponding to Algorithm C implies that there is a path in which all transitions are $0^k1^l \leftrightarrow 1^l0^k$ with $\min(k,l) \le 2$. Perhaps there is, in fact, a cycle with $\min(k,l)=1$ in each transition.

68. (a) $\{\emptyset\}$; (b) $\emptyset$.

69. The least N with $\kappa_t N < N$ is $\binom{2t-1}{t}+\binom{2t-3}{t-1}+\cdots+\binom{1}{1}+1=\frac{1}{2}\bigl(\binom{2t}{t}+\binom{2t-2}{t-1}+\cdots+\binom{0}{0}+1\bigr)$, because $\binom{n}{t-1} \le \binom{n}{t}$ if and only if $n \ge 2t-1$.

70. From the identity

$$\kappa_t(\binom{2t-3}{t}+N')-(\binom{2t-3}{t}+N') = \kappa_t(\binom{2t-2}{t}+N')-(\binom{2t-2}{t}+N') = \binom{2t-2}{t}\frac{1}{t-1}+\kappa_{t-1}N'-N'$$

when $N' < \binom{2t-3}{t}$, we conclude that the maximum is $\binom{2t-2}{t}\frac{1}{t} + \binom{2t-4}{t-1}\frac{1}{t-2} + \cdots + \binom{2}{2}\frac{1}{1}$, and it occurs at 2^{t-1} values of N when $t > 1$.

71. Let C_t be the t-cliques. The first $\binom{1414}{t} + \binom{1009}{t-1}$ t-combinations visited by Algorithm L define a graph on 1415 vertices with 1000000 edges. If $|C_t|$ were larger, $|\partial^{t-2}C_t|$ would exceed 1000000. Thus the single graph defined by $P_{(1000000)2}$ has the maximum number of t-cliques for all $t \geq 2$.

72. $M = \binom{m_s}{s} + \cdots + \binom{m_u}{u}$ for $m_s > \cdots > m_u \geq u \geq 1$, where $\{m_s, \ldots, m_u\} = \{s+t-1, \ldots, n_v\}\setminus\{n_t, \ldots, n_{v+1}\}$. (Compare with exercise 15, which gives $\binom{s+t}{t}-1-N$.)

If $\alpha = a_{n-1}\ldots a_0$ is the bit string corresponding to the combination $n_t \ldots n_1$, then v is 1 plus the number of trailing 1s in α, and u is the length of the rightmost run of 0s. For example, when $\alpha = 1010001111$ we have $s = 4$, $t = 6$, $M = \binom{8}{4} + \binom{7}{3}$, $u = 3$, $N = \binom{9}{6} + \binom{7}{5}$, $v = 5$.

73. A and B are cross-intersecting $\iff \alpha \not\subseteq U \setminus \beta$ for all $\alpha \in A$ and $\beta \in B \iff A \cap \partial^{n-s-t}B^- = \emptyset$, where $B^- = \{U \setminus \beta \mid \beta \in B\}$ is a set of $(n-t)$-combinations. Since $Q^-_{Nnt} = P_{N(n-t)}$, we have $|\partial^{n-s-t}B^-| \geq |\partial^{n-s-t}P_{N(n-t)}|$, and $\partial^{n-s-t}P_{N(n-t)} = P_{N's}$ where $N' = \kappa_{s+1}\ldots\kappa_{n-t}N$. Thus if A and B are cross-intersecting we have $M + N' \leq |A| + |\partial^{n-s-t}B^-| \leq \binom{n}{s}$, and $Q_{Mns} \cap P_{N's} = \emptyset$.

Conversely, if $Q_{Mns} \cap P_{N's} \neq \emptyset$ we have $\binom{n}{s} < M + N' \leq |A| + |\partial^{n-s-t}B^-|$, so A and B cannot be cross-intersecting.

74. $|\varrho Q_{Nnt}| = \kappa_{n-t}N$ (see exercise 94). Also, arguing as in (58) and (59), we find $\varrho P_{N5} = (n-1)P_{N5} \cup \cdots \cup 10P_{N5} \cup \{543210, \ldots, 987654\}$ in that particular case; and $|\varrho P_{Nt}| = (n+1-n_t)N + \binom{n_t+1}{t+1}$ in general.

75. The identity $\binom{n+1}{k} = \binom{n}{k} + \binom{n-1}{k-1} + \cdots + \binom{n-k}{0}$, Eq. 1.2.6–(10), gives another representation if $n_v > v$. But (60) is unaffected, since we have $\binom{n+1}{k-1} = \binom{n}{k-1} + \binom{n-1}{k-2} + \cdots + \binom{n-k+1}{0}$.

76. Represent $N+1$ by adding $\binom{v-1}{v-1}$ to (57); then use the previous exercise to deduce that $\kappa_t(N+1) - \kappa_t N = \binom{v-1}{v-2} = v - 1$.

77. [D. E. Daykin, *Nanta Math.* **8**, 2 (1975), 78–83.] We work with extended representations $M = \binom{m_t}{t} + \cdots + \binom{m_u}{u}$ and $N = \binom{n_t}{t} + \cdots + \binom{n_v}{v}$ as in exercise 75, calling them *improper* if the final index u or v is zero. Call N *flexible* if it has both proper and improper representations, that is, if $n_v > v > 0$.

(a) Given an integer S, find $M + N$ such that $M + N = S$ and $\kappa_t M + \kappa_t N$ is minimum, with M as large as possible. If $N = 0$, we're done. Otherwise the max-min operation preserves both $M + N$ and $\kappa_t M + \kappa_t N$, so we can assume that $v \geq u \geq 1$ in the proper representations of M and N. If N is inflexible, $\kappa_t(M+1) + \kappa_t(N-1) = (\kappa_t M + u - 1) + (\kappa_t N - v) < \kappa_t M + \kappa_t N$, by exercise 76; therefore N must be flexible. But then we can apply the max-min operation to M and the improper representation of N, increasing M: Contradiction.

This proof shows that equality holds if and only if $MN = 0$, a fact that was noted in 1927 by F. S. Macaulay.

(b) Now we try to minimize $\max(\kappa_t M, N) + \kappa_{t-1}N$ when $M + N = S$, this time representing N as $\binom{n_{t-1}}{t-1} + \cdots + \binom{n_v}{v}$. The max-min operation can still be used if $n_{t-1} < m_t$; leaving m_t unchanged, it preserves $M + N$ and $\kappa_t M + \kappa_{t-1}N$ as well as the relation $\kappa_t M > N$. We arrive at a contradiction as in (a) if $N \neq 0$, so we can assume that $n_{t-1} \geq m_t$.

If $n_{t-1} > m_t$ we have $N > \kappa_t M$ and also $\lambda_t N > M$; hence $M + N < \lambda_t N + N = \binom{n_{t-1}+1}{t} + \cdots + \binom{n_v+1}{v}$, and we have $\kappa_t(M+N) \leq \kappa_t(\lambda_t N + N) = N + \kappa_{t-1}N$.

Finally if $n_{t-1} = m_t = a$, let $M = \binom{a}{t} + M'$ and $N = \binom{a}{t-1} + N'$. Then $\kappa_t(M+N) = \binom{a+1}{t-1} + \kappa_{t-1}(M' + N')$, $\kappa_t M = \binom{a}{t-1} + \kappa_{t-1}M'$, and $\kappa_{t-1}N = \binom{a}{t-2} + \kappa_{t-2}N'$; the result follows by induction on t.

78. [J. Eckhoff and G. Wegner, *Periodica Math. Hung.* **6** (1975), 137–142; A. J. W. Hilton, *Periodica Math. Hung.* **10** (1979), 25–30.] Let $M = |A_1|$ and $N = |A_0|$; we can assume that $t > 0$ and $N > 0$. Then $|\partial A| = |\partial A_1 \cup A_0| + |\partial A_0| \geq \max(|\partial A_1|, |A_0|) + |\partial A_0| \geq \max(\kappa_t M, N) + \kappa_{t-1}N \geq \kappa_t(M+N) = |P_{|A|t}|$, by induction on $m + n + t$.

Conversely, let $A_1 = P_{Mt} + 1$ and $A_0 = P_{N(t-1)} + 1$; this notation means, for example, that $\{210, 320\} + 1 = \{321, 431\}$. Then $\kappa_t(M+N) \leq |\partial A| = |\partial A_1 \cup A_0| + |(\partial A_0)0| = \max(\kappa_t M, N) + \kappa_{t-1}N$, because $\partial A_1 = P_{(\kappa_t M)(t-1)} + 1$. [Schützenberger observed in 1959 that $\kappa_t(M+N) \leq \kappa_t M + \kappa_{t-1}N$ if and only if $\kappa_t M \geq N$.]

For the first inequality, let A and B be disjoint sets of t-combinations with $|A| = M$, $|\partial A| = \kappa_t M$, $|B| = N$, $|\partial B| = \kappa_t N$. Then $\kappa_t(M+N) = \kappa_t|A \cup B| \leq |\partial(A \cup B)| = |\partial A \cup \partial B| = |\partial A| + |\partial B| = \kappa_t M + \kappa_t N$.

79. In fact, $\mu_t(M + \lambda_{t-1}M) = M$, and $\mu_t N + \lambda_{t-1}\mu_t N = N + (n_2 - n_1)[v = 1]$ when N is given by (57).

80. If $N > 0$ and $t > 1$, represent N as in (57) and let $N = N_0 + N_1$, where

$$N_0 = \binom{n_t - 1}{t} + \cdots + \binom{n_v - 1}{v}, \qquad N_1 = \binom{n_t - 1}{t-1} + \cdots + \binom{n_v - 1}{v-1}.$$

Let $N_0 = \binom{y}{t}$ and $N_1 = \binom{z}{t-1}$. Then, by induction on t and $\lfloor x \rfloor$, we have $\binom{x}{t} = N_0 + \kappa_t N_0 \geq \binom{y}{t} + \binom{y}{t-1} = \binom{y+1}{t}$; $N_1 = \binom{x}{t} - \binom{y}{t} \geq \binom{x}{t} - \binom{x-1}{t} = \binom{x-1}{t-1}$; and $\kappa_t N = N_1 + \kappa_{t-1}N_1 \geq \binom{z}{t-1} + \binom{z}{t-2} = \binom{z+1}{t-1} \geq \binom{x}{t-1}$.

[Lovász actually proved a stronger result; see exercise 1.2.6–66. We have, similarly, $\mu_t N \geq \binom{x-1}{t-1}$; see Björner, Frankl, and Stanley, *Combinatorica* **7** (1987), 27–28.]

81. For example, if the largest element of $\widehat{P}_{N5}$ is 66433, we have

$$\widehat{P}_{N5} = \{00000, \ldots, 55555\} \cup \{60000, \ldots, 65555\} \cup \{66000, \ldots, 66333\} \cup \{66400, \ldots, 66433\}$$

so $N = \binom{10}{5} + \binom{9}{4} + \binom{6}{3} + \binom{5}{2}$. Its lower shadow is

$$\partial\widehat{P}_{N5} = \{0000, \ldots, 5555\} \cup \{6000, \ldots, 6555\} \cup \{6600, \ldots, 6633\} \cup \{6640, \ldots, 6643\},$$

of size $\binom{9}{4} + \binom{8}{3} + \binom{5}{2} + \binom{4}{1}$.

If the smallest element of Q_{N95} is 66433, we have

$$\widehat{Q}_{N95} = \{99999, \ldots, 70000\} \cup \{66666, \ldots, 66500\} \cup \{66444, \ldots, 66440\} \cup \{66433\}$$

so $N = (\binom{13}{9} + \binom{12}{8} + \binom{11}{7}) + (\binom{8}{6} + \binom{7}{5}) + \binom{5}{4} + \binom{3}{3}$. Its upper shadow is

$$\varrho\widehat{Q}_{N95} = \{999999, \ldots, 700000\} \cup \{666666, \ldots, 665000\} \cup \{664444, \ldots, 664400\} \cup \{664333, \ldots, 664330\},$$

of size $(\binom{14}{9} + \binom{13}{8} + \binom{12}{7}) + (\binom{9}{6} + \binom{8}{5}) + \binom{6}{4} + \binom{4}{3} = N + \kappa_9 N$. The size, t, of each combination is essentially irrelevant, as long as $N \leq \binom{s+t}{t}$; for example, the smallest element of $\widehat{Q}_{N98}$ is 99966433 in the case we have considered.

82. (a) The derivative would have to be $\sum_{k>0} r_k(x)$, but that series diverges.

[Informally, the graph of $\tau(x)$ shows "pits" of relative magnitude 2^{-k} at all odd multiples of 2^{-k}. Takagi's original publication, in *Proc. Physico-Math. Soc. Japan* (2) **1** (1903), 176–177, has been translated into English in his *Collected Papers* (Iwanami Shoten, 1973).]

(b) Since $r_k(1-t) = (-1)^{\lceil 2^k t\rceil}$ when $k > 0$, we have $\int_0^{1-x} r_k(t)\,dt = \int_x^1 r_k(1-u)\,du = -\int_x^1 r_k(u)\,du = \int_0^x r_k(u)\,du$. The second equation follows from the fact that $r_k(\frac{1}{2}t) = r_{k-1}(t)$. Part (d) shows that these two equations suffice to define $\tau(x)$ when x is rational.

(c) Since $\tau(2^{-a}x) = a2^{-a}x + 2^{-a}\tau(x)$ for $0 \le x \le 1$, we have $\tau(\epsilon) = a\epsilon + O(\epsilon)$ when $2^{-a-1} \le \epsilon \le 2^{-a}$. Therefore $\tau(\epsilon) = \epsilon \lg \frac{1}{\epsilon} + O(\epsilon)$ for $0 < \epsilon \le 1$.

(d) Suppose $0 \le p/q \le 1$. If $p/q \le 1/2$ we have $\tau(p/q) = p/q + \tau(2p/q)/2$; otherwise $\tau(p/q) = (q-p)/q + \tau(2(q-p)/q)/2$. Therefore we can assume that q is odd. When q is odd, let $p' = p/2$ when p is even, $p' = (q-p)/2$ when p is odd. Then $\tau(p/q) = 2\tau(p'/q) - 2p'/q$ for $0 < p < q$; this system of $q-1$ equations has a unique solution. For example, the values for $q = 3, 4, 5, 6, 7$ are 2/3, 2/3; 1/2, 1/2, 1/2; 8/15, 2/3, 2/3, 8/15; 1/2, 2/3, 1/2, 2/3, 1/2; 22/49, 30/49, 32/49, 32/49, 30/49, 22/49.

(e) The solutions $< \frac{1}{2}$ are $x = \frac{1}{4}, \frac{1}{4} - \frac{1}{16}, \frac{1}{4} - \frac{1}{16} - \frac{1}{64}, \frac{1}{4} - \frac{1}{16} - \frac{1}{64} - \frac{1}{256}, \ldots, \frac{1}{6}$.

(f) The value $\frac{2}{3}$ is achieved for $x = \frac{1}{2} \pm \frac{1}{8} \pm \frac{1}{32} \pm \frac{1}{128} \pm \cdots$, an uncountable set.

83. Given any integers $q > p > 0$, consider paths starting from 0 in the digraph

$$\begin{array}{ccccccccccccc} 0 & \leftarrow & 1 & \leftarrow & 2 & \leftarrow & 3 & \leftarrow & 4 & \leftarrow & 5 & \leftarrow & \cdots \\ \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & & \\ 1 & \rightarrow & 2 & \rightarrow & 3 & \rightarrow & 4 & \rightarrow & 5 & \rightarrow & 6 & \rightarrow & \cdots \end{array}$$

Compute an associated value v, starting with $v \leftarrow -p$; horizontal moves change $v \leftarrow 2v$, vertical moves from node a change $v \leftarrow 2(qa - v)$. The path stops if we reach a node twice with the same value v. Transitions are not allowed to upper node a if $v \le -q$ or $v \ge qa$ at that node; they are not allowed to lower node a with $v \le 0$ or $v \ge q(a+1)$. These restrictions force most steps of the path. (Node a in the upper row means, "Solve $\tau(x) = ax - v/q$"; in the lower row it means, "Solve $\tau(x) = v/q - ax$.") Empirical tests suggest that all such paths are finite. The equation $\tau(x) = p/q$ then has solutions $x = x_0$ defined by the sequence $x_0, x_1, x_2, \ldots$ where $x_k = \frac{1}{2}x_{k+1}$ on a horizontal step and $x_k = 1 - \frac{1}{2}x_{k+1}$ on a vertical step; eventually $x_k = x_j$ for some $j < k$. If $j > 0$ and if q is not a power of 2, these are all the solutions to $\tau(x) = p/q$ when $x > 1/2$.

For example, this procedure establishes that $\tau(x) = 1/5$ and $x > 1/2$ only when x is 83581/87040; the only path yields $x_0 = 1 - \frac{1}{2}x_1$, $x_1 = \frac{1}{2}x_2$, ..., $x_{18} = \frac{1}{2}x_{19}$, and $x_{19} = x_{11}$. There are, similarly, just two values $x > 1/2$ with $\tau(x) = 3/5$, having denominator $2^{46}(2^{56} - 1)/3$.

Moreover, it appears that all cycles in the digraph that pass through node 0 define values of p and q such that $\tau(x) = p/q$ has uncountably many solutions. Such values are, for example, 2/3, 8/15, 8/21, corresponding to the cycles (01), (0121), (012321). The value 32/63 corresponds to (012121) and also to (012101234545454321), as well as to two other paths that do not return to 0.

84. [Frankl, Matsumoto, Ruzsa, and Tokushige, *J. Combinatorial Theory* **A69** (1995), 125–148.] If $a \le b$ we have

$$\binom{2t-1-b}{t-a}\Big/T = t^{\underline{a}}(t-1)^{\underline{b-a}}/(2t-1)^{\underline{b}} = 2^{-b}\bigl(1 + f(a,b)t^{-1} + O(b^4/t^2)\bigr),$$

where $f(a,b) = a(1+b) - a^2 - b(1+b)/4 = f(a+1,b) - b + 2a$. Therefore if N has the combinatorial representation (57), and if we set $n_j = 2t - 1 - b_j$, we have

$$\frac{t}{T}\Big(\kappa_t N - N\Big) = \frac{b_t}{2^{b_t}} + \frac{b_{t-1}-2}{2^{b_{t-1}}} + \frac{b_{t-2}-4}{2^{b_{t-2}}} + \cdots + \frac{O(\log t)^3}{t},$$

the terms being negligible when b_j exceeds $2\lg t$. And one can show that

$$\tau\Big(\sum_{j=0}^{l} 2^{-e_j}\Big) = \sum_{j=0}^{l} (e_j - 2j)2^{-e_j}.$$

85. $N - \lambda_{t-1}N$ has the same asymptotic form as $\kappa_t N - N$, by (63), since $\tau(x) = \tau(1-x)$. So does $2\mu_t N - N$, up to $O(T(\log t)^3/t^2)$, because $\binom{2t-1-b}{t-a} = 2\binom{2t-2-b}{t-a}(1+O(\log t)/t)$ when $b < 2\lg t$.

86. $x \in X^{\circ\sim} \iff \bar{x} \notin X^\circ \iff \bar{x} \notin X$ or $\bar{x} \notin X + e_1$ or $\cdots$ or $\bar{x} \notin X + e_n \iff x \in X^\sim$ or $x \in X^\sim - e_1$ or $\cdots$ or $x \in X^\sim - e_n \iff x \in X^{\sim+}$.

87. All three are true, using the fact that $X \subseteq Y^\circ$ if and only if $X^+ \subseteq Y$: (a) $X \subseteq Y^\circ \iff X^\sim \supseteq Y^{\circ\sim} = Y^{\sim+} \iff Y^\sim \subseteq X^{\sim\circ}$. (b) $X^+ \subseteq X^+ \implies X \subseteq X^{+\circ}$; hence $X^\circ \subseteq X^{\circ+\circ}$. Also $X^\circ \subseteq X^\circ \implies X^{\circ+} \subseteq X$; hence $X^{\circ+\circ} \subseteq X^\circ$. (c) $\alpha M \le N \iff S_M^+ \subseteq S_N \iff S_M \subseteq S_N^\circ \iff M \le \beta N$.

88. If $\nu x < \nu y$ then $\nu(x - e_k) < \nu(y - e_j)$, so we can assume that $\nu x = \nu y$ and that $x > y$ in lexicographic order. We must have $y_j > 0$; otherwise $\nu(y - e_j)$ would exceed $\nu(x - e_k)$. If $x_i = y_i$ for $1 \le i \le j$, clearly $k > j$ and $x - e_k \prec y - e_j$. Otherwise $x_i > y_i$ for some $i \le j$; again we have $x - e_k \prec y - e_j$, unless $x - e_k = y - e_j$.

89. From the table

$j =$	0	1	2	3	4	5	6	7	8	9	10	11
$e_j + e_1 =$	e_1	e_0	e_4	e_5	e_2	e_3	e_8	e_9	e_6	e_7	e_{11}	e_{10}
$e_j + e_2 =$	e_2	e_4	e_0	e_6	e_1	e_8	e_3	e_{10}	e_5	e_{11}	e_7	e_9
$e_j + e_3 =$	e_3	e_5	e_6	e_7	e_8	e_9	e_{10}	e_0	e_{11}	e_1	e_2	e_4

we find $(\alpha 0, \alpha 1, \ldots, \alpha 12) = (0,4,6,7,8,9,10,11,11,12,12,12,12)$; $(\beta 0, \beta 1, \ldots, \beta 12) = (0,0,0,0,1,1,2,3,4,5,6,8,12)$.

90. Let $Y = X^+$ and $Z = C_k X$, and let $N_a = |X_k(a)|$ for $0 \le a < m_k$. Then

$$|Y| = \sum_{a=0}^{m_k-1} |Y_k(a)| = \sum_{a=0}^{m_k-1} \big|(X_k(a-1)+e_k) \cup (X_k(a)+E_k(0))\big|$$
$$\ge \sum_{a=0}^{m_k-1} \max(N_{a-1}, \alpha N_a),$$

where $a-1$ stands for $(a-1) \bmod m_k$ and the α function comes from the $(n-1)$-dimensional torus, because $|X_k(a) + E_k(0)| \ge \alpha N_a$ by induction. Also

$$|Z^+| = \sum_{a=0}^{m_k-1} |Z_k^+(a)| = \sum_{a=0}^{m_k-1} \big|(Z_k(a-1)+e_k) \cup (Z_k(a)+E_k(0))\big|$$
$$= \sum_{a=0}^{m_k-1} \max(N_{a-1}, \alpha N_a),$$

because both $Z_k(a-1)+e_k$ and $Z_k(a)+E_k(0)$ are standard in $n-1$ dimensions.

91. Let there be N_a points in row a of a totally compressed array, where row 0 is at the bottom; thus $l = N_{-1} \geq N_0 \geq \cdots \geq N_{m-1} \geq N_m = 0$. We show first that there is an optimum X for which the "bad" condition $N_a = N_{a+1}$ never occurs except when $N_a = 0$ or $N_a = l$. For if a is the smallest bad subscript, suppose $N_{a-1} > N_a = N_{a+1} = \cdots = N_{a+k} > N_{a+k+1}$. Then we can always decrease N_{a+k} by 1 and add 1 to some N_b for $b \leq a$ without increasing $|X^+|$, except in cases where $k = 1$ and $N_{a+2} = N_{a+1} - 1$ and $N_b = N_a + a - b < l$ for $0 \leq b \leq a$. Exploring such cases further, if $N_{c+1} < N_c = N_{c-1}$ for some $c > a+1$, we can set $N_c \leftarrow N_c - 1$ and $N_a \leftarrow N_a + 1$, thereby either decreasing a or increasing N_0. Otherwise we can find a subscript d such that $N_c = N_{a+1} + a + 1 - c > 0$ for $a < c < d$, and either $N_d = 0$ or $N_d < N_{d-1} - 1$. Then it is OK to decrease N_c by 1 for $a < c < d$ and subsequently to increase N_b by 1 for $0 \leq b < d - a - 1$. (It is important to note that if $N_d = 0$ we have $N_0 \geq d - 1$; hence $d = m$ implies $l = m$.)

Repeating such transformations until $N_a > N_{a+1}$ whenever $N_a \neq l$ and $N_{a+1} \neq 0$, we reach situation (86), and the proof can be completed as in the text.

92. Let $x + k$ denote the lexicographically smallest element of $T(m_1, \ldots, m_{n-1})$ that exceeds x and has weight $\nu x + k$, if any such element exists. For example, if $m_1 = m_2 = m_3 = 4$ and $x = 211$, we have $x+1 = 212$, $x+2 = 213$, $x+3 = 223$, $x+4 = 233$, $x+5 = 333$, and $x+6$ does not exist; in general, $x+k+1$ is obtained from $x+k$ by increasing the rightmost component that can be increased. If $x + k = (m_1 - 1, \ldots, m_{n-1} - 1)$, let us set $x + k + 1 = x + k$. Then if $S(k)$ is the set of all elements of $T(m_1, \ldots, m_{n-1})$ that are $\preceq x + k$, we have $S(k+1) = S(k)^+$. Furthermore, the elements of S that end in a are those whose first $n-1$ components are in $S(m-1-a)$.

The result of this exercise can be stated more intuitively: As we generate n-dimensional standard sets S_1, S_2, ..., the $(n-1)$-dimensional standard sets on each layer become spreads of each other just after each point is added to layer $m-1$. Similarly, they become cores of each other just before each point is added to layer 0.

93. (a) Suppose the parameters are $2 \leq m'_1 \leq m'_2 \leq \cdots \leq m'_n$ when sorted properly, and let k be minimal with $m_k \neq m'_k$. Then take $N = 1 + \mathrm{rank}(0, \ldots, 0, m'_k - 1, 0, \ldots, 0)$. (We must assume that $\min(m_1, \ldots, m_n) \geq 2$, since parameters equal to 1 can be placed anywhere.)

(b) Only in the proof for $n = 2$, buried inside the answer to exercise 91. That proof is incorporated by induction when n is larger.

94. Complementation reverses lexicographic order and changes ϱ to ∂.

95. For Theorem K, let $d = n - 1$ and $s_0 = \cdots = s_d = 1$. For Theorem M, let $d = s$ and $s_0 = \cdots = s_d = t + 1$.

96. In such a representation, N is the number of t-multicombinations of $\{s_0 \cdot 0, s_1 \cdot 1, s_2 \cdot 2, \ldots\}$ that precede $n_t n_{t-1} \ldots n_1$ in lexicographic order, because the generalized coefficient $\binom{S(n)}{t}$ counts the multicombinations whose leftmost component is $< n$.

If we truncate the representation by stopping at the rightmost nonzero term $\binom{S(n_v)}{v}$, we obtain a nice generalization of (60):

$$|\partial P_{Nt}| = \binom{S(n_t)}{t-1} + \binom{S(n_{t-1})}{t-2} + \cdots + \binom{S(n_v)}{v-1}.$$

[See G. F. Clements, *J. Combinatorial Theory* **A37** (1984), 91–97. The inequalities $s_0 \geq s_1 \geq \cdots \geq s_d$ are needed for the validity of Corollary C, but not for the calculation of $|\partial P_{Nt}|$. Some terms $\binom{S(n_k)}{k}$ for $t \geq k > v$ may be zero. For example, when $N = 1$, $t = 4$, $s_0 = 3$, and $s_1 = 2$, we have $N = \binom{S(1)}{4} + \binom{S(1)}{3} = 0 + 1$.]

97. (a) The tetrahedron has four vertices, six edges, four faces: $(N_0,\ldots,N_4) = (1,4,6,4,1)$. The octahedron, similarly, has $(N_0,\ldots,N_6) = (1,6,8,8,0,0,0)$, and the icosahedron has $(N_0,\ldots,N_{12}) = (1,12,30,20,0,\ldots,0)$. The hexahedron, aka the 3-cube, has eight vertices, 12 edges, and six square faces; perturbation breaks each square face into two triangles and introduces new edges, so we have $(N_0,\ldots,N_8) = (1,8,18,12,0,\ldots,0)$. Finally, the perturbed pentagonal faces of the dodecahedron lead to $(N_0,\ldots,N_{20}) = (1,20,54,36,0,\ldots,0)$.

(b) $\{210,310\} \cup \{10,20,21,30,31\} \cup \{0,1,2,3\} \cup \{\epsilon\}$.

(c) $0 \le N_t \le \binom{n}{t}$ for $0 \le t \le n$ and $N_{t-1} \ge \kappa_t N_t$ for $1 \le t \le n$. The second condition is equivalent to $\lambda_{t-1}N_{t-1} \ge N_t$ for $1 \le t \le n$, if we define $\lambda_0 1 = \infty$. These conditions are necessary for Theorem K, and sufficient if $A = \bigcup P_{N_t t}$.

(d) The complements of the elements not in a simplicial complex, namely the sets $\{\,\{0,\ldots,n-1\} \setminus \alpha \mid \alpha \notin C\,\}$, form a simplicial complex. (We can also verify that the necessary and sufficient condition holds: $N_{t-1} \ge \kappa_t N_t \iff \lambda_{t-1}N_{t-1} \ge N_t \iff \kappa_{n-t+1}\overline{N}_{n-t+1} \le \overline{N}_{n-t}$, because $\kappa_{n-t}\overline{N}_{n-t+1} = \binom{n}{t} - \lambda_{t-1}N_{t-1}$ by exercise 94.)

(e) 00000 $\leftrightarrow$ 14641; 10000 $\leftrightarrow$ 14640; 11000 $\leftrightarrow$ 14630; 12000 $\leftrightarrow$ 14620; 13000 $\leftrightarrow$ 14610; 14000 $\leftrightarrow$ 14600; 12100 $\leftrightarrow$ 14520; 13100 $\leftrightarrow$ 14510; 14100 $\leftrightarrow$ 14500; 13200 $\leftrightarrow$ 14410; 14200 $\leftrightarrow$ 14400; 13300 $\leftrightarrow$ 14400; and the self-dual cases 14300, 13310.

98. The following procedure by S. Linusson [*Combinatorica* **19** (1999), 255–266], who considered also the more general problem for multisets, is considerably faster than a more obvious approach. Let $L(n,h,l)$ count feasible vectors with $N_t = \binom{n}{t}$ for $0 \le t \le l$, $N_{l+1} < \binom{n}{l+1}$, and $N_t = 0$ for $t > h$. Then $L(n,h,l) = 0$ unless $-1 \le l \le h \le n$; also $L(n,h,h) = L(n,h,-1) = 1$, and $L(n,n,l) = L(n,n-1,l)$ for $l < n$. When $n > h \ge l \ge 0$ we can compute $L(n,h,l) = \sum_{j=l}^{h} L(n-1,h,j)L(n-1,j-1,l-1)$, a recurrence that follows from Theorem K. (Each size vector corresponds to the complex $\bigcup P_{N_t t}$, with $L(n-1,h,j)$ representing combinations that do not contain the maximum element $n-1$ and $L(n-1,j-1,l-1)$ representing those that do.) Finally the grand total is $L(n) = \sum_{l=1}^{n} L(n,n,l)$.

We have $L(0)$, $L(1)$, $L(2)$, $\ldots$ = 2, 3, 5, 10, 26, 96, 553, 5461, 100709, 3718354, 289725509, $\ldots$; $L(100) \approx 3.2299 \times 10^{1842}$.

99. The maximal elements of a simplicial complex form a clutter; conversely, the combinations contained in elements of a clutter form a simplicial complex. Thus the two concepts are essentially equivalent.

(a) If $(M_0,M_1,\ldots,M_n)$ is the size vector of a clutter, then $(N_0,N_1,\ldots,N_n)$ is the size vector of a simplicial complex if $N_n = M_n$ and $N_t = M_t + \kappa_{t+1}N_{t+1}$ for $0 \le t < n$. Conversely, every such $(N_0,\ldots,N_n)$ yields an $(M_0,\ldots,M_n)$ if we use the lexicographically first N_t t-combinations. [G. F. Clements extended this result to general multisets in *Discrete Math.* **4** (1973), 123–128.]

(b) In the order of answer 97(e) they are 00000, 00001, 10000, 00040, 01000, 00030, 02000, 00120, 03000, 00310, 04000, 00600, 00100, 00020, 01100, 00210, 02100, 00500, 00200, 00110, 01200, 00400, 00300, 01010, 01300, 00010. Notice that $(M_0,\ldots,M_n)$ is feasible if and only if $(M_n,\ldots,M_0)$ is feasible, so we have a different sort of duality in this interpretation.

100. Represent A as a subset of $T(m_1,\ldots,m_n)$ as in the proof of Corollary C. Then the maximum value of νA is obtained when A consists of the N lexicographically smallest points $x_1\ldots x_n$.

The proof starts by reducing to the case that A is compressed, in the sense that its t-multicombinations are $P_{|A\cap T_t|t}$ for each t. Then if y is the largest element $\in A$

and if x is the smallest element $\notin A$, we prove that $x < y$ implies $\nu x > \nu y$, hence $\nu(A \setminus \{y\} \cup \{x\}) > \nu A$. For if $\nu x = \nu y - k$ we could find an element of $\partial^k y$ that is greater than x, contradicting the assumption that A is compressed.

101. (a) In general, $F(p) = N_0 p^n + N_1 p^{n-1}(1-p) + \cdots + N_n(1-p)^n$ when $f(x_1, \ldots, x_n)$ is satisfied by exactly N_t binary strings $x_1 \ldots x_n$ of weight t. Thus we find $G(p) = p^4 + 3p^3(1-p) + p^2(1-p)^2$; $H(p) = p^4 + p^3(1-p) + p^2(1-p)^2$.

(b) A monotone formula f is equivalent to a simplicial complex C under the correspondence $f(x_1, \ldots, x_n) = 1 \iff \{j-1 \mid x_j = 0\} \in C$. Therefore the functions $f(p)$ of monotone Boolean functions are those that satisfy the condition of exercise 97(c), and we obtain a suitable function by choosing the lexicographically last N_{n-t} t-combinations (which are complements of the first N_s s-combinations): $\{3210\}$, $\{321, 320, 310\}$, $\{32\}$ gives $f(w,x,y,z) = wxyz \vee xyz \vee wyz \vee wxz \vee yz = wxz \vee yz$.

M. P. Schützenberger observed that we can find the parameters N_t easily from $f(p)$ by noting that $f\bigl(1/(1+u)\bigr) = (N_0 + N_1 u + \cdots + N_n u^n)/(1+u)^n$. One can show that $H(p)$ is not equivalent to a monotone formula in any number of variables, because $(1+u+u^2)/(1+u)^4 = (N_0 + N_1 u + \cdots + N_n u^n)/(1+u)^n$ implies that $N_1 = n-3$, $N_2 = \binom{n-3}{2} + 1$, and $\kappa_2 N_2 = n-2$.

But the task of deciding this question is not so simple in general. For example, the function $(1 + 5u + 5u^2 + 5u^3)/(1+u)^5$ does not match any monotone formula in five variables, because $\kappa_3 5 = 7$; but it equals $(1 + 6u + 10u^2 + 10u^3 + 5u^4)/(1+u)^6$, which works fine with six.

102. (a) Choose N_t linearly independent polynomials of degree t in I; order their terms lexicographically, and take linear combinations so that the lexicographically smallest terms are distinct monomials. Let I' consist of all multiples of those monomials.

(b) Each monomial of degree t in I' is essentially a t-multicombination; for example, $x_1^3 x_2 x_5^4$ corresponds to 55552111. If M_t is the set of independent monomials for degree t, the ideal property is equivalent to saying that $M_{t+1} \supseteq \varrho M_t$.

In the given example, $M_3 = \{x_0 x_1^2\}$; $M_4 = \varrho M_3 \cup \{x_0 x_1 x_2^2\}$; $M_5 = \varrho M_4 \cup \{x_1 x_2^4\}$, since $x_2^2(x_0 x_1^2 - 2x_1 x_2^2) - x_1(x_0 x_1 x_2^2) = -2x_1 x_2^4$; and $M_{t+1} = \varrho M_t$ thereafter.

(c) By Theorem M we can assume that $M_t = \widehat{Q}_{Mst}$. Let $N_t = \binom{n_{ts}}{s} + \cdots + \binom{n_{t2}}{2} + \binom{n_{t1}}{1}$, where $s + t \ge n_{ts} > \cdots > n_{t2} > n_{t1} \ge 0$; then $n_{ts} = s+t$ if and only if $n_{t(s-1)} = s-2, \ldots, n_{t1} = 0$. Furthermore we have

$$N_{t+1} \ge N_t + \kappa_s N_t = \binom{n_{ts} + [n_{ts} \ge s]}{s} + \cdots + \binom{n_{t2} + [n_{t2} \ge 2]}{2} + \binom{n_{t1} + [n_{t1} \ge 1]}{1}.$$

Therefore the sequence $(n_{ts} - t - \infty[n_{ts} < s], \ldots, n_{t2} - t - \infty[n_{t2} < 2], n_{t1} - t - \infty[n_{t1} < 1])$ is lexicographically nondecreasing as t increases, where we insert '$-\infty$' in components that have $n_{tj} = j-1$. Such a sequence cannot increase infinitely many times without exceeding the maximum value $(s, -\infty, \ldots, -\infty)$, by exercise 1.2.1–15(d).

103. Let P_{Nst} be the first N elements of a sequence determined as follows: For each binary string $x = x_{s+t-1} \ldots x_0$, in lexicographic order, write down $\binom{\nu x}{t}$ subcubes by changing t of the 1s to $*$s in all possible ways, in lexicographic order (considering $1 < *$). For example, if $x = 0101101$ and $t = 2$, we generate the subcubes 0101*0*, 010*10*, 010**01, 0*0110*, 0*01*01, 0*0*101.

[See B. Lindström, *Arkiv för Mat.* **8** (1971), 245–257; a generalization analogous to Corollary C appears in K. Engel, *Sperner Theory* (Cambridge Univ. Press, 1997), Theorem 8.1.1.]

104. The first N strings in cross order have the desired property. [T. N. Danh and D. E. Daykin, *J. London Math. Soc.* (2) **55** (1997), 417–426.]

Notes: Beginning with the observation that the "1-shadow" of the N lexicographically first strings of weight t (namely the strings obtained by deleting 1 bits only) consists of the first $\mu_t N$ strings of weight t, R. Ahlswede and N. Cai extended the Danh–Daykin theorem to allow insertion, deletion, and/or transposition of bits [*Combinatorica* **17** (1997), 11–29; *Applied Math. Letters* **11**, 5 (1998), 121–126]. Uwe Leck has proved that no total ordering of *ternary strings* has the analogous minimum-shadow property [Preprint 98/6 (Univ. Rostock, 1998), 6 pages].

105. Every number must occur the same number of times in the cycle. Equivalently, $\binom{n-1}{t-1}$ must be a multiple of t. This necessary condition appears to be sufficient as well, provided that n is not too small with respect to t; but such a result may well be true yet impossible to prove. [See Chung, Graham, and Diaconis, *Discrete Math.* **110** (1992), 55–57.]

The next few exercises consider the cases $t = 2$ and $t = 3$, for which elegant results are known. Similar but more complicated results have been derived for $t = 4$ and $t = 5$, and the case $t = 6$ has been partially resolved. The case $(n, t) = (12, 6)$ is currently the smallest for which the existence of a universal cycle is unknown.

106. Let the differences mod $(2m+1)$ be $1, 2, \ldots, m, 1, 2, \ldots, m, \ldots$, repeated $2m+1$ times; for example, the cycle for $m = 3$ is (013602561450346235124). This works because $1 + \cdots + m = \binom{m+1}{2}$ is relatively prime to $2m + 1$. [*J. École Polytechnique* **4**, Cahier 10 (1810), 16–48.]

107. The seven doubles , , ...,  can be inserted in 3^7 ways into any universal cycle of 3-combinations for $\{0, 1, 2, 3, 4, 5, 6\}$. The number of such universal cycles is the number of Eulerian trails of the complete graph K_7, which can be shown to be 129,976,320 if we regard $(a_0a_1 \ldots a_{20})$ as equivalent to $(a_1 \ldots a_{20}a_0)$ but not to the reverse-order cycle $(a_{20} \ldots a_1a_0)$. So the answer is 284,258,211,840.

[This problem was first solved in 1859 by M. Reiss, whose method was so complicated that people doubted the result; see *Nouvelles Annales de Mathématiques* **8** (1849), 74; **11** (1852), 115; *Annali di Matematica Pura ed Applicata* (2) **5** (1871–1873), 63–120. A considerably simpler solution, confirming Reiss's claim, was found by P. Jolivald and G. Tarry, who also enumerated the Eulerian trails of K_9; see *Comptes Rendus Association Française pour l'Avancement des Sciences* **15**, part 2 (1886), 49–53; É. Lucas, *Récréations Mathématiques* **4** (1894), 123–151. Brendan D. McKay and Robert W. Robinson found an approach that is better still, enabling them to continue the enumeration through K_{21} by using the fact that the number of trails is

$$(m-1)!^{2m+1}\,[z_0^{2m} z_1^{2m-2} \ldots z_{2m}^{2m-2}]\,\det(a_{jk}) \prod_{1 \le j < k \le 2m} (z_j^2 + z_k^2),$$

where $a_{jk} = -1/(z_j^2 + z_k^2)$ when $j \ne k$; $a_{jj} = -1/(2z_j^2) + \sum_{0 \le k \le 2m} 1/(z_j^2 + z_k^2)$; see *Combinatorics, Probability, and Computing* **7** (1998), 437–449.]

C. Flye Sainte-Marie, in L'*Intermédiaire des Mathématiciens* **1** (1894), 164–165, noted that the Eulerian trails of K_7 include 2×720 that have 7-fold symmetry under permutation of $\{0, 1, \ldots, 6\}$ (namely Poinsot's cycle and its reverse), plus 32×1680 with 3-fold symmetry, plus 25778×5040 cycles that are asymmetric.

108. No solution is possible for $n < 7$, except in the trivial case $n = 4$. When $n = 7$ there are $12{,}255{,}208 \times 7!$ universal cycles, not considering $(a_0a_1 \ldots a_{34})$ to be the same as $(a_1 \ldots a_{34}a_0)$, including cases with 5-fold symmetry like the example cycle in exercise 105.

When $n \geq 8$ we can proceed systematically as suggested by B. Jackson in *Discrete Math.* **117** (1993), 141–150; see also G. Hurlbert, *SIAM J. Disc. Math.* **7** (1994), 598–604: Put each 3-combination into the "standard cyclic order" $c_1c_2c_3$ where $c_2 = (c_1 + \delta) \bmod n$, $c_3 = (c_2 + \delta') \bmod n$, $0 < \delta, \delta' < n/2$, and either $\delta = \delta'$ or $\max(\delta, \delta') < n - \delta - \delta' \neq (n-1)/2$ or $(1 < \delta < n/4$ and $\delta' = (n-1)/2)$ or $(\delta = (n-1)/2$ and $1 < \delta' < n/4)$. For example, when $n = 8$ the allowable values of (δ, δ') are $(1,1)$, $(1,2)$, $(1,3)$, $(2,1)$, $(2,2)$, $(3,1)$, $(3,3)$; when $n = 11$ they are $(1,1)$, $(1,2)$, $(1,3)$, $(1,4)$, $(2,1)$, $(2,2)$, $(2,3)$, $(2,5)$, $(3,1)$, $(3,2)$, $(3,3)$, $(4,1)$, $(4,4)$, $(5,2)$, $(5,5)$. Then construct the digraph with vertices (c, δ) for $0 \leq c < n$ and $1 \leq \delta < n/2$, and with arcs $(c_1, \delta) \to (c_2, \delta')$ for every combination $c_1c_2c_3$ in standard cyclic order. This digraph is connected and balanced, so it has an Eulerian trail by Theorem 2.3.4.2D. (The peculiar rules about $(n-1)/2$ make the digraph connected when n is odd. The Eulerian trail can be chosen to have n-fold symmetry when $n = 8$, but not when $n = 12$.)

109. When $n = 1$ the cycle (000) is trivial; when $n = 2$ there is no cycle; and there are essentially only two when $n = 4$, namely (0001112223330202131 3) and (00011120203332221313). When $n \geq 5$, let the multicombination $d_1d_2d_3$ be in standard cyclic order if $d_2 = (d_1 + \delta - 1) \bmod n$, $d_3 = (d_2 + \delta' - 1) \bmod n$, and (δ, δ') is allowable for $n + 3$ in the previous answer. Construct the digraph with vertices (d, δ) for $0 \leq d < n$ and $1 \leq \delta < (n+3)/2$, and with arcs $(d_1, \delta) \to (d_2, \delta')$ for every multicombination $d_1d_2d_3$ in standard cyclic order; then find an Eulerian trail.

Perhaps a universal cycle of t-multicombinations exists for $\{0, 1, \ldots, n-1\}$ if and only if a universal cycle of t-combinations exists for $\{0, 1, \ldots, n+t-1\}$.

110. A nice way to check for runs is to compute the numbers $b(S) = \sum\{2^{p(c)} \mid c \in S\}$ where $(p(\mathtt{A}), \ldots, p(\mathtt{K})) = (1, \ldots, 13)$; then set $l \leftarrow b(S) \mathbin{\&} -b(S)$ and check that $b(S) + l = l \ll s$, and also that $((l \ll s) \mid (l \gg 1)) \mathbin{\&} a = 0$, where $a = 2^{p(c_1)} \mid \cdots \mid 2^{p(c_5)}$. The values of $b(S)$ and $\sum\{v(c) \mid c \in S\}$ are easily maintained as S runs through all 31 nonempty subsets in Gray-code order. The answers are (1009008, 99792, 2813796, 505008, 2855676, 697508, 1800268, 751324, 1137236, 361224, 388740, 51680, 317340, 19656, 90100, 9168, 58248, 11196, 2708, 0, 8068, 2496, 444, 356, 3680, 0, 0, 0, 76, 4) for $x = (0, \ldots, 29)$; thus the mean score is ≈ 4.769 and the variance is ≈ 9.768.

Hands without points are sometimes facetiously called nineteen,
as that number cannot be made by the cards.

— G. H. DAVIDSON, *Dee's Hand-Book of Cribbage* (1839)

Note: A four-card flush is not allowed in the "crib." Then the distribution is a bit easier to compute, and it turns out to be (1022208, 99792, 2839800, 508908, 2868960, 703496, 1787176, 755320, 1118336, 358368, 378240, 43880, 310956, 16548, 88132, 9072, 57288, 11196, 2264, 0, 7828, 2472, 444, 356, 3680, 0, 0, 0, 76, 4); the mean and variance decrease to approximately 4.735 and 9.667.

SECTION 7.2.1.4

1.

m^n	$m^{\underline{n}}$	$m!\left\{{n \atop m}\right\}$
$\binom{m+n-1}{n}$	$\binom{m}{n}$	$\binom{n-1}{n-m}$
$\left\{{n \atop 0}\right\}+\cdots+\left\{{n \atop m}\right\}$	$[m\geq n]$	$\left\{{n \atop m}\right\}$
$\left\lvert{m+n \atop m}\right\rvert$	$[m\geq n]$	$\left\lvert{n \atop m}\right\rvert$

2. In general, given any integers $x_1 \geq \cdots \geq x_m$, we obtain all integer m-tuples $a_1 \ldots a_m$ such that $a_1 \geq \cdots \geq a_m$, $a_1+\cdots+a_m = x_1+\cdots+x_m$, and $a_m \ldots a_1 \geq x_m \ldots x_1$ by initializing $a_1 \ldots a_m \leftarrow x_1 \ldots x_m$ and $a_{m+1} \leftarrow x_m - 2$. In particular, if c is any integer constant, we obtain all integer m-tuples such that $a_1 \geq \cdots \geq a_m \geq c$ and $a_1 + \cdots + a_m = n$ by initializing $a_1 \leftarrow n - mc + c$, $a_j \leftarrow c$ for $1 < j \leq m$, and $a_{m+1} \leftarrow c-2$, assuming that $n \geq cm$.

3. $a_j = \lfloor (n+m-j)/m \rfloor = \lceil (n+1-j)/m \rceil$, for $1 \leq j \leq m$; see *CMath* §3.4.

4. We must have $a_m \geq a_1 - 1$; therefore $a_j = \lfloor (n+m-j)/m \rfloor$ for $1 \leq j \leq m$, where m is the largest integer with $\lfloor n/m \rfloor \geq r$, namely $m = \lfloor n/r \rfloor$.

5. [See Eugene M. Klimko, *BIT* **13** (1973), 38–49.]

C1. [Initialize.] Set $c_0 \leftarrow 1$, $c_1 \leftarrow n$, $c_2 \ldots c_n \leftarrow 0 \ldots 0$, $l_0 \leftarrow 1$, $l_1 \leftarrow 0$. (We assume that $n > 0$.)

C2. [Visit.] Visit the partition represented by part counts $c_1 \ldots c_n$ and links $l_0 l_1 \ldots l_n$.

C3. [Branch.] Set $j \leftarrow l_0$ and $k \leftarrow l_j$. If $c_j = 1$, go to C6; otherwise, if $j > 1$, go to C5.

C4. [Change 1+1 to 2.] Set $c_1 \leftarrow c_1 - 2$, $c_2 \leftarrow c_2 + 1$. Then if $c_1 = 0$, set $l_0 \leftarrow 2$, and set $l_2 \leftarrow l_1$ if $k \neq 2$. If $c_1 > 0$ and $k \neq 2$, set $l_2 \leftarrow l_1$ and $l_1 \leftarrow 2$. Return to C2.

C5. [Change $j \cdot c_j$ to $(j+1) + 1 + \cdots + 1$.] Set $c_1 \leftarrow j(c_j - 1) - 1$ and go to C7.

C6. [Change $k \cdot c_k + j$ to $(k+1) + 1 + \cdots + 1$.] Terminate if $k = 0$. Otherwise set $c_j \leftarrow 0$; then set $c_1 \leftarrow k(c_k - 1) + j - 1$, $j \leftarrow k$, and $k \leftarrow l_k$.

C7. [Adjust links.] If $c_1 > 0$, set $l_0 \leftarrow 1$, $l_1 \leftarrow j+1$; otherwise set $l_0 \leftarrow j+1$. Then set $c_j \leftarrow 0$ and $c_{j+1} \leftarrow c_{j+1} + 1$. If $k \neq j+1$, set $l_{j+1} \leftarrow k$. Return to C2. ▌

Notice that this algorithm is *loopless*; but it isn't really faster than Algorithm P. Steps C4, C5, and C6 are performed respectively $p(n-2)$, $2p(n) - p(n+1) - p(n-2)$, and $p(n+1) - p(n)$ times; thus step C4 is most important when n is large. (See exercise 45 and the detailed analysis by Fenner and Loizou in *Acta Inf.* **16** (1981), 237–252.)

6. Set $k \leftarrow a_1$ and $j \leftarrow 1$. Then, while $k > a_{j+1}$, set $b_k \leftarrow j$ and $k \leftarrow k-1$ until $k = a_{j+1}$. If $k > 0$, set $j \leftarrow j+1$ and repeat until $k = 0$. (We have used (11) in the dual form $a_j - a_{j+1} = d_j$, where $d_1 \ldots d_n$ is the part-count representation of $b_1 b_2 \ldots$. Notice that the running time of this algorithm is essentially proportional to $a_1 + b_1$, the length of the output plus the length of the input.)

7. We have $b_1 \ldots b_n = n^{a_n}(n-1)^{a_{n-1}-a_n} \ldots 1^{a_1-a_2} 0^{n-a_1}$, by the dual of (11).

8. Transposing the Ferrers diagram corresponds to reflecting and complementing the bit string (15). So we simply interchange and reverse the p's and q's, getting the partition $b_1 b_2 \ldots = (q_t + \cdots + q_1)^{p_1}(q_t + \cdots + q_2)^{p_2} \ldots (q_t)^{p_t}$.

9. By induction: If $a_k = l-1$ and $b_l = k-1$, increasing a_k and b_l preserves equality.

10. (a) The left child of each node is obtained by appending '1'. The right child is obtained by increasing the rightmost digit; this child exists if and only if the parent node ends with unequal digits. All partitions of n appear on level n in lexicographic order.

(b) The left child is obtained by changing '11' to '2'; it exists if and only if the parent node contains at least two 1s. The right child is obtained by deleting a 1 and increasing the smallest part that exceeds 1; it exists if and only if there is at least one 1 and the smallest larger part appears exactly once. All partitions of n into m parts appear on level $n-m$ in lexicographic order; preorder of the entire tree gives lexicographic order of the whole. [T. I. Fenner and G. Loizou, *Comp. J.* **23** (1980), 332–337.]

11. $[z^{100}]\,1/((1-z)(1-z^2)(1-z^5)(1-z^{10})(1-z^{20})(1-z^{50})(1-z^{100})) = 4563$; and $[z^{100}]\,(1+z+z^2)(1+z^2+z^4)\dots(1+z^{100}+z^{200}) = 7$. [See G. Pólya, *AMM* **63** (1956), 689–697.] In the infinite series $\prod_{k\ge 1}(1+z^k+z^{2k})(1+z^{2k}+z^{4k})(1+z^{5k}+z^{10k})$, the coefficient of z^{10^n} is $2^{n+1}-1$, and the coefficient of z^{10^n-1} is 2^n.

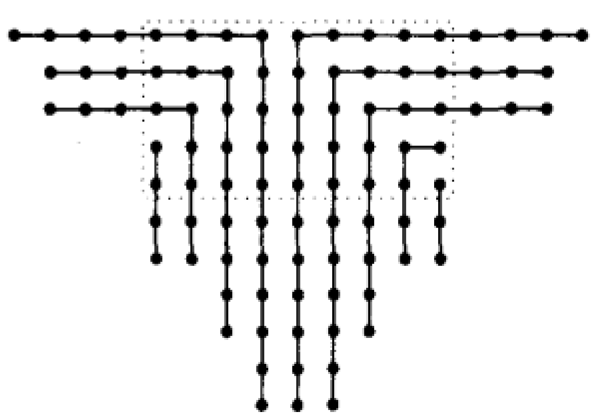
12. To prove that $(1+z)(1+z^2)(1+z^3)\dots = 1/((1-z)(1-z^3)(1-z^5)\dots)$, write the left-hand side as

$$\frac{(1-z^2)}{(1-z)}\frac{(1-z^4)}{(1-z^2)}\frac{(1-z^6)}{(1-z^3)}\cdots$$

and cancel common factors from numerator and denominator. Alternatively, replace z by z^1, z^3, z^5, $\dots$ in the identity $(1+z)(1+z^2)(1+z^4)(1+z^8)\dots = 1/(1-z)$ and multiply the results together. [*Novi Comment. Acad. Sci. Pet.* **3** (1750), 125–169, §47.]

13. Map the partition $c_1\cdot 1 + c_2\cdot 2 + \cdots$ into $\lfloor c_1/2\rfloor\cdot 2 + \lfloor c_2/2\rfloor\cdot 4 + \cdots + r_1\cdot 1 + r_3\cdot 3 + \cdots$, where $r_m = (c_m \bmod 2) + 2(c_{2m} \bmod 2) + 4(c_{4m} \bmod 2) + \cdots$. [*Johns Hopkins Univ. Circular* **2** (1882), 72.]

14. Sylvester's correspondence is best understood as a diagram in which the dots of the odd permutation are centered and divided into disjoint hooks. For example, the partition $17+15+15+9+9+9+9+5+5+3+3$, having five different odd parts, corresponds via the diagram



to the all-distinct partition $19+18+16+13+12+9+5+4+3$ with four gaps.

Conversely, a partition into $2t$ distinct nonnegative parts can be written uniquely in the form $(a_1+b_1-1)+(a_1+b_2-2)+(a_2+b_2-3)+(a_2+b_3-4)+\cdots+(a_{t-1}+b_t-2t+2)+(a_t+b_t-2t+1)+(a_t+b_{t+1}-2t)$ where $a_1 \ge a_2 \ge \cdots \ge a_t \ge t$ and $b_1 \ge b_2 \ge \cdots \ge b_t \ge b_{t+1} = t$. It corresponds to $(2a_1-1)+\cdots+(2a_t-1)+(2A_1-1)+\cdots+(2A_r-1)$, where $A_1+\cdots+A_r$ is the conjugate of $(b_1-t)+\cdots+(b_t-t)$. The value of t is essentially the size of a "Durfee rectangle."

The relevant odd-parts partitions when $n=10$ are $9+1$, $7+3$, $7+1+1+1$, $5+5$, $5+3+1+1$, $5+1+1+1+1+1$, $3+3+3+1$, $3+3+1+1+1+1$, $3+1+\cdots+1$, $1+\cdots+1$, corresponding respectively to the distinct-parts partitions $6+4$, $5+4+1$,

$7+3$, $4+3+2+1$, $6+3+1$, $8+2$, $5+3+2$, $7+2+1$, $9+1$, 10. [See Sylvester's remarkable paper in *Amer. J. Math.* **5** (1882), 251–330; **6** (1883), 334–336.]

15. Every self-conjugate partition of trace k corresponds to a partition of n into k distinct odd parts ("hooks"). Therefore we can write the generating function either as the product $(1+z)(1+z^3)(1+z^5)\ldots$ or as the sum $1+z^1/(1-z^2)+z^4/((1-z^2)(1-z^4))+z^9/((1-z^2)(1-z^4)(1-z^6))+\cdots$. [*Johns Hopkins Univ. Circular* **3** (1883), 42–43.]

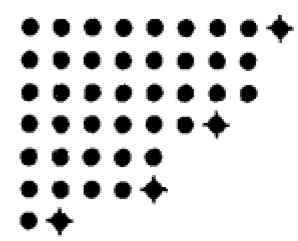
16. The Durfee square contains k^2 dots, and the remaining dots correspond to two independent partitions with largest part $\le k$. Thus, if we use w to count parts and z to count dots, we find

$$\prod_{m=1}^{\infty}\frac{1}{1-wz^m}=\sum_{k=0}^{\infty}\frac{w^k z^{k^2}}{(1-z)(1-z^2)\ldots(1-z^k)(1-wz)(1-wz^2)\ldots(1-wz^k)}.$$

[This impressive-looking formula turns out to be just the special case $x=y=0$ of the even more impressive identity of exercise 19.]

17. (a) $((1+uvz)(1+uvz^2)(1+uvz^3)\ldots)/((1-uz)(1-uz^2)(1-uz^3)\ldots)$.

(b) A joint partition can be represented by a generalized Ferrers diagram in which we merge all the parts together, putting a_i above b_j if $a_i \ge b_j$, then mark the rightmost dot of each b_j. For example, the joint partition $(8,8,5;\ 9,7,5,2)$ has the diagram illustrated here, with marked dots shown as '✦'. Marks appear only in corners; thus the transposed diagram corresponds to another joint partition, which in this case is $(7,6,6,4,3;\ 7,6,4,1)$. [See J. T. Joichi and D. Stanton, *Pacific J. Math.* **127** (1987), 103–120; S. Corteel and J. Lovejoy, *Trans. Amer. Math. Soc.* **356** (2004), 1623–1635; Igor Pak, "Partition bijections, a survey," to appear in *The Ramanujan Journal*.]



Every joint partition with $t>0$ parts corresponds in this way to a "conjugate" in which the largest part is t. And the generating function for such joint partitions is $((1+vz)\ldots(1+vz^{t-1}))/((1-z)\ldots(1-z^t))$ times (vz^t+z^t), where vz^t corresponds to the case that $b_1=t$, and z^t corresponds to the case that $r=0$ or $b_1<t$).

(c) Thus we obtain a form of the general z-nomial theorem in answer 1.2.6–58:

$$\frac{(1+uvz)}{(1-uz)}\frac{(1+uvz^2)}{(1-uz^2)}\frac{(1+uvz^3)}{(1-uz^3)}\ldots=\sum_{t=0}^{\infty}\frac{(1+v)}{(1-z)}\frac{(1+vz)}{(1-z^2)}\ldots\frac{(1+vz^{t-1})}{(1-z^t)}u^t z^t.$$

18. The equations obviously determine the a's and b's when the c's and d's are given, so we want to show that the c's and d's are uniquely determined from the a's and b's. The following algorithm determines the c's and d's from right to left:

A1. [Initialize.] Set $i\leftarrow r$, $j\leftarrow s$, $k\leftarrow 0$, and $a_0\leftarrow b_0\leftarrow\infty$.

A2. [Branch.] Stop if $i+j=0$. Otherwise go to A4 if $a_i\ge b_j-k$.

A3. [Absorb a_i.] Set $c_{i+j}\leftarrow a_i$, $d_{i+j}\leftarrow 0$, $i\leftarrow i-1$, $k\leftarrow k+1$, and return to A2.

A4. [Absorb b_j.] Set $c_{i+j}\leftarrow b_j-k$, $d_{i+j}\leftarrow 1$, $j\leftarrow j-1$, $k\leftarrow k+1$, and return to A2. ▌

There's also a left-to-right method:

B1. [Initialize.] Set $i\leftarrow 1$, $j\leftarrow 1$, $k\leftarrow r+s$, and $a_{r+1}\leftarrow b_{s+1}\leftarrow -\infty$.

B2. [Branch.] Stop if $k=0$. Otherwise set $k\leftarrow k-1$, then go to B4 if $a_i\le b_j-k$.

B3. [Absorb a_i.] Set $c_{i+j-1}\leftarrow a_i$, $d_{i+j-1}\leftarrow 0$, $i\leftarrow i+1$, and return to B2.

B4. [Absorb b_j.] Set $c_{i+j-1}\leftarrow b_j-k$, $d_{i+j-1}\leftarrow 1$, $j\leftarrow j+1$, and return to B2. ▌

In both cases the branching is forced and the resulting sequence satisfies $c_1 \ge \cdots \ge c_{r+s}$. Notice that $c_{r+s} = \min(a_r, b_s)$ and $c_1 = \max(a_1, b_1-r-s+1)$.

We have thereby proved the identity of exercise 17(c) in a different way. Extensions of this idea lead to a combinatorial proof of Ramanujan's "remarkable formula with many parameters,"

$$\sum_{n=-\infty}^{\infty} w^n \prod_{k=0}^{\infty} \frac{1-bz^{k+n}}{1-az^{k+n}} = \prod_{k=0}^{\infty} \frac{(1-a^{-1}bz^k)(1-a^{-1}w^{-1}z^{k+1})(1-awz^k)(1-z^{k+1})}{(1-a^{-1}bw^{-1}z^k)(1-a^{-1}z^{k+1})(1-az^k)(1-wz^k)}.$$

[*References:* G. H. Hardy, *Ramanujan* (1940), Eq. (12.12.2); D. Zeilberger, *Europ. J. Combinatorics* **8** (1987), 461–463; A. J. Yee, *J. Comb. Theory* **A105** (2004), 63–77.]

19. [*Crelle* **34** (1847), 285–328.] By exercise 17(c), the hinted sum over k is

$$\Bigl(\sum_{l\ge 0} v^l \frac{(z-bz)\ldots(z-bz^l)}{(1-z)\ldots(1-z^l)}\,\frac{(1-uz)\ldots(1-uz^l)}{(1-auz)\ldots(1-auz^l)}\Bigr)\cdot\prod_{m=1}^{\infty}\frac{1-auz^m}{1-uz^m};$$

and the sum over l is similar but with $u \leftrightarrow v$, $a \leftrightarrow b$, $k \leftrightarrow l$. Furthermore the sum over both k and l reduces to

$$\prod_{m=1}^{\infty}\frac{(1-uvz^{m+1})(1-auz^m)}{(1-uz^m)(1-vz^m)}$$

when $b = auz$. Now let $u = wxy$, $v = 1/(yz)$, $a = 1/x$, and $b = wyz$; equate this infinite product to the sum over l.

20. To get $p(n)$ we need to add or subtract approximately $\sqrt{8n/3}$ of the previous entries, and most of those entries are $\Theta(\sqrt{n})$ bits long. Therefore $p(n)$ is computed in $\Theta(n)$ steps and the total time is $\Theta(n^2)$.

(A straightforward use of (17) would take $\Theta(n^{5/2})$ steps.)

21. Since $\sum_{n=0}^{\infty} q(n)z^n = (1+z)(1+z^2)\ldots$ is equal to $(1-z^2)(1-z^4)\ldots P(z) = (1-z^2-z^4+z^{10}+z^{14}-z^{24}-\cdots)P(z)$, we have

$$q(n) = p(n)-p(n-2)-p(n-4)+p(n-10)+p(n-14)-p(n-24)-\cdots.$$

[There is also a "pure recurrence" in the q's alone, analogous to the recurrence for $\sigma(n)$ in the next exercise.]

22. From (21) we have $\sum_{n=1}^{\infty}\sigma(n)z^n = \sum_{m,n\ge 1} mz^{mn} = z\frac{d}{dz}\ln P(z) = (z+2z^2-5z^5-7z^7+\cdots)/(1-z-z^2+z^5+z^7+\cdots)$. [*Bibliothèque Impartiale* **3** (1751), 10–31.]

23. Set $u = w$ and $v = z/w$ to get

$$\begin{aligned}\prod_{k=1}^{\infty}(1-z^kw)(1-z^k/w)(1-z^k) &= \sum_{n=-\infty}^{\infty}(-1)^n w^n z^{n(n+1)/2}/(1-w)\\ &= \sum_{n=0}^{\infty}(-1)^n(w^{-n}-w^{n+1})z^{n(n+1)/2}/(1-w)\\ &= \sum_{n=0}^{\infty}(-1)^n(w^{-n}+\cdots+w^n)z^{n(n+1)/2}.\end{aligned}$$

These manipulations are legitimate when $|z|<1$ and w is near 1. Now set $w=1$.

[See §57 of Sylvester's paper cited in answer 14. Jacobi's proof is in §66 of his monograph *Fundamenta Nova Theoriæ Functionum Ellipticarum* (1829).]

24. (a) By (…) and exercise 23, $[z^n]\,A(z) = \sum(-1)^{j+k}(2k+1)[3j^2+j+k^2+k=2n]$, summed over all integers j and k. When $n \bmod 5 = 4$, the contributions all have $j \bmod 5 = 4$ and $k \bmod 5 = 2$; but then $(2k+1) \bmod 5 = 0$.

(b) $B(z)^p \equiv B(z^p)$ (modulo p) when p is prime, by Eq. 4.6.2–(5).

(c) Take $B(z) = P(z)$, since $A(z) = P(z)^{-4}$. [*Proc. Cambridge Philos. Soc.* **19** (1919), 207–210. A similar proof shows that $p(n)$ is a multiple of 7 when $n \bmod 7 = 5$. Ramanujan went on to obtain the beautiful formulas $p(5n+4)/5 = [z^n]\,P(z)^6/P(z^5)$; $p(7n+5)/7 = [z^n]\,(P(z)^4/P(z^7)^3 + 7zP(z)^8/P(z^7)^7)$. Atkin and Swinnerton-Dyer, in *Proc. London Math. Soc.* (3) **4** (1953), 84–106, showed that the partitions of $5n+4$ and $7n+5$ can be divided into equal-size classes according to the respective values of (largest part − number of parts) mod 5 or mod 7, as conjectured by F. Dyson. A slightly more complicated combinatorial statistic proves also that $p(n) \bmod 11 = 0$ when $n \bmod 11 = 6$; see F. G. Garvan, *Trans. Amer. Math. Soc.* **305** (1988), 47–77.]

25. [The hint can be proved by differentiating both sides of the stated identity. It is the special case $y = 1-x$ of a beautiful formula discovered by N. H. Abel in 1826:

$$\mathrm{Li}_2(x)+\mathrm{Li}_2(y) = \mathrm{Li}_2\Bigl(\frac{x}{1-y}\Bigr)+\mathrm{Li}_2\Bigl(\frac{y}{1-x}\Bigr)-\mathrm{Li}_2\Bigl(\frac{xy}{(1-x)(1-y)}\Bigr)-\ln(1-x)\ln(1-y).$$

See Abel's *Œuvres Complètes* **2** (Christiania: Grøndahl, 1881), 189–193.]

(a) Let $f(x) = \ln(1/(1-e^{-xt}))$. Then $\int_1^x f(x)\,dx = -\mathrm{Li}_2(e^{-tx})/t$ and $f^{(n)}(x) = (-t)^n e^{tx}\sum_k \left\langle {n-1 \atop k} \right\rangle e^{ktx}/(e^{tx}-1)^n$, so Euler's summation formula gives $\mathrm{Li}_2(e^{-t})/t + \frac12\ln(1/(1-e^{-t})) + O(1) = (\zeta(2) + t\ln(1-e^{-t}) - \mathrm{Li}_2(1-e^{-t}))/t - \frac12\ln t + O(1) = \zeta(2)/t + \frac12\ln t + O(1)$, as $t \to 0$.

(b) We have $\sum_{m,n\ge1} e^{-mnt}/n = \frac{1}{2\pi i}\sum_{m,n\ge1}\int_{1-i\infty}^{1+i\infty}(mnt)^{-z}\Gamma(z)\,dz/n$, which sums to $\frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty}\zeta(z+1)\zeta(z)t^{-z}\Gamma(z)\,dz$. The pole at $z=1$ gives $\zeta(2)/t$; the double pole at $z=0$ gives $-\zeta(0)\ln t + \zeta'(0) = \frac12\ln t - \frac12\ln 2\pi$; the pole at $z=-1$ gives $-\zeta(-1)\zeta(0)t = B_2B_1t = -t/24$. Zeros of $\zeta(z+1)\zeta(z)$ cancel the other poles of $\Gamma(z)$, so the result is $\ln P(e^{-t}) = \zeta(2)/t + \frac12\ln(t/2\pi) - t/24 + O(t^M)$ for arbitrarily large M.

26. Let $F(n) = \sum_{k=1}^{\infty} e^{-k^2/n}$. We can use (25) either with $f(x) = e^{-x^2/n}[x>0] + \frac12\delta_{x0}$, or with $f(x) = e^{-x^2/n}$ for all x because $2F(n)+1 = \sum_{k=-\infty}^{\infty} e^{-k^2/n}$. Let's choose the latter alternative; then the right-hand side of (25), for $\theta = 0$, is the rapidly convergent

$$\lim_{M\to\infty}\sum_{m=-M}^{M}\int_{-\infty}^{\infty} e^{-2\pi miy-y^2/n}\,dy \;=\; \sum_{m=-\infty}^{\infty} e^{-\pi^2m^2n^2}\int_{-\infty}^{\infty} e^{-u^2/n}\,du$$

if we substitute $u = y + \pi mni$; and the integral is $\sqrt{\pi n}$. [This result is formula (15) on page 420 of Poisson's original paper.]

27. Let $g_n = \sqrt{\pi/6t}\,e^{-n^2\pi^2/6t}\cos\frac{n\pi}{6}$. Then $\int_{-\infty}^{\infty} f(y)\cos 2\pi my\,dy = g_{2m+1} + g_{2m-1}$, so we have

$$\frac{e^{-t/24}}{P(e^{-t})} \;=\; g_1 + g_{-1} + 2\sum_{m=1}^{\infty}(g_{2m+1}+g_{2m-1}) \;=\; 2\sum_{m=-\infty}^{\infty} g_{2m+1}.$$

The terms g_{6n+1} and g_{-6n-1} combine to give the nth term of (30). [See M. I. Knopp, *Modular Functions in Analytic Number Theory* (1970), Chapter 3.]

28. (a,b,c,d) See *Trans. Amer. Math. Soc.* **43** (1938), 271–295. In fact, Lehmer found explicit formulas for $A_{p^e}(n)$, in terms of the Jacobi symbol of exercise 4.5.4–23:

$$A_{2^e}(n) = (-1)^e\left(\frac{-1}{m}\right)2^{e/2}\sin\frac{4\pi m}{2^{e+3}},\qquad \text{if } (3m)^2 \equiv 1-24n \text{ (modulo } 2^{e+3});$$

$$A_{3^e}(n) = (-1)^{e+1}\left(\frac{m}{3}\right)\frac{2}{\sqrt{3}}3^{e/2}\sin\frac{4\pi m}{3^{e+1}},\quad \text{if } (8m)^2 \equiv 1-24n \text{ (modulo } 3^{e+1});$$

$$A_{p^e}(n) = \begin{cases} 2\left(\dfrac{3}{p^e}\right)p^{e/2}\cos\dfrac{4\pi m}{p^e}, & \text{if } (24m)^2 \equiv 1-24n \text{ (modulo } p^e),\ p\ge 5, \\ & \quad\text{and } 24n \bmod p \ne 1; \\ \left(\dfrac{3}{p^e}\right)p^{e/2}[e=1], & \text{if } 24n \bmod p = 1 \text{ and } p \ge 5. \end{cases}$$

(e) If $n = 2^a3^bp_1^{e_1}\ldots p_t^{e_t}$ for $3 < p_1 < \cdots < p_t$ and $e_1\ldots e_t \ne 0$, the probability that $A_k(n) \ne 0$ is $2^{-t}(1+(-1)^{[e_1=1]}/p_1)\ldots(1+(-1)^{[e_t=1]}/p_t)$.

29. $z_1z_2\ldots z_m/((1-z_1)(1-z_1z_2)\ldots(1-z_1z_2\ldots z_m))$.

30. (a) $\left|{n+1 \atop m}\right|$ and (b) $\left|{m+n \atop m}\right|$, by (39).

31. *First solution* [Marshall Hall, Jr., *Combinatorial Theory* (1967), §4.1]: From the recurrence (39), we can show directly that, for $0 \le r < k!$, there is a polynomial $f_{k,r}(n) = n^{k-1}/(k!(k-1)!) + O(n^{k-2})$ such that $\left|{n \atop k}\right| = f_{n,n \bmod k!}(n)$.

Second solution: Since $(1-z)\ldots(1-z^m) = \prod_{p\perp q}(1-e^{2\pi ip/q}z)^{\lfloor m/q\rfloor}$, where the product is over all reduced fractions p/q with $0 \le p < q$, the coefficient of z^n in (41) can be expressed as a sum of roots of unity times polynomials in n, namely as $\sum_{p\perp q} e^{2\pi ipn/q}f_{pq}(n)$ where $f_{pq}(n)$ is a polynomial of degree less than m/q. Thus there exist constants such that $\left|{n \atop 2}\right| = a_1n + a_2 + (-1)^na_3$; $\left|{n \atop 3}\right| = b_1n^2 + b_2n + b_3 + (-1)^nb_4 + \omega^nb_5 + \omega^{-n}b_6$, where $\omega = e^{2\pi i/3}$; etc. The constants are determined by the values for small n, and the first two cases are

$$\left|{n \atop 2}\right| = \frac{1}{2}n - \frac{1}{4} + \frac{1}{4}(-1)^n;\qquad \left|{n \atop 3}\right| = \frac{1}{12}n^2 - \frac{7}{72} - \frac{1}{8}(-1)^n + \frac{1}{9}\omega^n + \frac{1}{9}\omega^{-n}.$$

It follows that $\left|{n \atop 3}\right|$ is the nearest integer to $n^2/12$. Similarly, $\left|{n \atop 4}\right|$ is the nearest integer to $(n^3 + 3n^2 - 9n\,[n\,\text{odd}])/144$.

[Exact formulas for $\left|{n \atop 2}\right|$, $\left|{n \atop 3}\right|$, and $\left|{n \atop 4}\right|$, without the simplification of floor functions, were first found by G. F. Malfatti, *Memorie di Mat. e Fis. Società Italiana* **3** (1786), 571–663. W. J. A. Colman, in *Fibonacci Quarterly* **21** (1983), 272–284, showed that $\left|{n \atop 5}\right|$ is the nearest integer to $(n^4+10n^3+10n^2-75n-45n(-1)^n)/2880$, and gave similar formulas for $\left|{n \atop 6}\right|$ and $\left|{n \atop 7}\right|$.]

32. Since $\left|{m+n \atop m}\right| \le p(n)$, with equality if and only if $m \ge n$, we have $\left|{n \atop m}\right| \le p(n-m)$ with equality if and only if $2m \ge n$.

33. A partition into m parts corresponds to at most $m!$ compositions; hence $\binom{n-1}{m-1} \le m!\left|{n \atop m}\right|$. Consequently $p(n) \ge (n-1)!/((n-m)!\,m!\,(m-1)!)$, and when $m = \sqrt{n}$ Stirling's approximation proves that $\ln p(n) \ge 2\sqrt{n} - \ln n - \frac{1}{2} - \ln 2\pi$.

34. $a_1 > a_2 > \cdots > a_m > 0$ if and only if $a_1-m+1 \ge a_2-m+2 \ge \cdots \ge a_m \ge 1$. And partitions into m distinct parts correspond to $m!$ compositions. Thus, by the previous answer, we have

$$\frac{1}{m!}\binom{n-1}{m-1} \le \left|{n \atop m}\right| \le \frac{1}{m!}\binom{n+m(m-1)/2}{m-1}.$$

[See H. Gupta, *Proc. Indian Acad. Sci.* **A16** (1942), 101–102. A detailed asymptotic formula for $\left|{n \atop m}\right|$ when $n = \Theta(m^3)$ appears in exercise 3.3.2–30.]

35. (a) $x = \frac{1}{C}\ln\frac{1}{C} \approx -0.194$.

(b) $x = \frac{1}{C}\ln\frac{1}{C} - \frac{1}{C}\ln\ln 2 \approx 0.092$; in general we have $x = \frac{1}{C}(\ln\frac{1}{C} - \ln\ln\frac{1}{F(x)})$.

(c) $\int_{-\infty}^{\infty} x\,dF(x) = \int_0^\infty (Cu)^{-2}(\ln u)e^{-1/(Cu)}\,du = -\frac{1}{C}\int_0^\infty(\ln C + \ln v)e^{-v}\,dv = (\gamma - \ln C)/C \approx 0.256$.

(d) Similarly, $\int_{-\infty}^{\infty} x^2 e^{-Cx}\exp(-e^{-Cx}/C)\,dx = (\gamma^2 + \zeta(2) - 2\gamma\ln C + (\ln C)^2)/C^2 \approx 1.0656$. So the variance is $\zeta(2)/C^2 = 1$, exactly(!).

[The probability distribution $e^{-e^{(a-x)/b}}$ is commonly called the Fisher–Tippett distribution; see *Proc. Cambridge Phil. Soc.* **24** (1928), 180–190.]

36. The sum over $j_r - (m+r-1) \ge \cdots \ge j_2 - (m+1) \ge j_1 - m \ge 1$ gives

$$\begin{aligned}\Sigma_r &= \sum_t \left|{t - rm - r(r-1)/2 \atop r}\right| \frac{p(n-t)}{p(n)}\\ &= \frac{\alpha}{1-\alpha}\frac{\alpha^2}{1-\alpha^2}\cdots\frac{\alpha^r}{1-\alpha^r}\alpha^{rm}(1 + O(n^{-1/2+2\epsilon})) + E\\ &= \frac{n^{-1/2}}{\alpha^{-1}-1}\frac{n^{-1/2}}{\alpha^{-2}-1}\cdots\frac{n^{-1/2}}{\alpha^{-r}-1}\exp(-Crx + O(rn^{-1/2+2\epsilon})) + E,\end{aligned}$$

where E is an error term that accounts for the cases $t > n^{1/2+\epsilon}$. The leading factor $n^{-1/2}/(\alpha^{-j}-1)$ is $\frac{1}{jC}(1+O(jn^{-1/2}))$. And it is easy to verify that $E = O(n^{\log n}e^{-Cn^\epsilon})$, even if we use the crude upper bound $\left|{t-rm-r(r-1)/2 \atop r}\right| \le t^r$, because

$$\sum_{t \ge xN} t^r e^{-t/N} = O\Big(\int_{xN}^{\infty} t^r e^{-t/N}\,dt\Big) = O(N^{r+1}x^r e^{-x}/(1 - r/x)),$$

where $N = \Theta(\sqrt{n})$, $x = \Theta(n^\epsilon)$, $r = O(\log n)$.

37. Such a partition is counted once in Σ_0, q times in Σ_1, $\binom{q}{2}$ times in Σ_2, ...; so it is counted exactly $\sum_{j=0}^{r}(-1)^j\binom{q}{j} = (-1)^r\binom{q-1}{r}$ times in the partial sum that ends with $(-1)^r\Sigma_r$. This count is at most δ_{q0} when r is odd, at least δ_{q0} when r is even. [A similar argument shows that the generalized principle of exercise 1.3.3–26 also has this bracketing property. *Reference:* C. Bonferroni, *Pubblicazioni del Reale Istituto Superiore de Scienze Economiche e Commerciale di Firenze* **8** (1936), 3–62.]

38. $z^{l+m-1}\binom{l+m-2}{m-1}_z = z^{l+m-1}(1-z^l)\ldots(1-z^{l+m-2})/((1-z)\ldots(1-z^{m-1}))$.

39. If $\alpha = a_1 \ldots a_m$ is a partition with at most m parts, let $f(\alpha) = \infty$ if $a_1 \le l$, otherwise $f(\alpha) = \min\{j \mid a_1 > l + a_{j+1}\}$. Let g_k be the generating function for partitions with $f(\alpha) > k$. Partitions with $f(\alpha) = k < \infty$ are characterized by the inequalities

$$a_1 \ge a_2 \ge \cdots \ge a_k \ge a_1 - l > a_{k+1} \ge \cdots \ge a_{m+1} = 0.$$

Thus $a_1a_2\ldots a_m = (b_k+l+1)(b_1+1)\ldots(b_{k-1}+1)b_{k+1}\ldots b_m$, where $f(b_1\ldots b_m) \ge k$; and the converse is also true. It follows that $g_k = g_{k-1} - z^{l+k}g_{k-1}$.

[See *American J. Math.* **5** (1882), 254–257.]

40. $z^{m(m+1)/2}\binom{l}{m}_z = (z-z^l)(z^2-z^l)\ldots(z^m-z^l)/((1-z)(1-z^2)\ldots(1-z^m))$. This formula is essentially the z-nomial theorem of exercise 1.2.6–58.

41. See G. Almkvist and G. E. Andrews, *J. Number Theory* **38** (1991), 135–144.

42. A. Vershik [*Functional Anal. Applic.* **30** (1996), 90–105, Theorem 4.7] has stated the formula

$$\frac{1-e^{-c\varphi}}{1-e^{-c(\theta+\varphi)}}e^{-ck/\sqrt{n}} + \frac{1-e^{-c\theta}}{1-e^{-c(\theta+\varphi)}}e^{-ca_k/\sqrt{n}} \approx 1,$$

where the constant c must be chosen as a function of θ and φ so that the area of the shape is n. This constant c is negative if $\theta\varphi < 2$, positive if $\theta\varphi > 2$; the shape reduces to a straight line

$$\frac{k}{\theta\sqrt{n}} + \frac{a_k}{\varphi\sqrt{n}} \approx 1$$

when $\theta\varphi = 2$. If $\varphi = \infty$ we have $c = \sqrt{\mathrm{Li}_2(t)}$ where t satisfies $\theta = (\ln\frac{1}{1-t})/\sqrt{\mathrm{Li}_2(t)}$.

43. We have $a_1 > a_2 > \cdots > a_k$ if and only if the conjugate partition includes each of the the parts 1, 2, ..., $k-1$ at least once. The number of such partitions is $p(n-k(k-1)/2)$; this total includes $\left|{n-(k-1)(k-2)/2 \atop k-1}\right|$ cases with $a_k = 0$.

44. Assume that $n > 0$. The number with smallest parts *unequal* (or with only one part) is $p(n+1)-p(n)$, the number of partitions of $n+1$ that don't end in 1, because we get the former from the latter by changing the smallest part. Therefore the answer is $2p(n)-p(n+1)$. [See R. J. Boscovich, *Giornale de' Letterati* (Rome, 1748), 15. The number of partitions whose smallest *three* parts are equal is $3p(n)-p(n+1)-2p(n+2)+p(n+3)$; similar formulas can be derived for other constraints on the smallest parts.]

45. By Eq. (37) we have $p(n-j)/p(n) = 1 - Cjn^{-1/2} + (C^2j^2+2j)/(2n) - (8C^3j^3 + 60Cj^2 + Cj + 12C^{-1}j)/(48n^{3/2}) + O(j^4n^{-2})$.

46. If $n > 1$, $T_2'(n) = p(n-1) - p(n-2) \le p(n) - p(n-1) = T_2''(n)$, because $p(n)-p(n-1)$ is the number of partitions of n that don't end in 1; every such partition of $n-1$ yields one for n if we increase the largest part. But the difference is rather small: $(T_2''(n) - T_2'(n))/p(n) = C^2/n + O(n^{-3/2})$.

47. The identity in the hint follows by differentiating (21); see exercise 22. The probability of obtaining the part-counts $c_1 \ldots c_n$ when $c_1 + 2c_2 + \cdots + nc_n = n$ is

$$\Pr(c_1\ldots c_n) = \sum_{k=1}^{n}\sum_{j=1}^{c_k}\frac{kp(n-jk)}{np(n)}\Pr(c_1\ldots c_{k-1}(c_k-j)c_{k+1}\ldots c_n)$$
$$= \sum_{k=1}^{n}\sum_{j=1}^{c_k}\frac{k}{np(n)} = \frac{1}{p(n)},$$

by induction on n. [*Combinatorial Algorithms* (Academic Press, 1975), Chapter 10.]

48. The probability that j has a particular fixed value in step N5 is $6/(\pi^2j^2) + O(n^{-1/2})$, and the average value of jk is order $\sqrt{n}$. The average time spent in step N4 is $\Theta(n)$, so the average running time is of order $n^{3/2}$. (A more precise analysis would be desirable.)

49. (a) We have $F(z) = \sum_{k=1}^{\infty}F_k(z)$, where $F_k(z)$ is the generating function for all partitions whose smallest part is $\ge k$, namely $1/((1-z^k)(1-z^{k+1})\ldots) - 1$.

(b) Let $f_k(n) = [z^n]\,F_k(z)/p(n)$. Then $f_1(n) = 1$; $f_2(n) = 1 - p(n-1)/p(n) = Cn^{-1/2} + O(n^{-1})$; $f_3(n) = (p(n)-p(n-1)-p(n-2)+p(n-3))/p(n) = 2C^2n^{-1} + O(n^{-3/2})$; and $f_4(n) = 6C^3n^{-3/2} + O(n^{-2})$. (See exercise 45.) It turns out that $f_{k+1}(n) = k!\,C^kn^{-k/2} + O(n^{-(k+1)/2})$; in particular, $f_5(n) = O(n^{-2})$. Hence $f_5(n) + \cdots + f_n(n) = O(n^{-1})$, because $f_{k+1}(n) \le f_k(n)$.

Adding everything up yields $[z^n]\,F(z) = p(n)(1 + C/\sqrt{n} + O(n^{-1}))$.

50. (a) $c_m(m+k) = c_{m-1}(m-1+k) + c_m(k) = m-1-k+c(k)+1$ by induction when $0 \le k < m$.

(b) Because $\left|{m+k \atop m}\right| = p(k)$ for $0 \le k \le m$.

(c) When $n = 2m$, Algorithm H essentially generates the partitions of m, and we know that $j-1$ is the second-smallest part in the conjugate of the partition just generated — except when $j-1 = m$, just after the partition $1 \ldots 1$ whose conjugate has only one part.

(d) If all parts of α exceed k, let $\alpha\, k^{q+1} j$ correspond to $\alpha\,(k{+}1)$.

(e) The generating function $G_k(z)$ for all partitions whose second-smallest part is $\ge k$ is $(z+\cdots+z^{k-1})F_k(z)+F_k(z)-z^k/(1-z) = F_{k+1}(z)/(1-z)$, where $F_k(z)$ is defined in the previous exercise. Consequently $C(z) = \bigl(F(z)-F_1(z)\bigr)/(1-z) + z/(1-z)^2$.

(f) We can show as in the previous exercise that $[z^n]\,G_k(n)/p(n) = O(n^{-k/2})$ for $k \le 5$; hence $c(m)/p(m) = 1 + O(m^{-1/2})$. The ratios $(c(m)+1)/p(m)$, which are readily computed for small m, reach a maximum of 2.6 at $m = 7$ and decrease steadily thereafter. So a rigorous attention to asymptotic error bounds will complete the proof.

Note: B. Fristedt [*Trans. Amer. Math. Soc.* **337** (1993), 703–735] has proved, among other things, that the number of k's in a random partition of n is greater than $Cx\sqrt{n}$ with asymptotic probability e^{-x}.

52. In lexicographic order, $\left|{64+13 \atop 13}\right|$ partitions of 64 have $a_1 \le 13$; $\left|{50+10 \atop 10}\right|$ of them have $a_1 = 14$ and $a_2 \le 10$; etc. Therefore, by the hint, the partition $14\ 11\ 9\ 6\ 4\ 3\ 2\ 1^{15}$ is preceded by exactly $p(64) - 1000000$ partitions in lexicographic order, making it the millionth in *reverse* lexicographic order.

53. As in the previous answer, $\left|{80 \atop 12}\right|$ partitions of 100 have $a_1 = 32$ and $a_2 \le 12$, etc.; so the lexicographically millionth partition in which $a_1 = 32$ is $32\ 13\ 12\ 8\ 7\ 6\ 5\ 5\ 1^{12}$. Algorithm H produces its conjugate, namely $20\ 8\ 8\ 8\ 6\ 5\ 4\ 3\ 3\ 3\ 3\ 2\ 1^{19}$.

54. (a) Obviously true. This question was just a warmup.

(b) True, but not so obvious. If $\alpha^T = a_1'a_2'\ldots$ we have

$$a_1+\cdots+a_k+a_1'+\cdots+a_k' \;\le\; n+kl \qquad \text{when } k \le a_l'$$

by considering the Ferrers diagram, with equality when $k = a_l'$. Thus if $\alpha \succeq \beta$ and $a_1'+\cdots+a_l' > b_1'+\cdots+b_l'$ for some l, with l minimum, we have $n+kl = b_1+\cdots+b_k+b_1'+\cdots+b_l' < a_1+\cdots+a_k+a_1'+\cdots+a_l' \le n+kl$ when $k = b_l'$, a contradiction.

(c) The recurrence $c_k = \min(a_1+\cdots+a_k,\ b_1+\cdots+b_k) - (c_1+\cdots+c_{k-1})$ clearly defines a greatest lower bound, if $c_1c_2\ldots$ is a partition. And it is; for if $c_1+\cdots+c_k = a_1+\cdots+a_k$ we have $0 \le \min(a_{k+1},\ b_{k+1}) \le \min(a_{k+1},\ b_{k+1}+b_1+\cdots+b_k-a_1-\cdots-a_k) = c_{k+1} \le a_{k+1} \le a_k = c_k + (c_1+\cdots+c_{k-1}) - (a_1+\cdots+a_{k-1}) \le c_k$.

(d) $\alpha \vee \beta = (\alpha^T \wedge \beta^T)^T$. (Double conjugation is needed because a max-oriented recurrence analogous to the one in part (c) can fail.)

(e) $\alpha \wedge \beta$ has $\max(l,m)$ parts and $\alpha \vee \beta$ has $\min(l,m)$ parts. (Consider the first components of their conjugates.)

(f) True for $\alpha \wedge \beta$, by the derivation in part (c). False for $\alpha \vee \beta$ (although true in Fig. 32); for example, $(17165432) \vee (179876) = (1716554)$.

Reference: T. Brylawski, *Discrete Mathematics* **6** (1973), 201–219.

55. (a) If $\alpha \succ\!\!\!\!\vdash \beta$ and $\alpha \succeq \gamma \succeq \beta$, where $\gamma = c_1c_2\ldots$, we have $a_1+\cdots+a_k = c_1+\cdots+c_k = b_1+\cdots+b_k$ for all k except $k = l$ and $k = l+1$; thus α covers β. Therefore β^T covers α^T.

Conversely, if $\alpha \succeq \beta$ and $\alpha \ne \beta$ we can find $\gamma \succeq \beta$ such that $\alpha \succ\!\!\!\!\vdash \gamma$ or $\gamma^T \succ\!\!\!\!\vdash \alpha^T$, as follows: Find the smallest k with $a_k > b_k$, and the smallest l with $a_k > a_{l+1}$. If

$a_l > a_{l+1}+1$, define $\gamma = c_1c_2\ldots$ by $c_k = a_k - [k=l]+[k=l+1]$. If $a_l = a_{l+1}+1$, find the smallest l' with $a_{l+1} > a_{l'+1}$ and let $c_k = a_k - [k=l']+[k=l'+1]$ if $a_{l'} > a_{l'+1}+1$, otherwise $c_k = a_k - [k=l]+[k=l'+1]$.

(b) Consider α and β to be strings of n 0s and n 1s, as in (15). Then $\alpha \succ \beta$ if and only if $\alpha \to \beta$, and $\beta^T \succ \alpha^T$ if and only if $\alpha \Rightarrow \beta$, where '$\to$' denotes replacing a substring of the form 011^q10 by 101^q01 and '$\Rightarrow$' denotes replacing a substring of the form 010^q10 by 100^q01, for some $q \ge 0$.

(c) A partition covers at most $[a_1 > a_2] + \cdots + [a_{m-1} > a_m] + [a_m \ge 2]$ others. The partition $\alpha = (n_2+n_1-1)(n_2-2)(n_2-3)\ldots 21$ maximizes this quantity in the case $a_m = 1$; cases with $a_m \ge 2$ give no improvement. (The conjugate partition, namely $(n_2-1)(n_2-2)\ldots 21^{n_1+1}$, is just as good. Therefore both α and α^T are also *covered by* the maximum number of others.)

(d) Equivalently, consecutive parts of μ differ by at most 1, and the smallest part is 1; the rim representation has no consecutive 1s.

(e) Use rim representations and replace $\succ$ by the relation $\to$. If $\alpha \to \alpha_1$ and $\alpha \to \alpha_1'$ we can easily show the existence of a string β such that $\alpha_1 \to \beta$ and $\alpha_1' \to \beta$; for example,

$$\begin{array}{ccccc} & \nearrow & 101^q0111^r10 & \searrow & \\ 011^q1011^r10 & & & & 101^q1011^r01. \\ & \searrow & 011^q1101^r01 & \nearrow & \end{array}$$

Let $\beta = \beta_2 \succ \cdots \succ \beta_m$ where β_m is minimal. Then, by induction on $\max(k,k')$, we have $k = m$ and $\alpha_k = \beta_m$; also $k' = m$ and $\alpha'_{k'} = \beta_m$.

(f) Set $\beta \leftarrow \alpha^T$; then repeatedly set $\beta \leftarrow \beta'$ until β is minimal, using any convenient partition β' such that $\beta \succ \beta'$. The desired partition is β^T.

Proof: Let $\mu(\alpha)$ be the common value $\alpha_k = \alpha'_{k'}$ in part (e); we must prove that $\alpha \succeq \beta$ implies $\mu(\alpha) \succeq \mu(\beta)$. There is a sequence $\alpha = \alpha_0, \ldots, \alpha_k = \beta$ where $\alpha_j \to \alpha_{j+1}$ or $\alpha_j \Rightarrow \alpha_{j+1}$ for $0 \le j < k$. If $\alpha_0 \to \alpha_1$ we have $\mu(\alpha) = \mu(\alpha_1)$; thus it suffices to prove that $\alpha \Rightarrow \beta$ and $\alpha \to \alpha'$ implies $\alpha' \succeq \mu(\beta)$. But we have, for example,

$$\begin{array}{ccccc} & \nearrow\!\!\!\nearrow & 100^q0111^r10 & \searrow & \\ 010^q1011^r10 & & & & 100^q1011^r01 \\ & \searrow & 010^q1101^r01 \to 010^{q-1}10011^r01 & \nearrow\!\!\!\nearrow & \end{array}$$

because we may assume that $q > 0$; and the other cases are similar.

(g) The parts of λ_n are $a_k = n_2 + [k \le n_1] - k$ for $1 \le k < n_2$; the parts of λ_n^T are $b_k = n_2 - k + [n_2 - k < n_1]$ for $1 \le k \le n_2$. The algorithm of (f) reaches λ_n^T from n^1 after $\binom{n_2+1}{3} - \binom{n_2-n_1}{2}$ steps, because each step increases $\sum kb_k = \sum \binom{a_k+1}{2}$ by 1.

(h) The path n, $(n-1)1$, $(n-2)2$, $(n-2)11$, $(n-3)21$, $\ldots$, 321^{n-5}, 31^{n-3}, 221^{n-4}, 21^{n-2}, 1^n, of length $2n-4$ when $n \ge 3$, is shortest.

It can be shown that the longest path has $m = 2\binom{n_2}{3} + n_1(n_2-1)$ steps. One such path has the form $\alpha_0, \ldots, \alpha_k, \ldots, \alpha_l, \ldots, \alpha_m$ where $\alpha_0 = n^1$; $\alpha_k = \lambda_n$; $\alpha_l = \lambda_n^T$; $\alpha_j \succ \alpha_{j+1}$ for $0 \le j < l$; and $\alpha_{j+1}^T \succ \alpha_j^T$ for $k \le j < m$.

Reference: C. Greene and D. J. Kleitman, *Europ. J. Combinatorics* **7** (1986), 1–10.

56. Suppose $\lambda = u_1 \ldots u_m$ and $\mu = v_1 \ldots v_m$. The following (unoptimized) algorithm applies the theory of exercise 54 to generate the partitions in colex order, maintaining $\alpha = a_1a_2\ldots a_m \preceq \mu$ as well as $\alpha^T = b_1b_2\ldots b_l \preceq \lambda^T$. To find the successor of α, we first find the largest j such that b_j can be increased. Then we have

$\beta = b_1 \dots b_{j-1}(b_j{+}1)1 \dots 1 \preceq \lambda^T$, hence the desired successor is $\beta^T \wedge \mu$. The algorithm maintains auxiliary tables $r_j = b_j + \cdots + b_l$, $s_j = v_1 + \cdots + v_j$, and $t_j = w_j + w_{j+1} + \cdots$, where $\lambda^T = w_1 w_2 \dots$.

M1. [Initialize.] Set $q \leftarrow 0$, $k \leftarrow u_1$. For $j = 1, \dots, m$, while $u_{j+1} < k$ set $t_k \leftarrow q \leftarrow q + j$ and $k \leftarrow k - 1$. Then set $q \leftarrow 0$ again, and for $j = 1, \dots, m$ set $a_j \leftarrow v_j$, $s_j \leftarrow q \leftarrow q + a_j$. Then set $q \leftarrow 0$ yet again, and $k \leftarrow l \leftarrow a_1$. For $j = 1, \dots, m$, while $a_{j+1} < k$ set $b_k \leftarrow j$, $r_k \leftarrow q \leftarrow q + j$, and $k \leftarrow k - 1$. Finally, set $t_1 \leftarrow 0$, $b_0 \leftarrow 0$, $b_{-1} \leftarrow -1$.

M2. [Visit.] Visit the partition $a_1 \dots a_m$ and/or its conjugate $b_1 \dots b_l$.

M3. [Find j.] Let j be the largest integer $< l$ such that $r_{j+1} > t_{j+1}$ and $b_j \neq b_{j-1}$. Terminate the algorithm if $j = 0$.

M4. [Increase b_j.] Set $x \leftarrow r_{j+1} - 1$, $k \leftarrow b_j$, $b_j \leftarrow k + 1$, and $a_{k+1} \leftarrow j$. (The previous value of a_{k+1} was $j - 1$. Now we're going to update $a_1 \dots a_k$ using essentially the method of exercise 54(c) to distribute x dots into columns $j + 1$, $j + 2$, ...)

M5. [Majorize.] Set $z \leftarrow 0$ and then do the following for $i = 1, \dots, k$: Set $x \leftarrow x + j$, $y \leftarrow \min(x, s_i)$, $a_i \leftarrow y - z$, $z \leftarrow y$; if $i = 1$ set $l \leftarrow p \leftarrow a_1$ and $q \leftarrow 0$; if $i > 1$ while $p > a_i$ set $b_p \leftarrow i - 1$, $r_p \leftarrow q \leftarrow q + i - 1$, $p \leftarrow p - 1$. Finally, while $p > j$ set $b_p \leftarrow k$, $r_p \leftarrow q \leftarrow q + k$, $p \leftarrow p - 1$. Return to M2. ▌

57. If $\lambda = \mu^T$ there obviously is only one such matrix, essentially the Ferrers diagram of λ. And the condition $\lambda \preceq \mu^T$ is necessary, for if $\mu^T = b_1 b_2 \dots$ we have $b_1 + \cdots + b_k = \min(c_1, k) + \min(c_2, k) + \cdots$, and this quantity must not be less than the number of 1s in the first k rows. Finally, if there is a matrix for λ and μ and if λ covers α, we can readily construct a matrix for α and μ by moving a 1 from any specified row to another that has fewer 1s.

Notes: This result is often called the Gale–Ryser theorem, because of well-known papers by D. Gale [*Pacific J. Math.* **7** (1957), 1073–1082] and H. J. Ryser [*Canadian J. Math.* **9** (1957), 371–377]. But the number of 0–1 matrices with row sums λ and column sums μ is the coefficient of the monomial symmetric function $\sum x_{i_1}^{c_1} x_{i_2}^{c_2} \dots$ in the product of elementary symmetric functions $e_{r_1} e_{r_2} \dots$, where

$$e_r = [z^r]\,(1 + x_1 z)(1 + x_2 z)(1 + x_3 z) \dots .$$

In this context the result has been known at least since the 1930s; see D. E. Littlewood's formula for $\prod_{m,n \ge 0}(1 + x_m y_n)$ in *Proc. London Math. Soc.* (2) **40** (1936), 40–70. [Cayley had shown much earlier, in *Philosophical Trans.* **147** (1857), 489–499, that the lexicographic condition $\lambda \le \mu^T$ is necessary.]

58. [R. F. Muirhead, *Proc. Edinburgh Math. Soc.* **21** (1903), 144–157.] The condition $\alpha \succeq \beta$ is necessary, because we can set $x_1 = \cdots = x_k = x$ and $x_{k+1} = \cdots = x_n = 1$ and let $x \to \infty$. It is sufficient because we need only prove it when α covers β. Then if, say, parts (a_1, a_2) become $(a_1 - 1, a_2 + 1)$, the left-hand side is the right-hand side plus the nonnegative quantity

$$\frac{1}{2m!} \sum x_{p_1}^{a_2} x_{p_2}^{a_2} \dots x_{p_m}^{a_m} (x_{p_1}^{a_1 - a_2 - 1} - x_{p_2}^{a_1 - a_2 - 1})(x_{p_1} - x_{p_2}).$$

[*Historical notes:* Muirhead's paper is the earliest known appearance of the concept now known as majorization; shortly afterward, an equivalent definition was given by M. O. Lorenz, *Quarterly Publ. Amer. Stat. Assoc.* **9** (1905), 209–219, who was interested in measuring nonuniform distribution of wealth. Yet another equivalent concept was formulated by I. Schur in *Sitzungsberichte Berliner Math. Gesellschaft* **22** (1923), 9–20. "Majorization" was named by Hardy, Littlewood, and Pólya, who established its most basic properties in *Messenger of Math.* **58** (1929), 145–152; see exercise 2.3.4.5–17. An excellent book, *Inequalities* by A. W. Marshall and I. Olkin (Academic Press, 1979), is entirely devoted to the subject.]

59. The unique paths for $n = 0$, 1, 2, 3, 4, and 6 must have the stated symmetry. There is one such path for $n = 5$, namely 11111, 2111, 221, 311, 32, 41, 5. And there are four for $n = 7$:

$$1111111, 211111, 22111, 2221, 322, 3211, 31111, 4111, 511, 421, 331, 43, 52, 61, 7;$$
$$1111111, 211111, 22111, 2221, 322, 421, 511, 4111, 31111, 3211, 331, 43, 52, 61, 7;$$
$$1111111, 211111, 31111, 22111, 2221, 322, 3211, 4111, 421, 331, 43, 52, 511, 61, 7;$$
$$1111111, 211111, 31111, 22111, 2221, 322, 421, 4111, 3211, 331, 43, 52, 511, 61, 7.$$

There are no others, because at least two self-conjugate partitions exist for all $n \geq 8$ (see exercise 16).

60. For $L(6,6)$, use (59); otherwise use $L'(4,6)$ and $L'(3,5)$ everywhere.
In $M(4,18)$, insert 444222, 4442211 between 443322 and 4432221.
In $M(5,11)$, insert 52211, 5222 between 62111 and 6221.
In $M(5,20)$, insert 5542211, 554222 between 5552111 and 555221.
In $M(6,13)$, insert 72211, 7222 between 62221 and 6322.
In $L(4,14)$, insert 44222, 442211 between 43322 and 432221.
In $L(5,15)$, insert 542211, 54222 between 552111 and 55221.
In $L(7,12)$, insert 62211, 6222 between 72111 and 7221.

62. The statement holds for $n = 7$, 8, and 9, except in two cases: $n = 8$, $m = 3$, $\alpha = 3221$; $n = 9$, $m = 4$, $\alpha = 432$.

64. If $n = 2^k q$ where q is odd, let ω_n denote the partition $(2^k)^q$, namely q parts equal to 2^k. The recursive rule

$$B(n) \;=\; B(n-1)^R 1,\ 2 \times B(n/2)$$

for $n > 0$, where $2 \times B(n/2)$ denotes doubling all parts of $B(n/2)$ (or the empty sequence if n is odd), defines a pleasant Gray path that begins with $\omega_{n-1}1$ and ends with ω_n, if we let $B(0)$ be the unique partition of 0. Thus,

$$B(1) = 1; \quad B(2) = 11, 2; \quad B(3) = 21, 111; \quad B(4) = 1111, 211, 22, 4.$$

Among the remarkable properties satisfied by this sequence is the fact that

$$B(n) = (2 \times B(0))1^n,\ (2 \times B(1))1^{n-2},\ (2 \times B(2))1^{n-4},\ \ldots,\ (2 \times B(n/2))1^0,$$

when n is even; for example,

$$B(8) = 11111111, 2111111, 221111, 41111, 4211, 22211, 2222, 422, 44, 8.$$

The following algorithm generates $B(n)$ looplessly when $n \geq 2$:

K1. [Initialize.] Set $c_0 \leftarrow p_0 \leftarrow 0$, $p_1 \leftarrow 1$. If n is even, set $c_1 \leftarrow n$, $t \leftarrow 1$; otherwise let $n - 1 = 2^k q$ where q is odd and set $c_1 \leftarrow 1$, $c_2 \leftarrow q$, $p_2 \leftarrow 2^k$, $t \leftarrow 2$.

K2. [Even visit.] Visit the partition $p_t^{c_t} \ldots p_1^{c_1}$. (Now $c_t + \cdots + c_1$ is even.)

K3. [Change the largest part.] If $c_t = 1$, split the largest part: If $p_t \neq 2p_{t-1}$, set $c_t \leftarrow 2$, $p_t \leftarrow p_t/2$, otherwise set $c_{t-1} \leftarrow c_{t-1} + 2$, $t \leftarrow t - 1$. But if $c_t > 1$, merge two of the largest parts: If $c_t = 2$, set $c_t \leftarrow 1$, $p_t \leftarrow 2p_t$, otherwise set $c_t \leftarrow c_t - 2$, $c_{t+1} \leftarrow 1$, $p_{t+1} \leftarrow 2p_t$, $t \leftarrow t + 1$.

K4. [Odd visit.] Visit the partition $p_t^{c_t} \ldots p_1^{c_1}$. (Now $c_t + \cdots + c_1$ is odd.)

K5. [Change the next-largest part.] Now we wish to apply the following transformation: "Remove $c_t - [t \text{ is even}]$ of the largest parts temporarily, then apply step K3, then restore the removed parts." More precisely, there are nine cases: (1a) If c_t is odd and $t = 1$, terminate. (1b1) If c_t is odd, $c_{t-1} = 1$, and $p_{t-1} = 2p_{t-2}$, set $c_{t-2} \leftarrow c_{t-2} + 2$, $c_{t-1} \leftarrow c_t$, $p_{t-1} \leftarrow p_t$, $t \leftarrow t - 1$. (1b2) If c_t is odd, $c_{t-1} = 1$, and $p_{t-1} \neq 2p_{t-2}$, set $c_{t-1} \leftarrow 2$, $p_{t-1} \leftarrow p_{t-1}/2$. (1c1) If c_t is odd, $c_{t-1} = 2$, and $p_t = 2p_{t-1}$, set $c_{t-1} \leftarrow c_t + 1$, $p_{t-1} \leftarrow p_t$, $t \leftarrow t - 1$. (1c2) If c_t is odd, $c_{t-1} = 2$, and $p_t \neq 2p_{t-1}$, set $c_{t-1} \leftarrow 1$, $p_{t-1} \leftarrow 2p_{t-1}$. (1d1) If c_t is odd, $c_{t-1} > 2$, and $p_t = 2p_{t-1}$, set $c_{t-1} \leftarrow c_{t-1} - 2$, $c_t \leftarrow c_t + 1$. (1d2) If c_t is odd, $c_{t-1} > 2$, and $p_t \neq 2p_{t-1}$, set $c_{t+1} \leftarrow c_t$, $p_{t+1} \leftarrow p_t$, $c_t \leftarrow 1$, $p_t \leftarrow 2p_{t-1}$, $c_{t-1} \leftarrow c_{t-1} - 2$, $t \leftarrow t + 1$. (2a) If c_t is even and $p_t = 2p_{t-1}$, set $c_t \leftarrow c_t - 1$, $c_{t-1} \leftarrow c_{t-1} + 2$. (2b) If c_t is even and $p_t \neq 2p_{t-1}$, set $c_{t+1} \leftarrow c_t - 1$, $p_{t+1} \leftarrow p_t$, $c_t \leftarrow 2$, $p_t \leftarrow p_t/2$, $t \leftarrow t + 1$. Return to K2. ▮

[The transformations in K3 and K5 undo themselves when performed twice in a row. This construction is due to T. Colthurst and M. Kleber, "A Gray path on binary partitions," to appear. Euler considered the number of such partitions in §50 of his paper in 1750.]

65. If $p_1^{e_1} \ldots p_r^{e_r}$ is the prime factorization of m, the number of such factorizations is $p(e_1) \ldots p(e_r)$, and we can let $n = \max(e_1, \ldots, e_r)$. Indeed, for each r-tuple $(x_1, \ldots, x_r)$ with $0 \le x_k < p(e_k)$ we can let $m_j = p_1^{a_{1j}} \ldots p_r^{a_{rj}}$, where $a_{k1} \ldots a_{kn}$ is the $(x_k + 1)$st partition of e_k. Thus we can use a reflected Gray code for r-tuples together with a Gray code for partitions.

66. Let $a_1 \ldots a_m$ be an m-tuple that satisfies the specified inequalities. We can sort it into nonincreasing order $a_{x_1} \ge \cdots \ge a_{x_m}$, where the permutation $x_1 \ldots x_m$ is uniquely determined if we require the sorting to be *stable*; see Eq. 5–(2).

If $j \prec k$, we have $a_j \ge a_k$, hence j appears to the left of k in the permutation $x_1 \ldots x_m$. Therefore $x_1 \ldots x_m$ is one of the permutations output by Algorithm 7.2.1.2V. Moreover, j will be left of k also when $a_j = a_k$ and $j < k$, by stability. Hence a_{x_i} is strictly greater than $a_{x_{i+1}}$ when $x_i > x_{i+1}$ is a "descent."

To generate all the relevant partitions of n, take each topological permutation $x_1 \ldots x_m$ and generate the partitions $y_1 \ldots y_m$ of $n - t$ where t is the *index* of $x_1 \ldots x_m$ (see Section 5.1.1). For $1 \le j \le m$ set $a_{x_j} \leftarrow y_j + t_j$, where t_j is the number of descents to the right of x_j in $x_1 \ldots x_m$.

For example, if $x_1 \ldots x_m = 314592687$ we want to generate all cases with $a_3 > a_1 \ge a_4 \ge a_5 \ge a_9 > a_2 \ge a_6 \ge a_8 > a_7$. In this case $t = 1 + 5 + 8 = 14$; so we set $a_1 \leftarrow y_2 + 2$, $a_2 \leftarrow y_6 + 1$, $a_3 \leftarrow y_1 + 3$, $a_4 \leftarrow y_3 + 2$, $a_5 \leftarrow y_4 + 2$, $a_6 \leftarrow y_7 + 1$, $a_7 \leftarrow y_9$, $a_8 \leftarrow y_8 + 1$, and $a_9 \leftarrow y_5 + 2$. The generalized generating function $\sum z_1^{a_1} \ldots z_9^{a_9}$ in the sense of exercise 29 is

$$\frac{z_1^2 z_2 z_3^3 z_4^2 z_5^2 z_6 z_8 z_9^2}{(1 - z_3)(1 - z_3 z_1)(1 - z_3 z_1 z_4)(1 - z_3 z_1 z_4 z_5) \ldots (1 - z_3 z_1 z_4 z_5 z_9 z_2 z_6 z_8 z_7)}.$$

When $\prec$ is any given partial ordering, the ordinary generating function for all such partitions of n is therefore $\sum z^{\operatorname{ind} \alpha}/((1 - z)(1 - z^2) \ldots (1 - z^m))$, where the sum is over all outputs α of Algorithm 7.2.1.2V.

[See R. P. Stanley, *Memoirs Amer. Math. Soc.* **119** (1972), for significant extensions and applications of these ideas. See also L. Carlitz, *Studies in Foundations and Combinatorics* (New York: Academic Press, 1978), 101–129, for information about up-down partitions.]

67. If $n+1 = q_1 \ldots q_r$, where the factors $q_1, \ldots, q_r$ are all ≥ 2, we get a perfect partition $\{(q_1-1)\cdot 1, (q_2-1)\cdot q_1, (q_3-1)\cdot q_1q_2, \ldots, (q_r-1)\cdot q_1 \ldots q_{r-1}\}$ that corresponds in an obvious way to mixed radix notation. (The order of the factors q_j is significant.)

Conversely, all perfect partitions arise in this way. Suppose the multiset $M = \{k_1 \cdot p_1, \ldots, k_m \cdot p_m\}$ is a perfect partition, where $p_1 < \cdots < p_m$; then we must have $p_j = (k_1+1)\ldots(k_{j-1}+1)$ for $1 \leq j \leq m$, because p_j is the smallest sum of a submultiset of M that is not a submultiset of $\{k_1 \cdot p_1, \ldots, k_{j-1} \cdot p_{j-1}\}$.

The perfect partitions of n with fewest elements occur if and only if the q_j are all prime, because $pq - 1 > (p-1) + (q-1)$ whenever $p > 1$ and $q > 1$. Thus, for example, the minimal perfect partitions of 11 correspond to the ordered factorizations $2\cdot 2\cdot 3$, $2\cdot 3\cdot 2$, and $3\cdot 2\cdot 2$. *Reference: Quarterly Journal of Mathematics* **21** (1886), 367–373.

68. (a) If $a_i + 1 \leq a_j - 1$ for some i and j we can change $\{a_i, a_j\}$ to $\{a_i+1, a_j-1\}$, thereby increasing the product by $a_j - a_i - 1 > 0$. Thus the optimum occurs only in the optimally balanced partition of exercise 3. [L. Oettinger and J. Derbès, *Nouv. Ann. Math.* **18** (1859), 442; **19** (1860), 117–118.]

(b) No part is 1; and if $a_j \geq 4$ we can change it to $2 + (a_j-2)$ without decreasing the product. Thus we can assume that all parts are 2 or 3. We get an improvement by changing $2+2+2$ to $3+3$, hence there are at most two 2s. The optimum therefore is $3^{n/3}$ when $n \bmod 3$ is 0; $4\cdot 3^{(n-4)/3} = 3^{(n-4)/3}\cdot 2\cdot 2 = (4/3^{4/3})3^{n/3}$ when $n \bmod 3$ is 1; $3^{(n-2)/3}\cdot 2 = (2/3^{2/3})3^{n/3}$ when $n \bmod 3$ is 2. [O. Meißner, *Mathematisch-naturwissenschaftliche Blätter* **4** (1907), 85.]

69. All $n > 2$ have the solution $(n, 2, 1, \ldots, 1)$. We can "sieve out" the other cases $\leq N$ by starting with $s_2 \ldots s_N \leftarrow 1 \ldots 1$ and then setting $s_{ak-b} \leftarrow 0$ whenever $ak - b \leq N$, where $a = x_1 \ldots x_t - 1$, $b = x_1 + \cdots + x_t - t - 1$, $k \geq x_1 \geq \cdots \geq x_t$, and $a > 1$, because $k + x_1 + \cdots + x_t + (ak - b - t - 1) = kx_1 \ldots x_t$. The sequence $(x_1, \ldots, x_t)$ needs to be considered only when $(x_1 \ldots x_t - 1)x_1 - (x_1 + \cdots + x_t) < N - t$; we can also continue to decrease N so that $s_N = 1$. In this way only $(32766, 1486539, 254887, 1511, 937, 478, 4)$ sequences $(x_1, \ldots, x_t)$ need to be tried when N is initially 2^{30}, and the only survivors turn out to be 2, 3, 4, 6, 24, 114, 174, and 444. [See E. Trost, *Elemente der Math.* **11** (1956), 135; M. Misiurewicz, *Elemente der Math.* **21** (1966), 90.]

Notes: No new survivors are likely as $N \to \infty$, but a new idea will be needed to rule them out. The simplest sequences $(x_1, \ldots, x_t) = (3)$ and $(2, 2)$ already exclude all $n > 5$ with $n \bmod 6 \neq 0$; this fact can be used to speed up the computation by a factor of 6. The sequences (6) and $(3, 2)$ exclude 40% of the remainder (namely all n of the forms $5k-4$ and $5k-2$); the sequences (8), $(4, 2)$, and $(2, 2, 2)$ exclude 3/7 of the remainder; the sequences with $t = 1$ imply that $n - 1$ must be prime; the sequences in which $x_1 \ldots x_t = 2^r$ exclude about $p(r)$ residues of $n \bmod (2^r - 1)$; sequences in which $x_1 \ldots x_t$ is the product of r distinct primes will exclude about ϖ_r residues of $n \bmod (x_1 \ldots x_t - 1)$.

70. Each step takes one partition of n into another, so we must eventually reach a repeating cycle. Many partitions simply perform a cyclic shift on each northeast-to-southwest diagonal of the Ferrers diagram, changing it

$$\text{from}\quad \begin{matrix} x_1 & x_2 & x_4 & x_7 & x_{11} & x_{16} & \cdots \\ x_3 & x_5 & x_8 & x_{12} & x_{17} & x_{23} & \cdots \\ x_6 & x_9 & x_{13} & x_{18} & x_{24} & x_{31} & \cdots \\ x_{10} & x_{14} & x_{19} & x_{25} & x_{32} & x_{40} & \cdots \\ x_{15} & x_{20} & x_{26} & x_{33} & x_{41} & x_{50} & \cdots \\ x_{21} & x_{27} & x_{34} & x_{42} & x_{51} & x_{61} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{matrix} \qquad \text{to}\quad \begin{matrix} x_1 & x_3 & x_6 & x_{10} & x_{15} & x_{21} & \cdots \\ x_2 & x_4 & x_7 & x_{11} & x_{16} & x_{22} & \cdots \\ x_5 & x_8 & x_{12} & x_{17} & x_{23} & x_{30} & \cdots \\ x_9 & x_{13} & x_{18} & x_{24} & x_{31} & x_{39} & \cdots \\ x_{14} & x_{19} & x_{25} & x_{32} & x_{40} & x_{49} & \cdots \\ x_{20} & x_{26} & x_{33} & x_{41} & x_{50} & x_{60} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{matrix}\;;$$

in other words, they apply the permutation $\rho = (1)(2\,3)(4\,5\,6)(7\,8\,9\,10)\ldots$ to the cells. Exceptions occur only when ρ introduces an empty cell above a dot; for example, x_{10} might be empty when x_{11} isn't. But we can get the correct new diagram by moving the top row down, sorting it into its proper place after applying ρ in such cases. Such a move always reduces the number of occupied diagonals, so it cannot be part of a cycle. Thus every cycle consists entirely of permutations by ρ.

If any element of a diagonal is empty in a cyclic partition, all elements of the next diagonal must be empty. For if, say, x_5 is empty, repeated application of ρ will make x_5 adjacent to each of the cells x_7, x_8, x_9, x_{10} of the next diagonal. Therefore if $n = \binom{n_2}{2} + \binom{n_1}{1}$ with $n_2 > n_1 \geq 0$ the cyclic states are precisely those with $n_2 - 1$ completely filled diagonals and n_1 dots in the next. [This result is due to J. Brandt, *Proc. Amer. Math. Soc.* **85** (1982), 483–486. The origin of the problem is unknown; see Martin Gardner, *The Last Recreations* (1997), Chapter 2.]

71. When $n = 1 + \cdots + m > 1$, the starting partition $(m-1)(m-1)(m-2)\ldots 211$ has distance $m(m-1)$ from the cyclic state, and this is maximum. [K. Igusa, *Math. Magazine* **58** (1985), 259–271; G. Etienne, *J. Combin. Theory* **A58** (1991), 181–197.] In the general case, Griggs and Ho [*Advances in Appl. Math.* **21** (1998), 205–227] have conjectured that the maximum distance to a cycle is $\max(2n+2-n_1(n_2+1), n+n_2+1, n_1(n_2+1)) - 2n_2$ for all $n > 1$; their conjecture has been verified for $n \leq 100$. Moreover, the worst-case starting partition appears to be unique when $n_2 = 2n_1 + \{-1, 0, 2\}$.

72. (a) Swap the jth occurrence of k in the partition $n = j \cdot k + \alpha$ with the kth occurrence of j in $k \cdot j + \alpha$, for every partition α of $n - jk$. For example, when $n = 6$ the swaps are

6,	51,	42,	411,	33,	321,	3111,	222,	2211,	21111,	111111.
a	b1	fg	c1g	hi	jk1	d1kh	n2i	m21n	e1mjf	1edcba

(b) $p(n-k) + p(n-2k) + p(n-3k) + \cdots$. [A. H. M. Hoare, *AMM* **93** (1986), 475–476.]

SECTION 7.2.1.5

1. Whenever m is set equal to r in step H6, change it back to $r - 1$.

2. **L1.** [Initialize.] Set $l_j \leftarrow j - 1$ and $a_j \leftarrow 0$ for $1 \leq j \leq n$. Also set $h_1 \leftarrow n$, $t \leftarrow 1$, and set l_0 to any convenient nonzero value.

L2. [Visit.] Visit the t-block partition represented by $l_1 \ldots l_n$ and $h_1 \ldots h_t$. (The restricted growth string corresponding to this partition is $a_1 \ldots a_n$.)

L3. [Find j.] Set $j \leftarrow n$; then, while $l_j = 0$, set $j \leftarrow j - 1$ and $t \leftarrow t - 1$.

L4. [Move j to the next block.] Terminate if $j = 0$. Otherwise set $k \leftarrow a_j + 1$, $h_k \leftarrow l_j$, $a_j \leftarrow k$. If $k = t$, set $t \leftarrow t + 1$ and $l_j \leftarrow 0$; otherwise set $l_j \leftarrow h_{k+1}$. Finally set $h_{k+1} \leftarrow j$.

L5. [Move $j+1$, …, n to block 1.] While $j < n$, set $j \leftarrow j + 1$, $l_j \leftarrow h_1$, $a_j \leftarrow 0$, and $h_1 \leftarrow j$. Return to L2. ▌

3. Let $\tau(k,n)$ be the number of strings $a_1 \ldots a_n$ that satisfy the condition $0 \leq a_j \leq 1 + \max(k-1, a_1, \ldots, a_{j-1})$ for $1 \leq j \leq n$; thus $\tau(k,0) = 1$, $\tau(0,n) = \varpi_n$, and $\tau(k,n) = k\tau(k,n-1) + \tau(k+1,n-1)$. [S. G. Williamson has called $\tau(k,n)$ a "tail coefficient"; see *SICOMP* **5** (1976), 602–617.] The number of strings that are generated by Algorithm H before a given restricted growth string $a_1 \ldots a_n$ is $\sum_{j=1}^{n} a_j \tau(b_j, n-j)$, where $b_j = 1 + \max(a_1, \ldots, a_{j-1})$. Working backwards with the help of a precomputed table of the tail coefficients, we find that this formula yields 999999 when $a_1 \ldots a_{12} = 010220345041$.

4. The most common representatives of each type, subscripted by the number of corresponding occurrences in the GraphBase, are $\texttt{zzzzz}_0$, $\texttt{ooohh}_0$, $\texttt{xxxix}_0$, $\texttt{xxxii}_0$, $\texttt{ooops}_0$, $\texttt{llull}_0$, $\texttt{llala}_0$, $\texttt{eeler}_0$, $\texttt{iitti}_0$, $\texttt{xxiii}_0$, $\texttt{ccxxv}_0$, $\texttt{eerie}_1$, $\texttt{llama}_1$, $\texttt{xxvii}_0$, $\texttt{oozed}_5$, $\texttt{uhuuu}_0$, $\texttt{mamma}_1$, $\texttt{puppy}_{28}$, $\texttt{anana}_0$, $\texttt{heheeo}_0$, $\texttt{vivid}_{15}$, $\texttt{rarer}_3$, $\texttt{etext}_1$, $\texttt{amass}_2$, $\texttt{again}_{137}$, $\texttt{ahhaa}_0$, $\texttt{esses}_1$, $\texttt{teeth}_{25}$, $\texttt{yaaay}_0$, $\texttt{ahhhh}_2$, $\texttt{pssst}_2$, $\texttt{seems}_7$, $\texttt{added}_6$, $\texttt{lxxii}_0$, $\texttt{books}_{184}$, $\texttt{swiss}_3$, $\texttt{sense}_{10}$, $\texttt{ended}_3$, $\texttt{check}_{160}$, $\texttt{level}_{18}$, $\texttt{tepee}_4$, $\texttt{slyly}_5$, $\texttt{never}_{154}$, $\texttt{sells}_6$, $\texttt{motto}_{21}$, $\texttt{whooo}_2$, $\texttt{trees}_{384}$, $\texttt{going}_{307}$, $\texttt{which}_{151}$, $\texttt{there}_{174}$, $\texttt{three}_{100}$, $\texttt{their}_{3834}$. (See S. Golomb, *Math. Mag.* **53** (1980), 219–221. Words with only two distinct letters are, of course, rare. The 18 representatives listed here with subscript 0 can be found in larger dictionaries or in English-language pages of the Internet.)

5. (a) $112 = \rho(0225)$. The sequence is $r(0)$, $r(1)$, $r(4)$, $r(9)$, $r(16)$, …, where $r(n)$ is obtained by expressing n in decimal notation (with one or more leading zeros), applying the ρ function of exercise 4, then deleting the leading zeros. Notice that $n/9 \le r(n) \le n$.

(b) $1012 = r(45^2)$. The sequence is the same as (a), but sorted into order and with duplicates removed. (Who knew that $88^2 = 7744$, $212^2 = 44944$, and $264^2 = 69696$?)

6. Use the topological sorting approach of Algorithm 7.2.1.2V, with an appropriate partial ordering: Include c_j chains of length j, with their least elements ordered. For example, if $n = 20$, $c_2 = 3$, and $c_3 = c_4 = 2$, we use that algorithm to find all permutations $a_1 \ldots a_{20}$ of $\{1, \ldots, 20\}$ such that $1 \prec 2$, $3 \prec 4$, $5 \prec 6$, $1 \prec 3 \prec 5$, $7 \prec 8 \prec 9$, $10 \prec 11 \prec 12$, $7 \prec 10$, $13 \prec 14 \prec 15 \prec 16$, $17 \prec 18 \prec 19 \prec 20$, $13 \prec 17$, forming the restricted growth strings $\rho(f(a_1) \ldots f(a_{20}))$, where ρ is defined in exercise 4 and $(f(1), \ldots, f(20)) = (1,1,2,2,3,3,4,4,4,5,5,5,6,6,6,6,7,7,7,7)$. The total number of outputs is, of course, given by (48).

7. Exactly ϖ_n. They are the permutations we get by reversing the left-right order of the blocks in (2) and dropping the '|' symbols: 1234, 4123, 3124, 3412, …, 4321. [See A. Claesson, *European J. Combinatorics* **22** (2001), 961–971. S. Kitaev, in "Partially ordered generalized patterns," *Discrete Math.*, to appear, has discovered a far-reaching generalization: Let π be a permutation of $\{0, \ldots, r\}$, let g_n be the number of permutations $a_1 \ldots a_n$ of $\{1, \ldots, n\}$ such that $a_{k-0\pi} > a_{k-1\pi} > \cdots > a_{k-r\pi} > a_j$ implies $j > k$, and let f_n be the number of permutations $a_1 \ldots a_n$ for which the pattern $a_{k-0\pi} > a_{k-1\pi} > \cdots > a_{k-r\pi}$ is avoided altogether for $r < k \le n$. Then $\sum_{n\ge0} g_n z^n/n! = \exp(\sum_{n\ge1} f_{n-1} z^n/n!)$.]

8. For each partition of $\{1, \ldots, n\}$ into m blocks, arrange the blocks in decreasing order of their smallest elements, and permute the non-smallest block elements in all possible ways. If $n = 9$ and $m = 3$, for example, the partition 126|38|4579 would yield 457938126 and eleven other cases obtained by permuting $\{5, 7, 9\}$ and $\{2, 6\}$ among themselves. (Essentially the same method generates all permutations that have exactly k cycles; see the "unusual correspondence" of Section 1.3.3.)

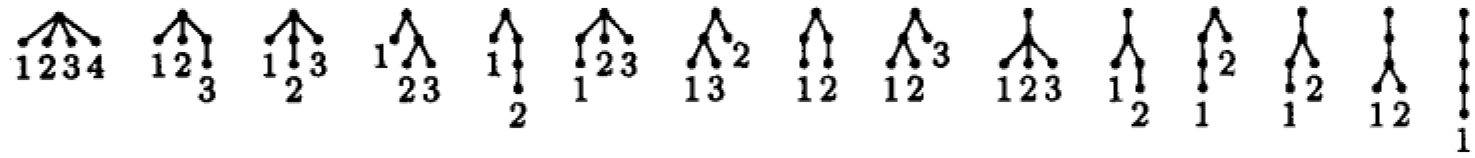
9. Among the permutations of the multiset $\{k_0 \cdot 0, k_1 \cdot 1, \ldots, k_{n-1} \cdot (n-1)\}$, exactly

$$\binom{k_0 + k_1 + \cdots + k_{n-1}}{k_0, k_1, \ldots, k_{n-1}} \frac{k_0}{(k_0 + k_1 + \cdots + k_{n-1})} \frac{k_1}{(k_1 + \cdots + k_{n-1})} \cdots \frac{k_{n-1}}{k_{n-1}}$$

have restricted growth, since $k_j/(k_j + \cdots + k_{n-1})$ is the probability that j precedes $\{j+1, \ldots, n-1\}$.

The average number of 0s, if $n > 0$, is $1 + (n-1)\varpi_{n-1}/\varpi_n = \Theta(\log n)$, because the total number of 0s among all ϖ_n cases is $\sum_{k=1}^{n} k\binom{n-1}{k-1}\varpi_{n-k} = \varpi_n + (n-1)\varpi_{n-1}$.

10. Given a partition of $\{1, \ldots, n\}$, construct an oriented tree on $\{0, 1, \ldots, n\}$ by letting $j - 1$ be the parent of all members of a block whose least member is j. Then relabel the leaves, preserving order, and erase the other labels. For example, the 15 partitions in (2) correspond respectively to



To reverse the process, take a semilabeled tree and assign new numbers to its nodes by considering the nodes first encountered on the path from the root to the smallest leaf, then on the path from the root to the second-smallest leaf, etc. The number of leaves is $n+1$ minus the number of blocks. [This construction is closely related to exercise 2.3.4.4–18 and to many enumerations in that section. See P. L. Erdős and L. A. Székely, *Advances in Applied Math.* **10** (1989), 488–496.]

11. We get pure alphametics from 900 of the 64855 set partitions into at most 10 blocks for which $\rho(a_1 \ldots a_{13}) = \rho(a_5 \ldots a_8 a_1 \ldots a_4 a_9 \ldots a_{13})$, and from 563,527 of the 13,788,536 for which $\rho(a_1 \ldots a_{13}) < \rho(a_5 \ldots a_8 a_1 \ldots a_4 a_9 \ldots a_{13})$. The first examples are **aaaa + aaaa = baaac**, **aaaa + aaaa = bbbbc**, and **aaaa + aaab = baaac**; the last are **abcd + efgd = dceab** (**goat + newt = tango**) and **abcd + efgd = dceaf** (**clad + nerd = dance**). [The idea of hooking a partition generator to an alphametic solver is due to Alan Sutcliffe.]

12. (a) Form $\rho((a_1a'_1)\ldots(a_na'_n))$, where ρ is defined in exercise 4, since we have $x \equiv y$ (modulo $\Pi \vee \Pi'$) if and only if $x \equiv y$ (modulo Π) and $x \equiv y$ (modulo Π').

(b) Represent Π by links as in exercise 2; represent Π' as in Algorithm 2.3.3E; and use that algorithm to make $j \equiv l_j$ whenever $l_j \neq 0$. (For efficiency, we can assume that Π has at least as many blocks as Π'.)

(c) When one block of Π has been split into two parts; that is, when two blocks of Π' have been merged together.

(d) $\binom{t}{2}$; (e) $(2^{s_1-1}-1)+\cdots+(2^{s_t-1}-1)$.

(f) True: Let $\Pi \vee \Pi'$ have blocks $B_1|B_2|\cdots|B_t$, where $\Pi = B_1B_2|B_3|\cdots|B_t$. Then Π' is essentially a partition of $\{B_1,\ldots,B_t\}$ with $B_1 \not\equiv B_2$, and $\Pi \wedge \Pi'$ is obtained by merging the block of Π' that contains B_1 with the block that contains B_2. [A finite lattice that satisfies this condition is called *lower semimodular*; see G. Birkhoff, *Lattice Theory* (1940), §I.8. The majorization lattice of exercise 7.2.1.4–54 does not have this property when, for example, $\alpha = 4111$ and $\alpha' = 331$.]

(g) False: For example, let $\Pi = 0011$, $\Pi' = 0101$.

(h) The blocks of Π and Π' are unions of the blocks of $\Pi \vee \Pi'$, so we can assume that $\Pi \vee \Pi' = \{1,\ldots,t\}$. As in part (b), merge j with l_j to get Π in r steps, when Π has $t-r$ blocks. These merges applied to Π' will each reduce the number of blocks by 0 or 1. Hence $b(\Pi') - b(\Pi \wedge \Pi') \le r = b(\Pi \vee \Pi') - b(\Pi)$.

[In *Algebra Universalis* **10** (1980), 74–95, P. Pudlák and J. Tůma proved that *every* finite lattice is a sublattice of the partition lattice of $\{1,\ldots,n\}$, for suitably large n.]

13. [See *Advances in Math.* **26** (1977), 290–305.] If the j largest elements of a t-block partition appear in singleton blocks, but the next element $n-j$ does not, let us say that the partition has order $t-j$. Define the "Stirling string" Σ_{nt} to be the sequence of orders of the t-block partitions Π_1, Π_2, …; for example, $\Sigma_{43} = 122333$. Then $\Sigma_{tt} = 0$, and we get $\Sigma_{(n+1)t}$ from Σ_{nt} by replacing each digit d in the latter by the string $d^d(d{+}1)^{d+1}\ldots t^t$ of length $\binom{t+1}{2} - \binom{d}{2}$; for example,

$$\Sigma_{53} = 122333_{\wedge}22333_{\wedge}22333_{\wedge}333_{\wedge}333_{\wedge}333.$$

The basic idea is to consider the lexicographic generation process of Algorithm H. Suppose $\Pi = a_1 \ldots a_n$ is a t-block partition of order j; then it is the lexicographically smallest t-block partition whose restricted growth string begins with $a_1 \ldots a_{n-t+j}$. The partitions covered by Π are, in lexicographic order, Π_{12}, Π_{13}, Π_{23}, Π_{14}, Π_{24}, Π_{34}, $\ldots$, $\Pi_{(t-1)t}$, where Π_{rs} means "coalesce blocks r and s of Π" (that is, "change all occurrences of $s-1$ to $r-1$ and then apply ρ to get a restricted growth string"). If Π' is any of the last $\binom{t}{2} - \binom{j}{2}$ of these, from $\Pi_{1(j+1)}$ onwards, then Π is the smallest t-block partition following Π'. For example, if $\Pi = 001012034$, then $n = 9$, $t = 5$, $j = 3$, and the relevant partitions Π' are $\rho(001012004)$, $\rho(001012014)$, $\rho(001012024)$, $\rho(001012030)$, $\rho(001012031)$, $\rho(001012032)$, $\rho(001012033)$.

Therefore $f_{nt}(N) = f_{nt}(N-1) + \binom{t}{2} - \binom{j}{2}$, where j is the Nth digit of Σ_{nt}.

14. **E1.** [Initialize.] Set $a_j \leftarrow 0$ and $b_j \leftarrow d_j \leftarrow 1$ for $1 \le j \le n$.

E2. [Visit.] Visit the restricted growth string $a_1 \ldots a_n$.

E3. [Find j.] Set $j \leftarrow n$; then, while $a_j = d_j$, set $d_j \leftarrow 1 - d_j$ and $j \leftarrow j - 1$.

E4. [Done?] Terminate if $j = 1$. Otherwise go to E6 if $d_j = 0$.

E5. [Move down.] If $a_j = 0$, set $a_j \leftarrow b_j$, $m \leftarrow a_j + 1$, and go to E7. Otherwise if $a_j = b_j$, set $a_j \leftarrow b_j - 1$, $m \leftarrow b_j$, and go to E7. Otherwise set $a_j \leftarrow a_j - 1$ and return to E2.

E6. [Move up.] If $a_j = b_j - 1$, set $a_j \leftarrow b_j$, $m \leftarrow a_j + 1$, and go to E7. Otherwise if $a_j = b_j$, set $a_j \leftarrow 0$, $m \leftarrow b_j$, and go to E7. Otherwise set $a_j \leftarrow a_j + 1$ and return to E2.

E7. [Fix $b_{j+1} \ldots b_n$.] Set $b_k \leftarrow m$ for $k = j+1, \ldots, n$. Return to E2. ▮

[This algorithm can be extensively optimized because, as in Algorithm H, j is almost always equal to n.]

15. It corresponds to the first n digits of the infinite binary string 01011011011..., because ϖ_{n-1} is even if and only if $n \bmod 3 = 0$ (see exercise 23).

16. 00012, 01012, 01112, 00112, 00102, 01102, 01002, 01202, 01212, 01222, 01022, 01122, 00122, 00121, 01121, 01021, 01221, 01211, 01201, 01200, 01210, 01220, 01020, 01120, 00120.

17. The following solution uses two mutually recursive procedures, $f(\mu,\nu,\sigma)$ and $b(\mu,\nu,\sigma)$, for "forward" and "backward" generation of $A_{\mu\nu}$ when $\sigma = 0$ and of $A'_{\mu\nu}$ when $\sigma = 1$. To start the process, assuming that $1 < m < n$, first set $a_j \leftarrow 0$ for $1 \le j \le n-m$ and $a_{n-m+j} \leftarrow j-1$ for $1 \le j \le m$, then call $f(m,n,0)$.

Procedure $f(\mu,\nu,\sigma)$: If $\mu = 2$, visit $a_1 \ldots a_n$; otherwise call $f(\mu-1, \nu-1, (\mu{+}\sigma) \bmod 2)$. Then, if $\nu = \mu + 1$, do the following: Change a_μ from 0 to $\mu - 1$, and visit $a_1 \ldots a_n$; repeatedly set $a_\nu \leftarrow a_\nu - 1$ and visit $a_1 \ldots a_n$, until $a_\nu = 0$. But if $\nu > \mu + 1$, change $a_{\nu-1}$ (if $\mu{+}\sigma$ is odd) or a_μ (if $\mu{+}\sigma$ is even) from 0 to $\mu - 1$; then call $b(\mu,\nu{-}1,0)$ if $a_\nu + \sigma$ is odd, $f(\mu,\nu{-}1,0)$ if $a_\nu + \sigma$ is even; and while $a_\nu > 0$, set $a_\nu \leftarrow a_\nu - 1$ and call $b(\mu,\nu{-}1,0)$ or $f(\mu,\nu{-}1,0)$ again in the same way until $a_\nu = 0$.

Procedure $b(\mu,\nu,\sigma)$: If $\nu = \mu + 1$, first do the following: Repeatedly visit $a_1 \ldots a_n$ and set $a_\nu \leftarrow a_\nu + 1$, until $a_\nu = \mu - 1$; then visit $a_1 \ldots a_n$ and change a_μ from $\mu - 1$ to 0. But if $\nu > \mu + 1$, call $f(\mu,\nu{-}1,0)$ if $a_\nu + \sigma$ is odd, $b(\mu,\nu{-}1,0)$ if $a_\nu + \sigma$ is even; then while $a_\nu < \mu - 1$, set $a_\nu \leftarrow a_\nu + 1$ and call $f(\mu,\nu{-}1,0)$ or $b(\mu,\nu{-}1,0)$ again in the same way until $a_\nu = \mu - 1$; finally change $a_{\nu-1}$ (if $\mu{+}\sigma$ is odd) or a_μ (if $\mu{+}\sigma$ is even) from $\mu - 1$ to 0. And finally, in both cases, if $\mu = 2$ visit $a_1 \ldots a_n$, otherwise call $b(\mu-1, \nu-1, (\mu{+}\sigma) \bmod 2)$.

Most of the running time is actually spent handing the case $\mu = 2$; faster routines based on Gray binary code (and deviating from Ruskey's actual sequences) could be substituted for this case. A streamlined procedure could also be used when $\mu = \nu - 1$.

18. The sequence must begin (or end) with $01\ldots(n-1)$. By exercise 32, no such Gray code can exist when $0 \neq \delta_n \neq (1)^{0+1+\cdots+(n-1)}$, namely when $n \bmod 12$ is 4, 6, 7, or 9.

The cases $n = 1$, 2, 3, are easily solved; and 1,927,683,326 solutions exist when $n = 5$. Thus there probably are zillions of solutions for all $n \geq 8$ except for the cases already excluded. Indeed, we can probably find such a Gray path through all ϖ_{nk} of the strings considered in answer 28(e) below, except when $n \equiv 2k + (2, 4, 5, 7)$ (modulo 12).

Note: The generalized Stirling number $\left\{ {n \atop m} \right\}_{-1}$ in exercise 30 exceeds 1 for $2 < m < n$, so there can be no such Gray code for the partitions of $\{1, \ldots, n\}$ into m blocks.

19. (a) Change (6) to the pattern 0, 2, ..., m, ..., 3, 1 or its reverse, as in endo-order (7.2.1.3–(45)).

(b) We can generalize (8) and (9) to obtain sequences $A_{mn\alpha}$ and A'_{mna} that begin with $0^{n-m}01\ldots(m-1)$ and end with $01\ldots(m-1)\alpha$ and $0^{n-m-1}01\ldots(m-1)a$, respectively, where $0 \leq a \leq m-2$ and α is any string $a_1 \ldots a_{n-m}$ with $0 \leq a_j \leq m-2$. When $2 < m < n$ the new rules are

$$A_{m(n+1)(\alpha a)} = \begin{cases} A_{(m-1)n(b\beta)}x_1, A^R_{mn\beta}x_1, A_{mn\alpha}x_2, \ldots, A_{mn\alpha}x_m, & \text{if } m \text{ is even;} \\ A'_{(m-1)nb}x_1, A_{mn\alpha}x_1, A^R_{mn\alpha}x_2, \ldots, A_{mn\alpha}x_m, & \text{if } m \text{ is odd;} \end{cases}$$

$$A'_{m(n+1)a} = \begin{cases} A'_{(m-1)nb}x_1, A_{mn\beta}x_1, A^R_{mn\beta}x_2, \ldots, A^R_{mn\beta}x_m, & \text{if } m \text{ is even;} \\ A_{(m-1)n(b\beta)}x_1, A^R_{mn\beta}x_1, A_{mn\beta}x_2, \ldots, A^R_{mn\beta}x_m, & \text{if } m \text{ is odd;} \end{cases}$$

here $b = m-3$, $\beta = b^{n-m}$, and $(x_1, \ldots, x_m)$ is a path from $x_1 = m-1$ to $x_m = a$.

20. 012323212122; in general $(a_1 \ldots a_n)^T = \rho(a_n \ldots a_1)$, in the notation of exercise 4.

21. The numbers $\langle s_0, s_1, s_2, \ldots \rangle = \langle 1, 1, 2, 3, 7, 12, 31, 59, 164, 339, 999, \ldots \rangle$ satisfy the recurrences $s_{2n+1} = \sum_k \binom{n}{k} s_{2n-2k}$, $s_{2n+2} = \sum_k \binom{n}{k}(2^k+1)s_{2n-2k}$, because of the way the middle elements relate to the others. Therefore $s_{2n} = n!\,[z^n]\exp((e^{2z}-1)/2+e^z-1)$ and $s_{2n+1} = n!\,[z^n]\exp((e^{2z}-1)/2+e^z+z-1)$. By considering set partitions on the first half we also have $s_{2n} = \sum_k \left\{ {n \atop k} \right\} x_k$ and $s_{2n+1} = \sum_k \left\{ {n+1 \atop k} \right\} x_{k-1}$, where $x_n = 2x_{n-1} + (n-1)x_{n-2} = n!\,[z^n]\exp(2z+z^2/2)$. [T. S. Motzkin considered the sequence $\langle s_{2n} \rangle$ in *Proc. Symp. Pure Math.* **19** (1971), 173.]

22. (a) $\sum_{k=0}^{\infty} k^n \Pr(X=k) = e^{-1}\sum_{k=0}^{\infty} k^n/k! = \varpi_n$ by (16). (b) $\sum_{k=0}^{\infty} k^n \Pr(X=k) = \sum_{k=0}^{\infty} k^n \sum_{j=0}^{m} \binom{j}{k}(-1)^{j-k}/j!$, and we can extend the inner sum to $j = \infty$ because $\sum_k \binom{j}{k}(-1)^k k^n = 0$ when $j > n$. Thus we get $\sum_{k=0}^{\infty}(k^n/k!)\sum_{l=0}^{\infty}(-1)^l/l! = \varpi_n$. [See J. O. Irwin, *J. Royal Stat. Soc.* **A118** (1955), 389–404; J. Pitman, *AMM* **104** (1997), 201–209.]

23. (a) The formula holds whenever $f(x) = x^n$, by (14), so it holds in general. (Thus we also have $\sum_{k=0}^{\infty} f(k)/k! = e f(\varpi)$, by (16).)

(b) Suppose we have proved the relation for k, and let $h(x) = (x-1)^{\underline{k}} f(x)$, $g(x) = f(x+1)$. Then $f(\varpi+k+1) = g(\varpi+k) = \varpi^{\underline{k}} g(\varpi) = h(\varpi+1) = \varpi h(\varpi) = \varpi^{\underline{k+1}} f(\varpi)$. [See J. Touchard, *Ann. Soc. Sci. Bruxelles* **53** (1933), 21–31. This symbolic "umbral calculus," invented by John Blissard in *Quart. J. Pure and Applied Math.* **4** (1861), 279–305, is quite useful; but it must be handled carefully because $f(\varpi) = g(\varpi)$ does not imply that $f(\varpi)h(\varpi) = g(\varpi)h(\varpi)$.]

(c) The hint is a special case of exercise 4.6.2–16(c). Setting $f(x) = x^n$ and $k = p$ in (b) then yields $\varpi_n \equiv \varpi_{p+n} - \varpi_{1+n}$.

(d) Modulo p, the polynomial $x^N - 1$ is divisible by $g(x) = x^p - x - 1$, because $x^{p^k} \equiv x + k$ and $x^N \equiv x^{\bar{p}} \equiv x^{\underline{p}} \equiv x^p - x \equiv 1$ (modulo $g(x)$ and p). Thus if $h(x) = (x^N - 1)x^n/g(x)$ we have $h(\varpi) \equiv h(\varpi + p) = \varpi^{\underline{p}} h(\varpi) \equiv (\varpi^p - \varpi)h(\varpi)$; and $0 \equiv g(\varpi)h(\varpi) = \varpi^{N+n} - \varpi^n$ (modulo p).

24. The hint follows by induction on e, because $x^{\underline{p^e}} = \prod_{k=0}^{p-1}(x - kp^{e-1})^{\underline{p^{e-1}}}$. We can also prove by induction on n that $x^n \equiv r_n(x)$ (modulo $g_1(x)$ and p) implies

$$x^{p^{e-1}n} \equiv r_n(x)^{p^{e-1}} \quad (\text{modulo } g_e(x),\ pg_{e-1}(x),\ \ldots,\ p^{e-1}g_1(x), \text{ and } p^e).$$

Hence $x^{p^{e-1}N} = 1 + h_0(x)g_e(x) + ph_1(x)g_{e-1}(x) + \cdots + p^{e-1}h_{e-1}(x)g_1(x) + p^e h_e(x)$ for certain polynomials $h_k(x)$ with integer coefficients. Modulo p^e we have $h_0(\varpi)\varpi^n \equiv h_0(\varpi + p^e)(\varpi + p^e)^n = \varpi^{\underline{p^e}} h_0(\varpi)\varpi^n \equiv (g_e(\varpi) + 1)h_0(\varpi)\varpi^n$; hence

$$\varpi^{p^{e-1}N+n} = \varpi^n + h_0(\varpi)g_e(\varpi)\varpi^n + ph_1(\varpi)g_{e-1}(\varpi)\varpi^n + \cdots \equiv \varpi^n.$$

[A similar derivation applies when $p = 2$, but we let $g_{j+1}(x) = g_j(x)^2 + 2[j=2]$, and we obtain $\varpi_n \equiv \varpi_{n+3\cdot 2^e}$ (modulo 2^e). These results are due to Marshall Hall; see *Bull. Amer. Math. Soc.* **40** (1934), 387; *Amer. J. Math.* **70** (1948), 387–388. For further information see W. F. Lunnon, P. A. B. Pleasants, and N. M. Stephens, *Acta Arith.* **35** (1979), 1–16.]

25. The first inequality follows by applying a much more general principle to the tree of restricted growth strings: In any tree for which deg(p) ≥ deg(parent(p)) for all non-root nodes p, we have $w_k/w_{k-1} \le w_{k+1}/w_k$ when w_k is the total number of nodes on level k. For if the $m = w_{k-1}$ nodes on level $k-1$ have respectively $a_1, \ldots, a_m$ children, they have at least $a_1^2 + \cdots + a_m^2$ grandchildren; hence $w_{k-1}w_{k+1} \ge m(a_1^2 + \cdots + a_m^2) \ge (a_1 + \cdots + a_m)^2 = w_k^2$.

For the second inequality, note that $\varpi_{n+1} - \varpi_n = \sum_{k=0}^{n}\bigl(\binom{n}{k} - \binom{n-1}{k-1}\bigr)\varpi_{n-k}$; thus

$$\frac{\varpi_{n+1}}{\varpi_n} - 1 = \sum_{k=0}^{n-1}\binom{n-1}{k}\frac{\varpi_{n-k}}{\varpi_n} \le \sum_{k=0}^{n-1}\binom{n-1}{k}\frac{\varpi_{n-k-1}}{\varpi_{n-1}} = \frac{\varpi_n}{\varpi_{n-1}}$$

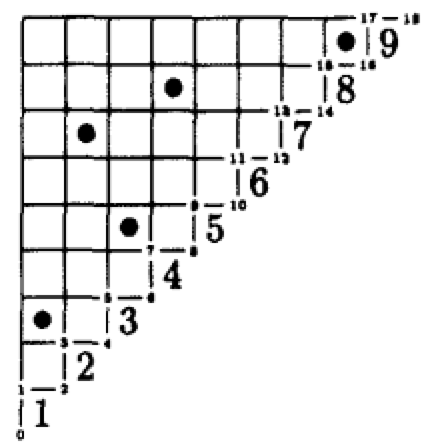
because, for example, $\varpi_{n-3}/\varpi_n = (\varpi_{n-3}/\varpi_{n-2})(\varpi_{n-2}/\varpi_{n-1})(\varpi_{n-1}/\varpi_n)$ is less than or equal to $(\varpi_{n-4}/\varpi_{n-3})(\varpi_{n-3}/\varpi_{n-2})(\varpi_{n-2}/\varpi_{n-1}) = \varpi_{n-4}/\varpi_{n-1}$.

26. There are $\binom{n-1}{n-t}$ rightward paths from (n1) to (tt); we can represent them by 0s and 1s, where 0 means "go right," 1 means "go up," and the positions of the 1s tell us which $n-t$ of the elements are in the block with 1. The next step, if $t > 1$, is to another vertex at the far left; so we continue with a path that defines a partition on the remaining $t-1$ elements. For example, the partition 14|2|3 corresponds to the path 0010 under these conventions, where the respective bits mean that $1 \not\equiv 2$, $1 \not\equiv 3$, $1 \equiv 4$, $2 \not\equiv 3$. [Many other interpretations are possible. The convention suggested here shows that ϖ_{nk} enumerates partitions with $1 \not\equiv 2, \ldots, 1 \not\equiv k$, a combinatorial property discovered by H. W. Becker; see *AMM* **51** (1944), 47, and *Mathematics Magazine* **22** (1948), 23–26.]

27. (a) In general, $\lambda_0 = \lambda_1 = \lambda_{2n-1} = \lambda_{2n} = 0$. The following list shows also the restricted growth strings that correspond to each loop via the algorithm of part (b):

0,0,0,0,0,0,0,0,0 0123	0,0,1,0,0,0,0,0,0 0012	0,0,1,1,1,0,0,0,0 0102
0,0,0,0,0,0,1,0,0 0122	0,0,1,0,0,0,1,0,0 0011	0,0,1,1,1,0,1,0,0 0100
0,0,0,0,1,0,0,0,0 0112	0,0,1,0,1,0,0,0,0 0001	0,0,1,1,1,1,1,0,0 0120
0,0,0,0,1,0,1,0,0 0111	0,0,1,0,1,0,1,0,0 0000	0,0,1,1,11,1,1,0,0 0101
0,0,0,0,1,1,1,0,0 0121	0,0,1,0,1,1,1,0,0 0010	0,0,1,1,2,1,1,0,0 0110

(b) The name "tableau" suggests a connection to Section 5.1.4, and indeed the theory developed there leads to an interesting one-to-one correspondence. We can represent set partitions on a triangular chessboard by putting a rook in column l_j of row $n+1-j$ whenever $l_j \neq 0$ in the linked list representation of exercise 2 (see the answer to exercise 5.1.3–19). For example, the rook representation of 135|27|489|6 is shown here. Equivalently, the nonzero links can be specified in a two-line array, such as $\binom{1\,2\,3\,4\,8}{3\,7\,5\,8\,9}$; see 5.1.4–(11).



Consider the path of length $2n$ that begins at the lower left corner of this triangular diagram and follows the right boundary edges, ending at the upper right corner: The points of this path are $z_k = (\lfloor k/2 \rfloor, \lceil k/2 \rceil)$ for $0 \le k \le 2n$. Moreover, the rectangle above and to the left of z_k contains precisely the rooks that contribute coordinate pairs $\frac{i}{j}$ to the two-line array when $i \le \lfloor k/2 \rfloor$ and $j > \lceil k/2 \rceil$; in our example, there are just two such rooks when $9 \le k \le 12$, namely $\binom{2\,4}{7\,8}$. Theorem 5.1.4A tells us that such two-line arrays are equivalent to tableaux (P_k, Q_k), where the elements of P_k come from the lower line and the elements of Q_k come from the upper line, and where both P_k and Q_k have the same shape. It is advantageous to use decreasing order in the P tableaux but increasing order in the Q tableaux, so that in our example they are respectively

k	P_k	Q_k	k	P_k	Q_k	k	P_k	Q_k
2	3	1	7	7 5	2 3	12	8 7	2 4
3	3	1	8	8 5 7	2 3 4	13	8	4
4	7 3	1 2	9	8 7	2 4	14	8	4
5	7	2	10	8 7	2 4	15	·	·
6	7 5	2 3	11	8 7	2 4	16	9	8

while P_k and Q_k are empty for $k = 0$, 1, 17, and 18.

In this way every set partition leads to a vacillating tableau loop $\lambda_0, \lambda_1, \ldots, \lambda_{2n}$, if we let λ_k be the integer partition that specifies the common shape of P_k and Q_k. (The loop is 0, 0, 1, 1, 11, 1, 2, 2, 21, 11, 11, 11, 11, 1, 1, 0, 1, 0, 0 in our example.) Moreover, $t_{2k-1} = 0$ if and only if row $n+1-k$ contains no rook, if and only if k is smallest in its block.

Conversely, the elements of P_k and Q_k can be uniquely reconstructed from the sequence of shapes λ_k. Namely, $Q_k = Q_{k-1}$ if $t_k = 0$. Otherwise, if k is even, Q_k is Q_{k-1}

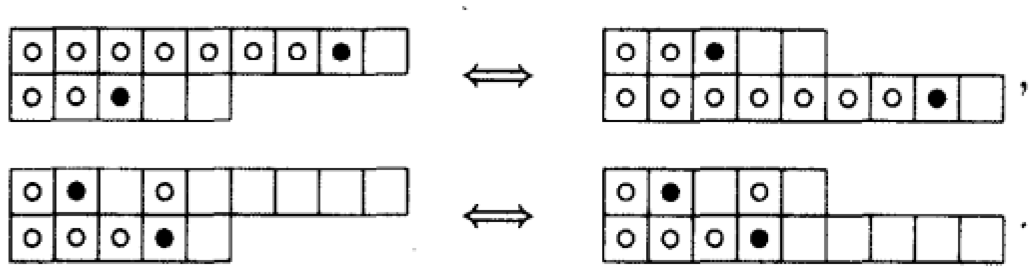
with the number $k/2$ placed in a new cell at the right of row t_k; if k is odd, Q_k is obtained from Q_{k-1} by using Algorithm 5.1.4D to delete the rightmost entry of row t_k. A similar procedure defines P_k from the values of P_{k+1} and t_{k+1}, so we can work back from P_{2n} to P_0. Thus the sequence of shapes λ_k is enough to tell us where to place the rooks.

Vacillating tableau loops were introduced in the paper "Crossings and nestings of matchings and partitions" by W. Y. C. Chen, E. Y. P. Deng, R. R. X. Du, R. P. Stanley, and C. H. Yan (preprint, 2005), who showed that the construction has significant (and surprising) consequences. For example, if the set partition Π corresponds to the vacillating tableau loop $\lambda_0, \lambda_1, \ldots, \lambda_{2n}$, let's say that its *dual* Π^D is the set partition that corresponds to the sequence of transposed shapes $\lambda_0^T, \lambda_1^T, \ldots, \lambda_{2n}^T$. Then, by exercise 5.1.4–7, Π contains a "k-crossing at l," namely a sequence of indices with $i_1 < \cdots < i_k \le l < j_1 < \cdots < j_k$ and $i_1 \equiv j_1, \ldots, i_k \equiv j_k$ (modulo Π), if and only if Π^D contains a "k-nesting at l," which is a sequence of indices with $i'_1 < \cdots < i'_k \le l < j'_k < \cdots < j'_1$ and $i'_1 \equiv j'_1, \ldots, i'_k \equiv j'_k$ (modulo Π^D). Notice also that an involution is essentially a set partition in which all blocks have size 1 or 2; the dual of an involution is an involution having the same singleton sets. In particular, the dual of a perfect matching (when there are no singleton sets) is a perfect matching.

Furthermore, an analogous construction applies to rook placements in *any* Ferrers diagram, not only in the stairstep shapes that correspond to set partitions. Given a Ferrers diagram that has at most m parts, all of size $\le n$, we simply consider the path $z_0 = (0,0), z_1, \ldots, z_{m+n} = (n,m)$ that hugs the right edge of the diagram, and stipulate that $\lambda_k = \lambda_{k-1} + e_{t_k}$ when $z_k = z_{k-1} + (1,0)$, $\lambda_k = \lambda_{k-1} - e_{t_k}$ when $z_k = z_{k-1} + (0,1)$. The proof we gave for stairstep shapes shows also that every placement of rooks in the Ferrers diagram, with at most one rook in each row and at most one in each column, corresponds to a unique tableau loop of this kind.

[And much more is true, besides! See S. Fomin, *J. Combin. Theory* **A72** (1995), 277–292; M. van Leeuwen, *Electronic J. Combinatorics* **3**, 2 (1996), paper #R15.]

28. (a) Define a one-to-one correspondence between rook placements, by interchanging the positions of rooks in rows j and $j+1$ if and only if there's a rook in the "panhandle" of the longer row:



(b) This relation is obvious from the definition, by transposing all the rooks.

(c) Suppose $a_1 \ge a_2 \ge \cdots$ and $a_k > a_{k+1}$. Then we have

$$R(a_1, a_2, \ldots) = xR(a_1-1, \ldots, a_{k-1}-1, a_{k+1}, \ldots) + yR(a_1, \ldots, a_{k-1}, a_k-1, a_{k+1}, \ldots)$$

because the first term counts cases where a rook is in row k and column a_k. Also $R(0) = 1$ because of the empty placement. From these recurrences we find

$$R(1) = x + y;\;\; R(2) = R(1,1) = x + xy + y^2;\;\; R(3) = R(1,1,1) = x + xy + xy^2 + y^3;$$
$$R(2,1) = x^2 + 2xy + xy^2 + y^3;$$
$$R(3,1) = R(2,2) = R(2,1,1) = x^2 + x^2y + xy + 2xy^2 + xy^3 + y^4;$$
$$R(3,1,1) = R(3,2) = R(2,2,1) = x^2 + 2x^2y + x^2y^2 + 2xy^2 + 2xy^3 + xy^4 + y^5;$$
$$R(3,2,1) = x^3 + 3x^2y + 3x^2y^2 + x^2y^3 + 3xy^3 + 2xy^4 + xy^5 + y^6.$$

(d) For example, the formula $\varpi_{73}(x,y) = x\varpi_{63}(x,y) + y\varpi_{74}(x,y)$ is equivalent to $R(5,4,4,3,2,1) = xR(4,3,3,2,1) + yR(5,4,3,3,2,1)$, a special case of (c); and $\varpi_{nn}(x,y) = R(n-2,\ldots,0)$ is obviously equal to $\varpi_{(n-1)1}(x,y) = R(n-2,\ldots,1)$.

(e) In fact $y^{k-1}\varpi_{nk}(x,y)$ is the stated sum over all restricted growth strings $a_1\ldots a_n$ for which $a_2 > 0, \ldots, a_k > 0$.

29. (a) If the rooks are respectively in columns $(c_1,\ldots,c_n)$, the number of free cells is the number of inversions of the permutation $(n+1-c_1)\ldots(n+1-c_n)$. [Rotate the right-hand example of Fig. 35 by 180° and compare the result to the illustration following Eq. 5.1.1–(5).]

(b) Each $r \times r$ configuration can be placed in, say, rows $i_1 < \cdots < i_r$ and columns $j_1 < \cdots < j_r$, yielding $(m-r)(n-r)$ free cells in the unchosen rows and columns; there are $(i_2-i_1+1) + 2(i_3-i_2-1) + \cdots + (r-1)(i_r-i_{r-1}-1) + r(m-i_r)$ in the unchosen rows and chosen columns, and a similar number in the chosen rows and unchosen columns. Furthermore

$$\sum_{1\le i_1<\cdots<i_r\le m} y^{(i_2-i_1+1)+2(i_3-i_2-1)+\cdots+(r-1)(i_r-i_{r-1}-1)+r(m-i_r)}$$

may be regarded as the sum of $y^{a_1+a_2+\cdots+a_{m-r}}$ over all partitions $r \ge a_1 \ge a_2 \ge \cdots \ge a_{m-r} \ge 0$, so it is $\binom{m}{r}_y$ by Theorem C. The polynomial $r!_y$ generates free cells for the chosen rows and columns, by (a). Therefore the answer is $y^{(m-r)(n-r)}\binom{m}{r}_y\binom{n}{r}_y r!_y = y^{(m-r)(n-r)}m!_y n!_y/((m-r)!_y(n-r)!_y r!_y)$.

(c) The left-hand side is the generating function $R_m(t+a_1,\ldots,t+a_m)$ for the Ferrers diagram with t additional columns of height m. For there are $t+a_m$ ways to put a rook in row m, yielding $1+y+\cdots+y^{t+a_m-1} = (1-y^{t+a_m})/(1-y)$ free cells with respect to those choices; then there are $t+a_{m-1}-1$ available cells in row $m-1$, etc.

The right-hand side, likewise, equals $R_m(t+a_1,\ldots,t+a_m)$. For if $m-k$ rooks are placed into columns $> t$, we must put k rooks into columns $\le t$ of the k unused rows; and we have seen that $t!_y/(t-k)!_y$ is the generating function for free cells when k rooks are placed on a $k \times t$ board.

[The formula proved here can be regarded as a polynomial identity in the variables y and y^t; therefore it is valid for arbitrary t, although our proof assumed that t is a nonnegative integer. This result was discovered in the case $y = 1$ by J. Goldman, J. Joichi, and D. White, *Proc. Amer. Math. Soc.* **52** (1975), 485–492. The general case was established by A. M. Garsia and J. B. Remmel, *J. Combinatorial Theory* **A41** (1986), 246–275, who used a similar argument to prove the additional formula

$$\sum_{t=0}^{\infty} z^t \prod_{j=1}^{m} \frac{1-y^{a_j+m-j+t}}{1-y} = \sum_{k=0}^{n} k!_y \left(\frac{z}{1-yz}\right)\cdots\left(\frac{z}{1-y^k z}\right) R_{m-k}(a_1,\ldots,a_m).$$

(d) This statement, which follows immediately from (c), also implies that we have $R(a_1,\ldots,a_m) = R(a'_1,\ldots,a'_m)$ if and only if equality holds for all x and for any nonzero value of y. The Peirce polynomial $\varpi_{nk}(x,y)$ of exercise 28(d) is the rook polynomial for $\binom{n-1}{k-1}$ different Ferrers diagrams; for example, $\varpi_{63}(x,y)$ enumerates rook placements for the shapes 43321, 44221, 44311, 4432, 53221, 53311, 5332, 54211, 5422, and 5431.

30. (a) We have $\varpi_n(x,y) = \sum_m x^{n-m}A_{mn}$, where $A_{mn} = R_{n-m}(n-1,\ldots,1)$ satisfies a simple law: If we don't place a rook in row 1 of the shape $(n-1,\ldots,1)$, that row has $m-1$ free cells because of the $n-m$ rooks in other rows. But if we do put a rook there, we leave 0 or 1 or $\cdots$ or $m-1$ of its cells free. Hence $A_{mn} = y^{m-1}A_{(m-1)(n-1)} + (1+y+\cdots+y^{m-1})A_{m(n-1)}$, and it follows by induction that $A_{mn} = y^{m(m-1)/2}\left\{{n \atop m}\right\}_y$.

(b) The formula $\varpi_{n+1}(x,y) = \sum_k \binom{n}{k} x^{n-k} y^k \varpi_k(x,y)$ yields

$$A_{m(n+1)} = \sum_k \binom{n}{k} y^k A_{(m-1)k}.$$

(c) From (a) and (b) we have

$$\frac{z^n}{(1-z)(1-(1+q)z)\ldots(1-(1+q+\cdots+q^{n-1})z)} = \sum_k \left\{{k \atop n}\right\}_q z^k;$$

$$\sum_k \binom{n}{k}_q (-1)^k q^{\binom{k}{2}} e^{(1+q+\cdots+q^{n-k-1})z} = q^{\binom{n}{2}}\, n!_q \sum_k \left\{{k \atop n}\right\}_q \frac{z^k}{k!}.$$

[The second formula is proved by induction on n, because both sides satisfy the differential equation $G'_{n+1}(z) = (1+q+\cdots+q^n)e^z G_n(qz)$; exercise 1.2.6–58 proves equality when $z=0$.]

Historical note: Leonard Carlitz introduced q-Stirling numbers in *Transactions of the Amer. Math. Soc.* **33** (1933), 127–129. Then in *Duke Math. J.* **15** (1948), 987–1000, he derived (among other things) an appropriate generalization of Eq. 1.2.6–(45):

$$(1+q+\cdots+q^{m-1})^n = \sum_k \left\{{n \atop k}\right\}_q q^{\binom{k}{2}} \frac{m!_q}{(m-k)!_q}.$$

31. $\exp(e^{w+z}+w-1)$; therefore $\varpi_{nk} = (\varpi+1)^{n-k}\varpi^{k-1} = \varpi^{n+1-k}(\varpi-1)^{k-1}$ in the umbral notation of exercise 23. [L. Moser and M. Wyman, *Trans. Royal Soc. Canada* (3) **43** (1954), Section 3, 31–37.] In fact, the numbers $\varpi_{nk}(x,1)$ of exercise 28(d) are generated by $\exp((e^{xw+xz}-1)/x+xw)$.

32. We have $\delta_n = \varpi_n(1,-1)$, and a simple pattern is easily perceived in the generalized Peirce triangle of exercise 28(d) when $x=1$ and $y=-1$: We have $|\varpi_{nk}(1,-1)| \le 1$ and $\varpi_{n(k+1)}(1,-1) \equiv \varpi_{nk}(1,-1) + (-1)^n$ (modulo 3) for $1 \le k < n$. [In *JACM* **20** (1973), 512–513, Gideon Ehrlich gave a combinatorial proof of an equivalent result.]

33. Representing set partitions by rook placements as in answer 27 leads to the answer ϖ_{nk}, by setting $x=y=1$ in exercise 28(d). [The case $k=n$ was discovered by H. Prodinger, *Fibonacci Quarterly* **19** (1981), 463–465.]

34. (a) Guittone's *Sonetti* included 149 of scheme 01010101232323, 64 of scheme 01010101234234, two of scheme 01010101234342, seven with schemes used only once (like 01100110234432), and 29 poems that we would no longer consider to be sonnets because they do not have 14 lines.

(b) Petrarch's *Canzoniere* included 115 sonnets of scheme 01100110234234, 109 of scheme 01100110232323, 66 of scheme 01100110234324, 7 of scheme 01100110232232, and 20 others of schemes like 01010101232323 used at most three times each.

(c) In Spenser's *Amoretti*, 88 of 89 sonnets used the scheme 01011212232344; the exception (number 8) was "Shakespearean."

(d) Shakespeare's 154 sonnets all used the rather easy scheme 01012323454566, except that two of them (99 and 126) didn't have 14 lines.

(e) Browning's 44 *Sonnets From the Portuguese* obeyed the Petrarchan scheme 01100110232323.

Sometimes the lines would rhyme (by chance?) even when they didn't need to; for example, Browning's final sonnet actually had the scheme 01100110121212.

Incidentally, the lengthy cantos in Dante's *Divine Comedy* used an interlocking scheme of rhymes in which $1 \equiv 3$ and $3n-1 \equiv 3n+1 \equiv 3n+3$ for $n = 1, 2, \ldots$.

35. Every incomplete n-line rhyme scheme Π corresponds to a singleton-free partition of $\{1,\ldots,n{+}1\}$ in which $(n{+}1)$ is grouped with all of Π's singletons. [H. W. Becker gave an algebraic proof in *AMM* **48** (1941), 702. Notice that $\varpi'_n = \sum_k \binom{n}{k}(-1)^{n-k}\varpi_k$, by the principle of inclusion and exclusion, and $\varpi_n = \sum_k \binom{n}{k}\varpi'_k$; we can in fact write $\varpi' = \varpi - 1$ in the umbral notation of exercise 23. J. O. Shallit has suggested extending Peirce's triangle by setting $\varpi_{n(n+1)} = \varpi'_n$; see exercises 38(e) and 33. In fact, ϖ_{nk} is the number of partitions of $\{1,\ldots,n\}$ with the property that 1, ..., $k-1$ are not singletons; see H. W. Becker, *Bull. Amer. Math. Soc.* **58** (1954), 63.]

36. $\exp(e^z - 1 - z)$. (In general, if ϑ_n is the number of partitions of $\{1,\ldots,n\}$ into subsets of allowable sizes $s_1 < s_2 < \cdots$, the exponential generating function $\sum_n \vartheta_n z^n/n!$ is $\exp(z^{s_1}/s_1! + z^{s_2}/s_2! + \cdots)$, because $(z^{s_1}/s_1! + z^{s_2}/s_2! + \cdots)^k$ is the exponential generating function for partitions into exactly k parts.)

37. There are $\sum_k \binom{n}{k}\varpi'_k\varpi'_{n-k}$ possibilities of length n, hence 784,071,966 when $n = 14$. (But Pushkin's scheme is hard to beat.)

38. (a) Imagine starting with $x_1x_2\ldots x_n = 01\ldots(n{-}1)$, then successively removing some element b_j and placing it at the left, for $j = 1, 2, \ldots, n$. Then x_k will be the kth most recently moved element, for $1 \le k \le |\{b_1,\ldots,b_n\}|$; see exercise 5.2.3–36. Consequently the array $x_1\ldots x_n$ will return to its original state if and only if $b_n\ldots b_1$ is a restricted growth string. [Robbins and Bolker, *Æquat. Math.* **22** (1981), 281–282.]

In other words, let $a_1\ldots a_n$ be a restricted growth string. Set $b_{-j} \leftarrow j$ and $b_{j+1} \leftarrow a_{n-j}$ for $0 \le j < n$. Then for $1 \le j \le n$, define k_j by the rule that b_j is the k_jth distinct element of the sequence $b_{j-1}, b_{j-2}, \ldots$. For example, the string $a_1\ldots a_{16} =$ 0123032303456745 corresponds in this way to the σ-cycle 6688448628232384.

(b) Such paths correspond to restricted growth strings with $\max(a_1,\ldots,a_n) \le m$, so the answer is ${n \brace 0} + {n \brace 1} + \cdots + {n \brace m}$.

(c) We may assume that $i = 1$, because the sequence $k_2\ldots k_nk_1$ is a σ-cycle whenever $k_1k_2\ldots k_n$ is. Thus the answer is the number of restricted growth strings with $a_n = j-1$, namely ${n-1 \brace j-1} + {n-1 \brace j} + {n-1 \brace j+1} + \cdots$.

(d) If the answer is f_n we must have $\sum_k \binom{n}{k} f_k = \varpi_n$, since σ_1 is the identity permutation. Therefore $f_n = \varpi'_n$, the number of set partitions without singletons (exercise 35).

(e) Again ϖ'_n, by (a) and (d). [Consequently $\varpi'_p \bmod p = 1$ when p is prime.]

39. Set $u = t^{p+1}$ to obtain $\frac{1}{p+1}\int_0^\infty e^{-u}u^{(q-p)/(p+1)}\,du = \frac{1}{p+1}\Gamma\bigl(\frac{q+1}{p+1}\bigr)$.

40. We have $g(z) = cz - n\ln z$, so the saddle point occurs at n/c. The rectangular path now has corners at $\pm n/c \pm mi/c$; and $\exp g(n/c + it) = (e^nc^n/n^n)\exp(-t^2c^2/(2n) + it^3c^3/(3n^2) + \cdots)$. The final result is $e^n(c/n)^{n-1}/\sqrt{2\pi n}$ times $1 + n/12 + O(n^{-2})$.

(Of course we could have obtained this result more quickly by letting $w = cz$ in the integral. But the answer given here applies the saddle point method mechanically, without attempting to be clever.)

41. Again the net result is just to multiply (21) by c^{n-1}; but in this case the *left* edge of the rectangular path is significant instead of the right edge. (Incidentally, when $c = -1$ we cannot derive an analog of (22) using Hankel's contour when x is real and positive, because the integral on that path diverges. But with the usual definition of z^z, a suitable path of integration does yield the formula $-(\cos \pi x)/\Gamma(x)$ when $n = x > 0$.)

42. We have $\oint e^{z^2}\,dz/z^n = 0$ when n is even. Otherwise both left and right edges of the rectangle with corners $\pm\sqrt{n/2} \pm in$ contribute approximately

$$\frac{e^{n/2}}{2\pi(n/2)^{n/2}} \int_{-\infty}^{\infty} \exp\Bigl(-2t^2 - \frac{(-it)^3}{3}\frac{2^{3/2}}{n^{1/2}} + \frac{(it)^4}{n} - \cdots\Bigr)\,dt,$$

when n is large. We can restrict $|t| \le n^\epsilon$ to show that this integral is $I_0 + (I_4 - \frac{4}{9}I_6)/n$ with relative error $O(n^{9\epsilon-3/2})$, where $I_k = \int_{-\infty}^{\infty} e^{-2t^2}t^k\,dt$. As before, the relative error is actually $O(n^{-2})$; we deduce the answer

$$\frac{1}{((n-1)/2)!} = \frac{e^{n/2}}{\sqrt{2\pi}(n/2)^{n/2}}\Bigl(1 + \frac{1}{12n} + O\Bigl(\frac{1}{n^2}\Bigr)\Bigr), \qquad n \text{ odd.}$$

(The analog of (22) is $(\sin \frac{\pi x}{2})^2/\Gamma((x-1)/2)$ when $n = x > 0$.)

43. Let $f(z) = e^{e^z}/z^n$. When $z = -n + it$ we have $|f(z)| < en^{-n}$; when $z = t + 2\pi in + i\pi/2$ we have $|f(z)| = |z|^{-n} < (2\pi n)^{-n}$. So the integral is negligible except on a path $z = \xi + it$; and on that path $|f|$ decreases as $|t|$ increases from 0 to π. Already when $t = n^{\epsilon-1/2}$ we have $|f(z)|/f(\xi) = O(\exp(-n^{2\epsilon}/(\log n)^2))$. And when $|t| > \pi$ we have $|f(z)|/f(\xi) < 1/|1 + i\pi/\xi|^n = \exp(-\frac{n}{2}\ln(1 + \pi^2/\xi^2))$.

44. Set $u = na_2t^2$ in (25) to obtain $\Re\int_0^\infty e^{-u}\exp(n^{-1/2}c_3(-u)^{3/2} + n^{-1}c_4(-u)^2 + n^{-3/2}c_5(-u)^{5/2} + \cdots)\,du/\sqrt{na_2u}$ where $c_k = (2/(\xi+1))^{k/2}(\xi^{k-1} + (-1)^k(k-1)!)/k! = a_k/a_2^{k/2}$. This expression leads to

$$b_l = \sum_{\substack{k_1+2k_2+3k_3+\cdots=2l\\ k_1+k_2+k_3+\cdots=m\\ k_1,k_2,k_3,\ldots\ge 0}} \Bigl(-\frac{1}{2}\Bigr)^{\underline{l+m}} \frac{c_3^{k_1}}{k_1!}\frac{c_4^{k_2}}{k_2!}\frac{c_5^{k_3}}{k_3!}\ldots,$$

a sum over partitions of $2l$. For example, $b_1 = \frac{3}{4}c_4 - \frac{15}{16}c_3^2$.

45. To get $\varpi_n/n!$ we replace $g(z)$ by $e^z - (n+1)\ln z$ in the derivation of (26). This change multiplies the integrand in the previous answer by $1/(1 + it/\xi)$, which is $1/(1 - n^{-1/2}a(-u)^{1/2})$ where $a = -\sqrt{2/(\xi+1)}$. Thus we get

$$b_l' = \sum_{\substack{k+k_1+2k_2+3k_3+\cdots=2l\\ k_1+k_2+k_3+\cdots=m\\ k,k_1,k_2,k_3,\ldots\ge 0}} \Bigl(-\frac{1}{2}\Bigr)^{\underline{l+m}} a^k\,\frac{c_3^{k_1}}{k_1!}\frac{c_4^{k_2}}{k_2!}\frac{c_5^{k_3}}{k_3!}\ldots,$$

a sum of $p(2l) + p(2l-1) + \cdots + p(0)$ terms; $b_1' = \frac{3}{4}c_4 - \frac{15}{16}c_3^2 + \frac{3}{4}ac_3 - \frac{1}{2}a^2$. [The coefficient b_1' was obtained in a different way by L. Moser and M. Wyman, *Trans. Royal Soc. Canada* (3) **49**, Section 3 (1955), 49–54, who were the first to deduce an asymptotic series for ϖ_n. Their approximation is slightly less accurate than the result of (26) with n changed to $n+1$, because it doesn't pass exactly through the saddle point. Formula (26) is due to I. J. Good, *Iranian J. Science and Tech.* **4** (1975), 77–83.]

46. Eqs. (13) and (31) show that $\varpi_{nk} = (1 - \xi/n)^k\varpi_n(1 + O(n^{-1}))$ for fixed k as $n \to \infty$. And this approximation also holds when $k = n$, but with relative error $O((\log n)^2/n)$.

47. Steps (H1, ..., H6) are performed respectively $(1, \varpi_n, \varpi_n - \varpi_{n-1}, \varpi_{n-1}, \varpi_{n-1}, \varpi_{n-1} - 1)$ times. The loop in H4 sets $j \leftarrow j - 1$ a total of $\varpi_{n-2} + \varpi_{n-3} + \cdots + \varpi_1$ times; the loop in H6 sets $b_j \leftarrow m$ a total of $(\varpi_{n-2} - 1) + \cdots + (\varpi_1 - 1)$ times. The ratio ϖ_{n-1}/ϖ_n is approximately $(\ln n)/n$, and $(\varpi_{n-2} + \cdots + \varpi_1)/\varpi_n \approx (\ln n)^2/n^2$.

48. We can easily verify the interchange of summation and integration in

$$\frac{e\varpi_x}{\Gamma(x+1)} = \frac{1}{2\pi i}\oint \frac{e^{e^z}}{z^{x+1}}\,dz = \frac{1}{2\pi i}\oint \sum_{k=0}^{\infty} \frac{e^{kz}}{k!\,z^{x+1}}\,dz$$
$$= \sum_{k=0}^{\infty} \frac{1}{k!}\frac{1}{2\pi i}\oint \frac{e^{kz}}{z^{x+1}}\,dz = \sum_{k=0}^{\infty} \frac{1}{k!}\frac{k^x}{\Gamma(x+1)}.$$

49. If $\xi = \ln n - \ln\ln n + x$, we have $\beta = 1 - e^{-x} - \alpha x$. Therefore by Lagrange's inversion formula (exercise 4.7–8),

$$x = \sum_{k=1}^{\infty} \frac{\beta^k}{k}\,[t^{k-1}]\Bigl(\frac{f(t)}{1-\alpha f(t)}\Bigr)^k = \sum_{k=1}^{\infty}\sum_{j=0}^{\infty} \frac{\beta^k}{k}\alpha^j \binom{k+j-1}{j}\,[t^{k-1}]\,f(t)^{j+k},$$

where $f(t) = t/(1 - e^{-t})$. So the result follows from the handy identity

$$\Bigl(\frac{z}{1-e^{-z}}\Bigr)^m = \sum_{n=0}^{\infty} \left[{m \atop m-n}\right] \frac{z^n}{(m-1)(m-2)\dots(m-n)}.$$

(This identity should be interpreted carefully when $n \ge m$; the coefficient of z^n is a polynomial in m of degree n, as explained in *CMath* equation (7.59).)

The formula in this exercise is due to L. Comtet, *Comptes Rendus Acad. Sci.* (A) **270** (Paris, 1970), 1085–1088, who identified the coefficients previously computed by N. G. de Bruijn, *Asymptotic Methods in Analysis* (1958), 25–28. Convergence for $n \ge e$ was shown by Jeffrey, Corless, Hare, and Knuth, *Comptes Rendus Acad. Sci.* (I) **320** (1995), 1449–1452, who also derived a formula that converges somewhat faster.

(The equation $\xi e^\xi = n$ has complex roots as well. We can obtain them all by using $\ln n + 2\pi i m$ in place of $\ln n$ in the formula of this exercise; the sum converges rapidly when $m \ne 0$. See Corless, Gonnet, Hare, Jeffrey, and Knuth, *Advances in Computational Math.* **5** (1996), 347–350.)

50. Let $\xi = \xi(n)$. Then $\xi'(n) = \xi/((\xi+1)n)$, and the Taylor series

$$\xi(n+k) = \xi + k\xi'(n) + \frac{k^2}{2}\xi''(n) + \cdots$$

can be shown to converge for $|k| < n + 1/e$.

Indeed, much more is true, because the function $\xi(n) = -T(-n)$ is obtained from the tree function $T(z)$ by analytic continuation to the negative real axis. (The tree function has a quadratic singularity at $z = e^{-1}$; after going around this singularity we encounter a logarithmic singularity at $z = 0$, as part of an interesting multi-level Riemann surface on which the quadratic singularity appears only at level 0.) The derivatives of the tree function satisfy $z^k T^{(k)}(z) = R(z)^k p_k(R(z))$, where $R(z) = T(z)/(1 - T(z))$ and $p_k(x)$ is the polynomial of degree $k-1$ defined by $p_{k+1}(x) = (1+x)^2 p_k'(x) + k(2+x)p_k(x)$. For example,

$$p_1(x) = 1,\ \ p_2(x) = 2 + x,\ \ p_3(x) = 9 + 10x + 3x^2,\ \ p_4(x) = 64 + 113x + 70x^2 + 15x^3.$$

(The coefficients of $p_k(x)$, incidentally, enumerate certain phylogenetic trees called Greg trees: $[x^j]\,p_k(x)$ is the number of oriented trees with j unlabeled nodes and k labeled nodes, where leaves must be labeled and unlabeled nodes must have at least two children. See J. Felsenstein, *Systematic Zoology* **27** (1978), 27–33; L. R. Foulds and R. W. Robinson, *Lecture Notes in Math.* **829** (1980), 110–126; C. Flight, *Manuscripta* **34** (1990), 122–128.) If $q_k(x) = p_k(-x)$, we can prove by induction that $(-1)^m q_k^{(m)}(x) \ge 0$ for $0 \le x \le 1$. Therefore $q_k(x)$ decreases monotonically from k^{k-1} to $(k-1)!$ as x goes from 0 to 1, for all $k, m \ge 1$. It follows that

$$\xi(n+k) = \xi + \frac{kx}{n} - \left(\frac{kx}{n}\right)^2 \frac{q_2(x)}{2!} + \left(\frac{kx}{n}\right)^3 \frac{q_3(x)}{3!} - \cdots, \qquad x = \frac{\xi}{\xi+1},$$

where the partial sums alternately overshoot and undershoot the correct value if $k > 0$.

51. There are two saddle points, $\sigma = \sqrt{n+5/4} - 1/2$ and $\sigma' = -1-\sigma$. Integration on a rectangular path with corners at $\sigma \pm im$ and $\sigma' \pm im$ shows that only σ is relevant as $n \to \infty$ (although σ' contributes a relative error of roughly $e^{-\sqrt{n}}$, which can be significant when n is small). Arguing almost as in (25), but with $g(z) = z + z^2/2 - (n+1)\ln z$, we find that t_n is well approximated by

$$\frac{n!}{2\pi}\int_{-n^\epsilon}^{n^\epsilon} e^{g(\sigma) - a_2 t^2 + a_3 i t^3 + \cdots + a_l(-it)^l + O(n^{(l+1)\epsilon - (l-1)/2})}\,dt, \qquad a_k = \frac{\sigma+1}{k\sigma^{k-1}} + \frac{[k=2]}{2}.$$

The integral expands as in exercise 44 to

$$\frac{n!\,e^{(n+\sigma)/2}}{2\sigma^{n+1}\sqrt{\pi a_2}}\bigl(1 + b_1 + b_2 + \cdots + b_m + O(n^{-m-1})\bigr).$$

This time $c_k = (\sigma+1)\sigma^{1-k}(1+1/(2\sigma))^{-k/2}/k$ for $k \ge 3$, hence $(2\sigma+1)^{3k}\sigma^k b_k$ is a polynomial in σ of degree $2k$; for example,

$$b_1 \;=\; \frac{3}{4}c_4 - \frac{15}{16}c_3^2 \;=\; \frac{8\sigma^2 + 7\sigma - 1}{12\sigma(2\sigma+1)^3}.$$

In particular, Stirling's approximation and the b_1 term yield

$$t_n = \frac{1}{\sqrt{2}} n^{n/2} e^{-n/2+\sqrt{n}-1/4}\left(1 + \frac{7}{24}n^{-1/2} - \frac{119}{1152}n^{-1} - \frac{7933}{414720}n^{-3/2} + O(n^{-2})\right)$$

after we plug in the formula for σ — a result substantially more accurate than equation 5.1.4–(53), and obtained with considerably less labor.

52. Let $G(z) = \sum_k \Pr(X=k)z^k$, so that the jth cumulant κ_j is $j!\,[t^j]\ln G(e^t)$. In case (a) we have $G(z) = e^{e^{\xi z} - e^\xi}$; hence

$$\ln G(e^t) = e^{\xi e^t} - e^\xi = e^\xi(e^{\xi(e^t-1)} - 1) = e^\xi \sum_{k=1}^{\infty} (e^t-1)^k \frac{\xi^k}{k!}, \quad \kappa_j = e^\xi \sum_k \left\{ {k \atop j} \right\} \xi^k [j \ne 0].$$

Case (b) is sort of a dual situation: Here $\kappa = j = \varpi_j\,[j \ne 0]$ because

$$G(z) = e^{e^{-1}-1} \sum_{j,k} \left\{ {k \atop j} \right\} e^{-j} \frac{z^k}{k!} = e^{e^{-1}-1} \sum_j \frac{(e^{z-1} - e^{-1})^j}{j!} = e^{e^{z-1}-1}.$$

[If $\xi e^\xi = 1$ in case (a) we have $\kappa_j = e\varpi\,[j \ne 0]$. But if $\xi e^\xi = n$ in that case, the mean is $\kappa_1 = n$ and the variance σ^2 is $(\xi+1)n$. Thus, the formula in exercise 45 states that the mean value n occurs with approximate probability $1/\sqrt{2\pi\sigma}$ and relative error $O(1/n)$. This observation leads to another way to prove that formula.]

53. We can write $\ln G(e^t) = \mu t + \sigma^2 t^2/2 + \kappa_3 t^3/3! + \cdots$ as in Eq. 1.2.10–(23), and there is a positive constant δ such that $\sum_{j=3}^{\infty} |\kappa_j| t^j/j! < \sigma^2 t^2/6$ when $|t| \le \delta$. Hence, if $0 < \epsilon < 1/2$, we can prove that

$$[z^{\mu n+r}]\, G(z)^n = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{G(e^{it})^n\, dt}{e^{it(\mu n+r)}}$$
$$= \frac{1}{2\pi}\int_{-n^{\epsilon-1/2}}^{n^{\epsilon-1/2}} \exp\Bigl(-irt - \frac{\sigma^2 t^2 n}{2} + O(n^{3\epsilon-1/2})\Bigr)\, dt + O(e^{-cn^{2\epsilon}})$$

as $n \to \infty$, for some constant $c > 0$: The integrand for $n^{\epsilon-1/2} \le |t| \le \delta$ is bounded in absolute value by $\exp(-\sigma^2 n^{2\epsilon}/3)$; and when $\delta \le |t| \le \pi$ its magnitude is at most α^n, where $\alpha = \max|G(e^{it})|$ is less than 1 because the individual terms $p_k e^{kit}$ don't all lie on a straight line by our assumption. Thus

$$[z^{\mu n+r}]\, G(z)^n = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\Bigl(-irt - \frac{\sigma^2 t^2 n}{2} + O(n^{3\epsilon-1/2})\Bigr)\, dt + O(e^{-cn^{2\epsilon}})$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\Bigl(-\frac{\sigma^2 n}{2}\Bigl(t + \frac{ir}{\sigma^2 n}\Bigr)^2 - \frac{r^2}{2\sigma^2 n} + O(n^{3\epsilon-1/2})\Bigr)\, dt + O(e^{-cn^{2\epsilon}})$$
$$= \frac{e^{-r^2/(2\sigma^2 n)}}{\sigma\sqrt{2\pi n}} + O(n^{3\epsilon-1}).$$

By taking account of $\kappa_3, \kappa_4, \ldots$ in a similar way we can refine the estimate to $O(n^{-m})$ for arbitrarily large m; thus the result is valid also for $\epsilon = 0$. [In fact, such refinements lead to the "Edgeworth expansion," according to which $[z^{\mu n+r}]\, G(z)^n$ is asymptotic to

$$\frac{e^{-r^2/(2\sigma^2 n)}}{\sigma\sqrt{2\pi n}} \sum_{\substack{k_1+2k_2+3k_3+\cdots=m\\ k_1+k_2+k_3+\cdots=l\\ k_1,k_2,k_3,\ldots\ge 0\\ 0\le s\le l+m/2}} \frac{(-1)^s (2l+m)^{\underline{2s}}}{\sigma^{4l+2m-2s}\, 2^s s!}\, \frac{r^{2l+m-2s}}{n^{l+m-s}}\, \frac{1}{k_1!\, k_2!\, \ldots} \Bigl(\frac{\kappa_3}{3!}\Bigr)^{k_1} \Bigl(\frac{\kappa_4}{4!}\Bigr)^{k_2} \ldots;$$

the absolute error is $O(n^{-p/2})$, where the constant hidden in the O depends only on p and G but not on r or n, if we restrict the sum to cases with $m < p - 1$. For example, when $p = 3$ we get

$$[z^{\mu n+r}]\, G(z)^n = \frac{e^{-r^2/(2\sigma^2 n)}}{\sigma\sqrt{2\pi n}} \Bigl(1 - \frac{\kappa_3}{2\sigma^4}\Bigl(\frac{r}{n}\Bigr) + \frac{\kappa_3}{6\sigma^6}\Bigl(\frac{r^3}{n^2}\Bigr)\Bigr) + O\Bigl(\frac{1}{n^{3/2}}\Bigr),$$

and there are seven more terms when $p = 4$. See P. L. Chebyshev, *Zapiski Imp. Akad. Nauk* **55** (1887), No. 6, 1–16; *Acta Math.* **14** (1890), 305–315; F. Y. Edgeworth, *Trans. Cambridge Phil. Soc.* **20** (1905), 36–65, 113–141; H. Cramér, *Skandinavisk Aktuarietidsskrift* **11** (1928), 13–74, 141–180.]

54. Formula (40) is equivalent to $\alpha = s\coth s + s$, $\beta = s\coth s - s$.

55. Let $c = \alpha e^{-\alpha}$. The Newtonian iteration $\beta_0 = c$, $\beta_{k+1} = (1-\beta_k)ce^{\beta_k}/(1-ce^{-\beta_k})$ rises rapidly to the correct value, unless α is extremely close to 1. For example, β_7 differs from $\ln 2$ by less than 10^{-75} when $\alpha = \ln 4$.

56. (a) By induction on n, $g^{(n+1)}(z) = (-1)^n \Bigl(\dfrac{\sum_{k=0}^{n} \langle {n \atop k} \rangle e^{(n-k)z}}{\alpha(e^z-1)^{n+1}} - \dfrac{n!}{z^{n+1}}\Bigr)$.

(b) $\sum_{k=0}^{n} \left\langle{n \atop k}\right\rangle e^{k\sigma}/n! = \int_0^1 \cdots \int_0^1 \exp(\lfloor u_1 + \cdots + u_n \rfloor \sigma)\, du_1 \ldots du_n$
$< \int_0^1 \cdots \int_0^1 \exp((u_1 + \cdots + u_n)\sigma)\, du_1 \ldots du_n = (e^\sigma - 1)^n/\sigma^n$.
The lower bound is similar, since $\lfloor u_1 + \cdots + u_n \rfloor > u_1 + \cdots + u_n - 1$.

(c) Thus $n!\,(1-\beta/\alpha) < (-\sigma)^n g^{(n+1)}(\sigma) < 0$, and we need only verify that $1-\beta/\alpha < 2(1-\beta)$, namely that $2\alpha\beta < \alpha + \beta$. But $\alpha\beta < 1$ and $\alpha + \beta > 2$, by exercise 54.

57. (a) $n + 1 - m = (n+1)(1 - 1/\alpha) < (n+1)(1 - \beta/\alpha) = (n+1)\sigma/\alpha \le 2N$ as in answer 56(c). (b) The quantity $\alpha + \alpha\beta$ increases as α increases, because its derivative with respect to α is $1 + \beta + \beta(1-\alpha)/(1-\beta) = (1-\alpha\beta)/(1-\beta) + \beta > 0$. Therefore $1 - \beta < 2(1 - 1/\alpha)$.

58. (a) The derivative of $|e^{\sigma+it} - 1|^2/|\sigma + it|^2 = (e^{\sigma+it} - 1)(e^{\sigma-it} - 1)/(\sigma^2 + t^2)$ with respect to t is $(\sigma^2 + t^2)\sin t - t(2\sin\frac{t}{2})^2 - (2\sinh\frac{\sigma}{2})^2 t$ times a positive function. This derivative is always negative for $0 < t \le 2\pi$, because it is less than $t^2 \sin t - t(2\sin\frac{t}{2})^2 = 8u \sin u \cos u(u - \tan u)$ where $t = 2u$.

Let $s = 2\sinh\frac{\sigma}{2}$. When $\sigma \ge \pi$ and $2\pi \le t \le 4\pi$, the derivative is still negative, because we have $t \le 4\pi \le s^2 - \sigma^2/(2\pi) \le s^2 - \sigma^2/t$. Similarly, when $\sigma \ge 2\pi$ the derivative remains negative for $4\pi \le t \le 168\pi$; the proof gets easier and easier.

(b) Let $t = u\sigma/\sqrt{N}$. Then (41) and (42) prove that

$$\int_{-\tau}^{\tau} e^{(n+1)g(\sigma+it)}\, dt =$$

$$\frac{(e^\sigma - 1)^m}{\sigma^n\sqrt{N}} \int_{-N^\epsilon}^{N^\epsilon} \exp\Bigl(-\frac{u^2}{2} + \frac{(-iu)^3 a_3}{N^{1/2}} + \cdots + \frac{(-iu)^l a_l}{N^{l/2-1}} + O(N^{(l+1)\epsilon-(l-1)/2})\Bigr)\, du,$$

where $(1-\beta)a_k$ is a polynomial of degree $k-1$ in α and β, with $0 \le a_k \le 2/k$. (For example, $6a_3 = (2 - \beta(\alpha+\beta))/(1-\beta)$ and $24a_4 = (6 - \beta(\alpha^2 + 4\alpha\beta + \beta^2))/(1-\beta)$.) The monotonicity of the integrand shows that the integral over the rest of the range is negligible. Now trade tails, extend the integral over $-\infty < u < \infty$, and use the formula of answer 44 with $c_k = 2^{k/2} a_k$ to define $b_1, b_2, \ldots$.

(c) We will prove that $|e^z - 1|^m \sigma^{n+1}/((e^\sigma - 1)^m |z|^{n+1})$ is exponentially small on those three paths. If $\sigma \le 1$, this quantity is less than $1/(2\pi)^{n+1}$ (because, for example, $e^\sigma - 1 > \sigma$). If $\sigma > 1$, we have $\sigma < 2|z|$ and $|e^z - 1| \le e^\sigma - 1$.

59. In this extreme case, $\alpha = 1 + n^{-1}$ and $\beta = 1 - n^{-1} + \frac{2}{3}n^{-2} + O(n^{-3})$; hence $N = 1 + \frac{1}{3}n^{-1} + O(n^{-2})$. The leading term $\beta^{-n}/\sqrt{2\pi N}$ is $e/\sqrt{2\pi}$ times $1 - \frac{1}{3}n^{-1} + O(n^{-2})$. (Notice that $e/\sqrt{2\pi} \approx 1.0844$.) The quantity a_k in answer 58(b) turns out to be $1/k + O(n^{-1})$. So the correction terms, to first order, are

$$\frac{b_j}{N^j} = [z^j] \exp\Bigl(-\sum_{k=1}^{\infty} \frac{B_{2k} z^{2k-1}}{2k(2k-1)}\Bigr) + O\Bigl(\frac{1}{n}\Bigr),$$

namely the terms in the (divergent) series corresponding to Stirling's approximation

$$\frac{1}{1!} \sim \frac{e}{\sqrt{2\pi}}\Bigl(1 - \frac{1}{12} + \frac{1}{288} + \frac{139}{51840} - \frac{571}{2488320} - \cdots\Bigr).$$

60. (a) The number of m-ary strings of length n in which all m digits appear is $m!\left\{{n \atop m}\right\}$, and the inclusion-exclusion principle expresses this quantity as $\binom{m}{0}m^n - \binom{m}{1}(m-1)^n + \cdots$. Now see exercise 7.2.1.4–37.

(b) We have $(m-1)^n/(m-1)! = (m^n/m!)m\exp(n\ln(1-1/m))$, and $\ln(1-1/m)$ is less than $-n^{\epsilon-1}$.

(c) In this case $\alpha > n^\epsilon$ and $\beta = \alpha e^{-\alpha}e^{\beta} < \alpha e^{1-\alpha}$. Therefore $1 < (1-\beta/\alpha)^{m-n} < \exp(nO(e^{-\alpha}))$; and $1 > e^{-\beta m} = e^{-(n+1)\beta/\alpha} > \exp(-nO(e^{-\alpha}))$. So (45) becomes $(m^n/m!)(1+O(n^{-1})+O(ne^{-n^\epsilon}))$.

61. Now $\alpha = 1+\frac{r}{n}+O(n^{2\epsilon-2})$ and $\beta = 1-\frac{r}{n}+O(n^{2\epsilon-2})$. Thus $N = r+O(n^{2\epsilon-1})$, and the case $l=0$ of Eq. (43) reduces to

$$n^{\underline{r}}\left(\frac{n}{2}\right)^r\frac{e^r}{r^r\sqrt{2\pi r}}\left(1+O(n^{2\epsilon-1})+O\left(\frac{1}{r}\right)\right).$$

(This approximation meshes well with identities such as $\left\{{n \atop n-1}\right\} = \binom{n}{2}$ and $\left\{{n \atop n-2}\right\} = 2\binom{n}{4}+\binom{n+1}{4}$; indeed, we have

$$\left\{{n \atop n-r}\right\} = \frac{n^{2r}}{2^r r!}\left(1+O\left(\frac{1}{n}\right)\right) \qquad \text{as } n\to\infty$$

when r is constant, according to formulas (6.42) and (6.43) of *CMath.*)

62. The assertion is true for $1 \le n \le 10000$ (with $m = \lfloor e^\xi - 1\rfloor$ in 5648 of those cases). E. R. Canfield and C. Pomerance, in a paper that nicely surveys previous work on related problems, have shown that the statement holds for all sufficiently large n, and that the maximum occurs in *both* cases only if $e^\xi \bmod 1$ is extremely close to $\frac{1}{2}$. [*Integers* **2** (2002), A1, 1–13.]

63. (a) The result holds when $p_1 = \cdots = p_n = p$, because $a_{k-1}/a_k = (k/(n+1-k)) \times ((n-\mu)/\mu) \le (n-\mu)/(n+1-\mu) < 1$. It is also true by induction when $p_n = 0$ or 1. For the general case, consider the minimum of $a_k - a_{k-1}$ over all choices of $(p_1,\ldots,p_n)$ with $p_1+\cdots+p_n = \mu$: If $0 < p_1 < p_2 < 1$, let $p_1' = p_1-\delta$ and $p_2' = p_2+\delta$, and notice that $a_k' - a_{k-1}' = a_k - a_{k-1} + \delta(p_1-p_2-\delta)\alpha$ for some α depending only on $p_3, \ldots, p_n$. At a minimum point we must have $\alpha = 0$; thus we can choose δ so that either $p_1' = 0$ or $p_2' = 1$. The minimum can therefore be achieved when all p_j have one of three values $\{0,1,p\}$. But we have proved that $a_k - a_{k-1} > 0$ in such cases.

(b) Changing each p_j to $1-p_j$ changes μ to $n-\mu$ and a_k to a_{n-k}.

(c) No roots of $f(x)$ are positive. Hence $f(z)/f(1)$ has the form in (a) and (b).

(d) Let $C(f)$ be the number of sign changes in the sequence of coefficients of f; we want to show that $C((1-x)^2f) = 2$. In fact, $C((1-x)^mf) = m$ for all $m \ge 0$. For $C((1-x)^m) = m$, and $C((a+bx)f) \le C(f)$ when a and b are positive; hence $C((1-x)^mf) \le m$. And if $f(x)$ is any nonzero polynomial whatsoever, $C((1-x)f) > C(f)$; hence $C((1-x)^mf) \ge m$.

(e) Since $\sum_k \left[{n \atop k}\right]x^k = x(x+1)\ldots(x+n-1)$, part (c) applies directly with $\mu = H_n$. And for the polynomials $f_n(x) = \sum_k \left\{{n \atop k}\right\}x^k$, we can use part (c) with $\mu = \varpi_{n+1}/\varpi_n - 1$, if $f_n(x)$ has n real roots. The latter statement follows by induction because $f_{n+1}(x) = x(f_n(x)+f_n'(x))$: If $a > 0$ and if $f(x)$ has n real roots, so does the function $g(x) = e^{ax}f(x)$. And $g(x)\to 0$ as $x\to-\infty$; hence $g'(x) = e^{ax}(af(x)+f'(x))$ also has n real roots (namely, one at the far left, and $n-1$ between the roots of $g(x)$).

[See E. Laguerre, *J. de Math.* (3) **9** (1883), 99–146; W. Hoeffding, *Annals Math. Stat.* **27** (1956), 713–721; J. N. Darroch, *Annals Math. Stat.* **35** (1964), 1317–1321; J. Pitman, *J. Combinatorial Theory* **A77** (1997), 297–303.]

64. We need only use computer algebra to subtract $\ln\varpi_n$ from $\ln\varpi_{n-k}$.

65. It is ϖ_n^{-1} times the number of occurrences of k-blocks plus the number of occurrences of ordered pairs of k-blocks in the list of all set partitions, namely $\bigl(\binom{n}{k}\varpi_{n-k} + \binom{n}{k}\binom{n-k}{k}\varpi_{n-2k}\bigr)/\varpi_n$, minus the square of (49). Asymptotically, $(\xi^k/k!)(1+O(n^{4\epsilon-1}))$.

66. (The maximum of (48) when $n = 100$ is achieved for the partitions $7^1 6^2 5^4 4^6 3^7 2^6 1^4$ and $7^1 6^2 5^4 4^6 3^8 2^5 1^3$.)

67. The expected value of M^k is ϖ_{n+k}/ϖ_n. By (50), the mean is therefore $\varpi_{n+1}/\varpi_n = n/\xi + \xi/(2(\xi+1)^2) + O(n^{-1})$, and the variance is

$$\frac{\varpi_{n+2}}{\varpi_n} - \frac{\varpi_{n+1}^2}{\varpi_n^2} = \left(\frac{n}{\xi}\right)^2\left(1+\frac{\xi(2\xi+1)}{(\xi+1)^2 n} - 1 - \frac{\xi^2}{(\xi+1)^2 n} + O\left(\frac{1}{n^2}\right)\right) = \frac{n}{\xi(\xi+1)} + O(1).$$

68. The maximum number of nonzero components in all parts of a partition is $n = n_1 + \cdots + n_m$; it occurs if and only if all component parts are 0 or 1. The maximum level is also equal to n.

69. At the beginning of step M3, if $k > b$ and $l = r-1$, go to M5. In step M5, if $j = a$ and $(v_j - 1)(r-l) < u_j$, go to M6 instead of decreasing v_j.

70. (a) $\left|{n-1 \atop r-1}\right| + \left|{n-2 \atop r-1}\right| + \cdots + \left|{r-1 \atop r-1}\right|$, since $\left|{n-k \atop r-1}\right|$ contain the block $\{0,\ldots,0,1\}$ with k 0s. The total, also known as $p(n-1,1)$, is $p(n-1) + \cdots + p(1) + p(0)$.

(b) Exactly $N = {n-1 \brace r} + {n-2 \brace r-2}$ of the r-block partitions of $\{1,\ldots,n-1,n\}$ are the same if we interchange $n-1 \leftrightarrow n$. So the answer is $N + \frac{1}{2}({n \brace r} - N) = \frac{1}{2}({n \brace r} + N)$, which is also the number of restricted growth strings $a_1 \ldots a_n$ with $\max(a_1,\ldots,a_n) = r-1$ and $a_{n-1} \le a_n$. And the total is $\frac{1}{2}(\varpi_n + \varpi_{n-1} + \varpi_{n-2})$.

71. $\lfloor \frac{1}{2}(n_1+1)\ldots(n_m+1) - \frac{1}{2} \rfloor$, because there are $(n_1+1)\ldots(n_m+1)-2$ *compositions* into two parts, and half of those compositions fail to be in lexicographic order unless all n_j are even. (See exercise 7.2.1.4–31. Formulas for up to 5 parts have been worked out by E. M. Wright, *Proc. London Math. Soc.* (3) **11** (1961), 499–510.)

72. Yes. The following algorithm computes $a_{jk} = p(j,k)$ for $0 \le j,k \le n$ in $\Theta(n^4)$ steps: Start with $a_{jk} \leftarrow 1$ for all j and k. Then for $l = 0, 1, \ldots, n$ and $m = 0, 1, \ldots, n$ (in any order), if $l + m > 1$ set $a_{jk} \leftarrow a_{jk} + a_{(j-l)(k-m)}$ for $j = l, \ldots, n$ and $k = m, \ldots, n$ (in increasing order).

(See Table A-1. A similar method computes $p(n_1,\ldots,n_m)$ in $O(n_1\ldots n_m)^2$ steps. Cheema and Motzkin, in the cited paper, have derived the recurrence relation

$$n_1 p(n_1,\ldots,n_m) = \sum_{l=1}^{\infty} \sum_{k_1,\ldots,k_m \ge 0} k_1 p(n_1 - k_1 l, \ldots, n_m - k_m l),$$

but this interesting formula is helpful for computation only in certain cases.)

Table A-1

MULTIPARTITION NUMBERS

n	0	1	2	3	4	5	6
$p(0,n)$	1	1	2	3	5	7	11
$p(1,n)$	1	2	4	7	12	19	30
$p(2,n)$	2	4	9	16	29	47	77
$p(3,n)$	3	7	16	31	57	97	162
$p(4,n)$	5	12	29	57	109	189	323
$p(5,n)$	7	19	47	97	189	339	589

n	0	1	2	3	4	5
$P(0,n)$	1	2	9	66	712	10457
$P(1,n)$	1	4	26	249	3274	56135
$P(2,n)$	2	11	92	1075	16601	325269
$P(3,n)$	5	36	371	5133	91226	2014321
$P(4,n)$	15	135	1663	26683	537813	13241402
$P(5,n)$	52	566	8155	149410	3376696	91914202

73. Yes. Let $P(m,n) = p(1,\ldots,1,2,\ldots,2)$ when there are m 1s and n 2s; then $P(m,0) = \varpi_m$, and we can use the recurrence

$$2P(m,n+1) = P(m+2,n) + P(m+1,n) + \sum_k \binom{n}{k} P(m,k).$$

This recurrence can be proved by considering what happens when we replace a pair of x's in the multiset for $P(m,n+1)$ by two distinct elements x and x'. We get $2P(m,n+1)$ partitions, representing $P(m+2,n)$, except in the $P(m+1,n)$ cases where x and x' belong to the same block, or in $\binom{n}{k}P(m,n-k)$ cases where the blocks containing x and x' are identical and have k additional elements.

Notes: See Table A-1. Another recurrence, less useful for computation, is

$$P(m+1,n) = \sum_{j,k} \binom{n}{k}\binom{n-k+m}{j} P(j,k).$$

The sequence $P(0,n)$ was first investigated by E. K. Lloyd, *Proc. Cambridge Philos. Soc.* **103** (1988), 277–284, and by G. Labelle, *Discrete Math.* **217** (2000), 237–248, who computed it by completely different methods. Exercise 70(b) showed that $P(m,1) = (\varpi_m + \varpi_{m+1} + \varpi_{m+2})/2$; in general $P(m,n)$ can be written in the umbral notation $\varpi^m q_n(\varpi)$, where $q_n(x)$ is a polynomial of degree $2n$ defined by the generating function $\sum_{n=0}^{\infty} q_n(x) z^n/n! = \exp\bigl((e^z + (x+x^2)z - 1)/2\bigr)$. Thus, by exercise 31,

$$\sum_{n=0}^{\infty} P(m,n)\frac{z^n}{n!} = e^{(e^z-1)/2} \sum_{k=0}^{\infty} \frac{\varpi_{(2k+m+1)(k+m+1)}}{2^k}\frac{z^k}{k!}.$$

Labelle proved, as a special case of much more general results, that the number of partitions of $\{1,1,\ldots,n,n\}$ into exactly r blocks is

$$n!\,[x^r z^n]\, e^{-x+x^2(e^z-1)/2} \sum_{k=0}^{\infty} e^{zk(k+1)/2}\frac{x^k}{k!}.$$

75. The saddle point method yields $Ce^{An^{2/3}+Bn^{1/3}}/n^{55/36}$, where $A = 3\zeta(3)^{1/3}$, $B = \pi^2\zeta(3)^{-1/3}/2$, and $C = \zeta(3)^{19/36}(2\pi)^{-5/6}3^{-1/2}\exp(1/3 + B^2/4 + \zeta'(2)/(2\pi^2) - \gamma/12)$. [F. C. Auluck, *Proc. Cambridge Philos. Soc.* **49** (1953), 72–83; E. M. Wright, *American J. Math.* **80** (1958), 643–658.]

76. Using the fact that $p(n_1,n_2,n_3,\ldots) \ge p(n_1+n_2,n_3,\ldots)$, hence $P(m+2,n) \ge P(m,n+1)$, one can prove by induction that $P(m,n+1) \ge (m+n+1)P(m,n)$. Thus

$$2P(m,n) \le P(m+2,n-1) + P(m+1,n-1) + eP(m,n-1).$$

Iterating this inequality shows that $2^n P(0,n) = (\varpi^2+\varpi)^n + O(n(\varpi^2+\varpi)^{n-1}) = (n\varpi_{2n-1}+\varpi_{2n})(1+O((\log n)^3/n))$. (A more precise asymptotic formula can be obtained from the generating function in the answer to exercise 75.)

78. 3 3 3 3 2 1 0 0 0
1 0 0 0 2 2 3 2 0 (because the encoded partitions
2 2 1 0 0 2 1 0 2 must all be (000000000))
2 1 0 2 2 0 0 1 3

79. There are 432 such cycles. But they yield only 304 different cycles of set partitions, since different cycles might describe the same sequence of partitions. For example, (000012022332321) and (000012022112123) are partitionwise equivalent.

80. [See F. Chung, P. Diaconis, and R. Graham, *Discrete Mathematics* **110** (1992), 52–55.] Construct a digraph with ϖ_{n-1} vertices and ϖ_n arcs; each restricted growth string $a_1 \ldots a_n$ defines an arc from vertex $a_1 \ldots a_{n-1}$ to vertex $\rho(a_2 \ldots a_n)$, where ρ is the function of exercise 4. (For example, arc 01001213 runs from 0100121 to 0110203.) Every universal cycle defines an Eulerian trail in this digraph; conversely, every Eulerian trail can be used to define one or more universal sequences of restricted growth on the elements $\{0, 1, \ldots, n-1\}$.

An Eulerian trail exists by the method of Section 2.3.4.2, if we let the last exit from every nonzero vertex $a_1 \ldots a_{n-1}$ be through arc $a_1 \ldots a_{n-1} a_{n-1}$. The sequence might not be cyclic, however. For example, no universal cycle exists when $n < 4$; and when $n = 4$ the universal sequence 000012030110100222 defines a cycle of set partitions that does not correspond to any universal cycle.

The existence of a cycle can be proved for $n \geq 6$ if we start with an Eulerian trail that begins $0^n x y x^{n-3} u (uv)^{\lfloor (n-2)/2 \rfloor} u^{[n \text{ odd}]}$ for some distinct elements $\{u, v, x, y\}$. This pattern is possible if we alter the last exit of $0^k 1 2 1^{n-3-k}$ from $0^{k-1} 1 2 1^{n-2-k}$ to $0^{k-1} 1 2 1^{n-3-k} 2$ for $2 \leq k \leq n-4$, and let the last exits of 0121^{n-4} and $01^{n-3}2$ be respectively $010^{n-4}1$ and $0^{n-3}10$. Now if we choose numbers of the cycle *backwards*, thereby determining u and v, we can let x and y be the smallest elements distinct from $\{0, u, v\}$.

We can conclude in fact that the number of universal cycles having this extremely special type is huge — at least

$$\Bigl(\prod_{k=2}^{n-1} \bigl(k!\,(n-k)\bigr)^{\left\{ {n-1 \atop k} \right\}}\Bigr) \Big/ \bigl((n-1)!\,(n-2)^3 3^{2n-5} 2^2\bigr), \qquad \text{when } n \geq 6.$$

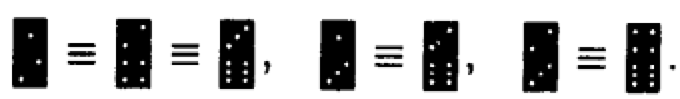
Yet none of them are known to be readily decodable. See below for the case $n = 5$.

81. Noting that $\varpi_5 = 52$, we use a universal cycle for $\{1, 2, 3, 4, 5\}$ in which the elements are 13 clubs, 13 diamonds, 13 hearts, 12 spades, and a joker. One such cycle, found by trial and error using Eulerian trails as in the previous answer, is

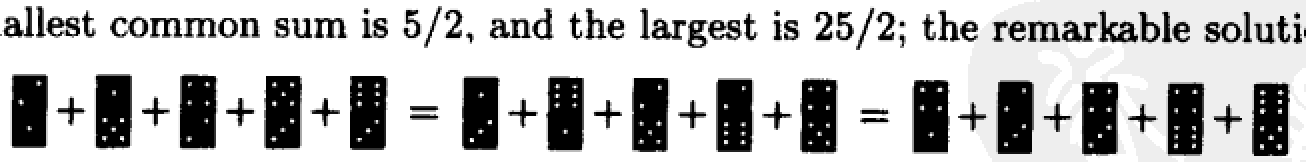
(♠♠♠♠♠♣♢♡J♣♡♢♡♠♣♣♢♡♣♢♣♡♢♢♣♡♣♡♣♠♣♣♢♣♢♢♣♢♠♠♢♡♡♡♠♡♡♡♠♠♢♢).

(In fact, there are essentially 114,056 such cycles if we branch to $a_k = a_{k-1}$ as a last resort and if we introduce the joker as soon as possible.) The trick still works with probability $\frac{47}{52}$ if we call the joker a spade.

82. There are 13644 solutions, although this number reduces to 1981 if we regard



The smallest common sum is 5/2, and the largest is 25/2; the remarkable solution



is one of only two essentially distinct ways to get the common sum 118/15. [This problem was posed by B. A. Kordemsky in *Matematicheskaiâ Smekalka* (1954); it is number 78 in the English translation, *The Moscow Puzzles* (1972).]

计算机程序设计艺术

第4卷 第3册

生成所有组合和分划

译 者 序

金秋华夏喜频传，神舟太空威尽现。在结束了第1卷第1册的翻译工作之后，我又立即投入了高德纳的第4卷第2、3册（现在已出版的即是这两册）的翻译工作。时间也从炎夏进入盛秋，而金秋时节的和风丽日，加上华夏大地频频传来的喜讯，给我的工作带来了强大的动力。

高德纳的这部巨著，始于1968年，而且很快，他就把前三卷完成了。人们原来以为，接下来出版的自然应该是第4卷了，然后顺理成章的是第5卷、第6卷，直到第7卷。然而，事实并非如此。近40年过去了，他对第1卷做过多次修改并再版。第2、3卷，也都分别至少做了一次修改并再版。而且如他所宣称的那样，每次修改，不论是哪一卷，都涉及几乎所有页面。工作量之大，作者为此付出之匠心，实在令人叹为观止。在此期间，他又发表了大量的论著，特别是完成了METAFONT和T_EX，为计算机排版技术做出巨大贡献。完成了著名的《公理与外壳》和《具体数学》等著述。但是对于第4卷，却迟迟未闻有任何信息。只是后来才听说，第4卷将分为三个分卷出版，但也未见任何一卷问世。直到此前，人们才终于见到我现在译出的这第2册和第3册。至于后边还有多少分册，仍不得而知。因此，它的出场，真可谓是"犹抱琵琶半遮面"了。

但是，也正因为是经过几十年才出来的东西，因此它就不同凡响，是名副其实的专著和经典之作。作者在前言写道："我怀着非常喜悦的心情来写这部分内容，就像许多年前写第2卷时我感觉到的激动那样。如同在第2卷中那样，在那里我高兴地看到，初等概率论和数论的基本原理很自然地出现在关于随机数生成和算术运算的研究中。而在准备7.2.1节的过程中我了解到，当我们研究组合生成的算法时，初等组合学的基本原理也自然地出现，并且具有高度启发性。"他感到，这里有着许多美丽的故事，等待他来讲述。

他说的确实是真的。我想，在这方面，至少有三个方面值得我们从中学习。我们不仅要学习其中的这些内容，还要学习作者的治学精神。

1. 首先是关于生成基本的组合模式的问题。

乍一看去，无论是生成所有n元组，还是生成排列，这些都不过是很简单的问题，并没有什么高深的理论，一说谁都知道，属于已经解决了的问题，再无什么遗留问题需要研究。然而，经过作者点拨，才发现事实上问题绝非我们所想像的那么简单。作者在提出对于这个问题的研究动机时指出，没有一个聪明人愿意通过以几千张纸来打印出{0，1，…，9}的所有10!=3 628 800个排列的清单，甚至也没有人愿意在一个计算机文件中把它们全写出来。但是，我们需要的是，当确实用得着它们时，在顷刻之间就以某种数据结构把它们提供出来，使得一个程序可

以一次一个地来考察每个排列。正是出于这个目的，人们就还要考虑这样一个问题。而且就这一点而言，它确实是远未解决的问题。

作者在研究中所体现的严谨性表现在，不仅考虑二进制，还考虑十进制和混合进制；不仅考虑集合的元素，还考虑集合的子集、多重集合等。因此，它就体现为既有问题的深度，又有问题的广度。

2. 关于组合模式同一些问题的关系

前边所说的深度，指的是如何以最快速度和最少内存访问来生成所需元组或排列。这样一个问题，竟是同图论有关，同哈密顿通路或循环有关，也同树形的遍历有关。因而作者在这里向我们揭示了这种关系，使我们懂得原来组合模式的问题还有这种背景，或者说可以从那些问题的求解中获得解决问题的理论基础。

记得几十年前当人工智能在我国掀起一股热潮时，一些人曾经从人工智能的角度研究过九链环（又称九龙环）的问题。然而，就译者所知，并没有任何一个中国学者，深入地去研究九链环问题的历史以及它的发展过程，更没有人去探讨它和其他问题的联系。但在这本书里，作者告诉我们，早在1872年法国人刘易斯·格罗斯在一本题为《步行理论》的小册子中，就揭示了九链环同二进制数之间的关系，比如我们以0表示环与杆分离的状态，而以1表示环在杆上的状态，则原来环锁在杆上的状态就是9个1的状态。问题是要通过从右边开始（或从左边开始），每次改换一位，最终使9个1变成为9个0。因此作者说，格罗斯才是格雷二进制码真正的发明者，也是九链环问题真正透彻的最初研究者。

所以，我们对于几十年前在我国人工智能学界曾经有过的研究九链环的热潮，不得不作一些反思，那就是我们对于问题的研究，是否确实应更着重于深度和广度。为什么不是我们，而是外国人来发现原本是我们提出的问题的理论基础，从而对它给出真正的解呢？

高德纳还列举了组合模式生成与欧洲教堂的洪钟鸣响模式的联系，揭示了各种钟鸣模式与生成排列的关系，这同样给人以巨大的启示。

3. 关于组合模式同字母算术或密码数学的关系

在国外，有些人或者生来就喜欢标新立异、独出心裁，或者有闲情逸致来这样做。因而很早以前就有人提出字母算术的概念。如亨利·厄思尼斯特·达德尼在1924年就提出这样一个著名问题：如果每个字母代表着不同的十进制数字，问要使

$$\begin{array}{r} \text{SEND} \\ +\text{MORE} \\ \hline \text{MONEY} \end{array}$$

表示一个正确的求和，则每个字母应当分别表示什么数？

这看似纯粹游戏的题目，却引发了人们的思考。假若要传送的是数字信息，用字母来对它们加密，这不就成了密码了吗？因而在1931年，西蒙·瓦特利宽特就给它起了另一个名称“密码数学”。

在他们的开创下，字母算术或密码数学就蓬勃发展起来。开头，人们关注于具有惟一解的问题。而后，人们开始考虑它们能有的各种解，乃至研究它有多少解。有惟一解的情况，称为纯的字母算术或纯的密码数学。而有的问题，不仅是字母上有意义，而且以数值解代入后仍正确。例如

VIOLIN+VIOLIN+VIOLA=TRIO+SUNATA

（小提琴+小提琴+中提琴=三重合奏）

ZEROES + ONES = BINARY

（诸0+诸1=二进制）

COUPLE+COUPLE=QUARTET

（两个+两个=四个）

等等。这些通过加法给出的字母算术，称为加法性字母算术。还可以有乘法性字母算术，例如

TWO × TWO=SQUARE

PI × R × R=AREA

等等。依照这些问题，我们也可以提出，如何代入数字，使

工业化+农业化+科学化=现代化

和

立党为公+执政为民=为人民服务

以及

更快 × 更高 × 更强=奥林匹克精神

成立?

除了字母算术外，也还有像在0与9这些数字之间，如何加上适当的加减乘除号，使得结果成为一个数，如100。在这方面的研究，不仅仅在于给出正确答案，还要研究它们可以有多少个解，如对于123456789，可以有12种方法来插入+和－，使其和为100，如100=1+23－4+5+6+78－9=123－45－67+89=－1+2－3+4+5+6+78+9。又如，在12345678987654321中，插入+和－的一个方式为100=－1234－5－6+7898－7－6543－2－1。

为了鼓励这方面的研究，国外出版了多种杂志，出版的专门书籍也不少。我们所知道的杂志就有*J. Recreational Math*（娱乐数学杂志）、*Recreational Math. Magazine*（娱乐数学期刊）等。这样，也就使这方面的研究长盛不衰，一直延续下来。它也推动了人们特别是青少年对于科学的兴趣和追求。

我们说，现代化的科学技术源自于基础理论。有了基础理论的深厚功底和深刻探索，才带来了现代科学技术的发展。最雄辩的例子是爱因斯坦的理论，乃是现代科学技术用之不竭的思想源泉。有一项研究表明，现代科学技术中直接利用爱因斯坦的理论成果的就达2300多项。因此我们可以斩钉截铁地说，没有基础数

学的研究成果，也就没有今天的计算机科学和经济学等的辉煌成果。我国今天取得的航天领域的成就，也是我们在基础理论和综合科学技术方面的成就。然而，在我国仍然存在忽视基础理论研究的倾向，或者更确切地说，我们在基础理论研究上的投入和从事这方面研究的人数，同我们这样一个大国，是不大相称的。为了我国今后科学的持续发展，特别是为了使我国早日成为不仅是经济强国、军事强国，更是科学强国，我们希望上至党中央国务院，下至各地方党委政府，还有我国人民，都能从振兴中华，从民族的未来发展上重视这个问题。如果我们能引导青少年一代，首先重视自己的思想、素质的成长，然后是去专注于包括娱乐数学、娱乐物理这种难题的研究，从而激发对科学本身的兴趣，而不是沉溺于网上那些无聊的活动，那我们民族的科学素质就会大为改观。而到那时，科学大家的出现，诺贝尔奖、图灵奖的获得，都是指日可待的。

因此，我想，我们出版高德纳的书，其深远意义也应包括这些，而不仅仅在于计算机的那些算法本身上。

我们等待着祖国科学辉煌的明天。

译　者

2006年2月于广西梧州

前　言

在我的第1版的前言中，
我请求读者不要去注意错误，
我现在希望我不曾这样做，
而对于忽略我请求的少数读者我谨致谢意。
——施图亚特·苏德兰德，*The International Dictionary of Psychology* (1996)

本册是《计算机程序设计艺术 第4卷 组合算法》的第3册，如同在第1卷第1册的前言中所说明的那样，我以这种预备形式来传播这个材料，因为我知道，完成第4卷的任务将花费许多年。我急不可待地让人们开始来读迄今我已写成的材料，并且提供有价值的反馈。

从整卷的情况来看，这一册包含关于组合查找的这一冗长一章的7.2.1.3节、7.2.1.4节和7.2.1.5节。假设我能维持自己的健康状态，第7章将最终归入三卷中(即卷4A、4B和4C)。它将以关于图论的一个简短复习开始，并且强调在斯坦福图库中一些重要图形的某些要点，从这出发我将引出许多例子。然后是7.1节，它涉及按位操作并讨论有关布尔函数的一些算法。7.2节是关于生成所有可能性的，而且它由7.2.1节“生成基本的组合模式”开始。关于生成n元组的各种有用方法的细节在7.2.1.1节中，而在7.2.1.2节中讨论排列的生成。这就设置了本册的主要内容的框架，即7.2.1.3节(把这些思想推广到一次从n个事物中取t个组合)；7.2.1.4节(关于一个整数的分划)；以及7.2.1.5节(关于一个集合的分划)。然后是在第4册中给出7.2.1.6节(关于树形)和7.2.1.7节(关于组合生成的历史)。7.2.2节将一般地讨论回溯。因此如果一切进展顺利，它将继续下去。作为当前想像的整个第7章的轮廓，可参见封底所给出的taocp网页。

我怀着非常喜悦的心情来写这部分内容，就像许多年前写第2卷时我感觉到的激动那样。如同在第2卷中那样，在那里我高兴地看到，初等概率论和数论的基本原理很自然地出现在关于随机数生成和算术运算的研究中。而在准备7.2.1节的过程中我了解到，当我们研究组合生成的算法时，初等组合学的基本原理也自然地出现，并且具有高度启发性。因此，我再次发现，一个美丽的故事就在那里等候着被讲述。

例如，在本册中，我们发现由组合形成的许多美丽的模式，有的有重复，有的没有重复，而且还知道，它们如何同外部组合学的著名定理相关联。于是，这就成了我来讲述分划的不同寻常的故事的机会了；确实，关于分划的理论是所有数学中最为优美的章节之一，而且在7.2.1.5节中，由查·圣·珀西(C.S.Peirce)所发现的鲜为人知的数的三角形证明，是简化和统一化集合分划的研究，是另一个重要的课题。沿着这一方面，我还对算法分析中极为重要的两个数学技术进行了

剖析：即泊松(Poisson)的求和公式和强有力的马鞍点方法。如同在以前的分册中那样，也还有游戏和难题。

我原来打算简单介绍这些课题，但当我看到，这些想法对于一般的组合研究是何等基本时，我就懂得了，除非我十分透彻地讲解这些基础，否则我绝不会感到快乐。因此我就尽最大努力来构筑理论和实践的思想的一个坚实基础，它们将支持许多类型的可靠的超级结构。

我要向弗朗克·拉斯基表示感谢，感谢他大胆地向大学生们讲授本书的初稿并向我讲述他的授课经验。还有许多其他读者也帮助我检查最初的一些草稿，我还要特别感谢乔治·克莱门特斯(George Clements)和斯万特·简森(Svante Janson)的深刻透彻的评论。

我还将高兴地为头一个向我报告本分册中的每一个错误的人支付2.56美元的酬金，无论这个错误是印刷上的、技术上的还是历史上的。对于我忘了放进索引的那些条目，也有同样的酬金。而对于正文的有价值的改进建议，每一个将获32美分的奖励。(而且，如果你找到一个习题更好的解答，我将通过在最后出版的书中公布你的名字来实际上以永久的荣誉来奖励你，而不仅仅是钱。)

这里所使用的记号以及未予说明的记号可以在第1、2或3卷末尾的记号索引中找到。这些索引指向可以获得进一步信息的那些位置。当然，某天第4卷也将包含它自己的记号索引。

在《计算机程序设计艺术》所有未来版本中的机器语言的例子，都将基于MMIX计算机，它在第1卷第1册中介绍过了。

在以下的篇幅中，对于尚未完成的内容的交叉参考有时以“00”来表示；这个不可能的值是以后要予以提供的实际号码的位置表示。

祝阅读愉快！

D.E.K.

加利福尼亚，斯坦福

2005年6月

第7章

组 合 查 找

说明：本章的开始部分将出现于第4卷第0册和第4卷第1册中，计划于2006年和2007年出版。

7.2 生成所有可能性

7.2.1 生成基本的组合模式

在这一节中，我们的目标是研究在某个组合世界中，跑遍所有可能性的方法，因为我们经常面对这样一些问题，其中有必要并希望对于所有情况作穷尽的考察……

7.2.1.1 生成所有n元组

我们由简单的开始，首先考虑如何跑遍由n个二进制数字组成的所有2^n个字符串……

7.2.1.2 生成所有排列

在n元组之后，对于组合生成，几乎每个人都希望列出的下一个最重要的项目是，访问某个给定集合或多重集合的所有排列……

关于7.2.1.1节和7.2.1.2节的完整的正文，可在2005年2月第一次出版的第4卷第
0 2册中找到。

7.2.1.3 生成所有组合

组合数学通常被描述为是关于"排列、组合等的研究"，因此我们现在把注意力转向组合。*n个事物一次取t个的组合*，通常简单地称作n个事物的t个组合，是从大小为n的一个给定集合中选择大小为t的一个子集的方法，从等式1.2.6-(2)可知，恰好有$\binom{n}{t}$个方法来做这件事；而且在3.4.2节中我们已学习了如何随机地选择t个组合。

在n个对象中选择t个等价于选择不选的$n-t$个。贯穿于这个讨论，通过令

$$n=s+t \tag{1}$$

我们强调这一对称性。而且我们通常将把n个事物的t组合称作"(s,t)组合"。因此，一个(s,t)组合是把$s+t$个对象分成大小为s和t的两个集合的方法。

> 如果我问从25中取21有多少种组合，实际上我在问可以有多少种取4个的组合。因为取21个的方法就如同取剩下4个的方法一样多。
>
> ——奥古斯托斯 · 德 · 摩根(Augustus de Morgan),
> *An Essay on Probabilities* (1838)

表示(s,t)组合有两种主要的方法，我们可以列出已被选择的元素$c_t \cdots c_2 c_1$，或者可以以二进制串$a_{n-1} \cdots a_1 a_0$来进行工作。对于它

$$a_{n-1}+\cdots+a_1+a_0=t \tag{2}$$

这后一表示有s个0和t个1，对应于未被选择或被选择的元素。如果我们令诸元素为集合$\{0,1,\cdots,n-1\}$的成员，而且如果我们以*递减*顺序来列出它们：

$$n>c_t>\cdots>c_2>c_1\geqslant 0 \tag{3}$$

则表示$c_t \cdots c_2 c_1$趋向于最好地工作。二进制记号可很好地联系这两种表示。因为项目列表$c_t \cdots c_2 c_1$对应于和数

$$2^{c_t} + \cdots + 2^{c_2} + 2^{c_1} = \sum_{k=0}^{n-1} a_k 2^k = (a_{n-1} \cdots a_1 a_0)_2 \qquad (4)$$ 1

当然我们也可以列出在$a_{n-1} \cdots a_1 a_0$中为0的位置$b_s \cdots b_2 b_1$，其中

$$n > b_s > \cdots > b_2 > b_1 \geqslant 0 \qquad (5)$$

组合是重要的，不仅仅是因为子集在数学中随处可见，而且是因为它们等价于许多其他的配置。例如，每个(s, t)组合对应于从$s+1$个事物中每次取t个且允许重复的组合，这也称为一个多重组合(multicombination)，即是整数$d_t \cdots d_2 d_1$的一个序列且

$$s \geqslant d_t \geqslant \cdots \geqslant d_2 \geqslant d_1 \geqslant 0 \qquad (6)$$

一个原因是$d_t \cdots d_2 d_1$解(6)当且仅当$c_t \cdots c_2 c_1$解(3)，其中

$$c_t = d_t + t - 1,\ \cdots,\ c_2 = d_2 + 1,\ c_1 = d_1 \qquad (7)$$

(参见习题1.2.6－60。)还有另外一个把带重复的组合同通常的组合关联起来的有用方法。它是由索罗门·戈罗姆布(Solomon Golomb)[*AMM* 75(1968), 530-531]提出的，即定义

$$e_j = \begin{cases} c_j & \text{如果} c_j \leqslant s \\ e_{c_j - s} & \text{如果} c_j > s \end{cases} \qquad (8)$$

在此形式下，数$e_t \cdots e_1$不必以递减顺序出现，但多重集合$\{e_1, e_2, \cdots, e_t\}$等价于$\{c_1, c_2, \cdots, c_t\}$，当且仅当$\{e_1, e_2, \cdots, e_t\}$是一个集合。(参见表1和习题1。)

一个(s, t)组合也等价于$n+1$到$t+1$部分的一个合成，即一个有序的和

$$n+1 = p_t + \cdots + p_1 + p_0,\ \text{其中} p_t, \cdots, p_1, p_0 \geqslant 1 \qquad (9)$$

同(3)的联系现在成为

$$p_t = n - c_t,\ p_{t-1} = c_t - c_{t-1},\ \cdots,\ p_1 = c_2 - c_1,\ p_0 = c_1 + 1 \qquad (10)$$

等价地，如果$q_j = p_j - 1$，我们有s到$t+1$非负部分的一个合成

$$s = q_t + \cdots + q_1 + q_0 \qquad \text{其中} q_t, \cdots, q_1, q_0 \geqslant 0 \qquad (11)$$

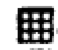
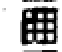
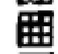
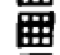
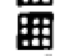
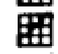
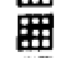
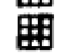
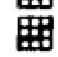
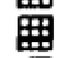
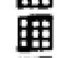
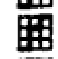
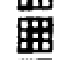
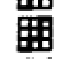
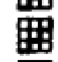
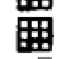
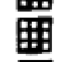
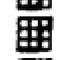
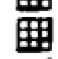
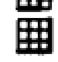
这是通过设置

$$q_t = s - d_t,\ q_{t-1} = d_t - d_{t-1},\ \cdots,\ q_1 = d_2 - d_1,\ q_0 = d_1 \qquad (12)$$

与(6)相关的。

其次，容易看出，一个(s, t)组合等价于一个$s \times t$的栅格从角到角的长度为$s+t$的一条通路。因为这样一条通路包含s个垂直的步骤和t个水平的步骤。因此，至少可以有8个不同的形式来研究组合。表1示出，在$s=t=3$的情况下所有$\binom{6}{3}=20$种可能性。

表1 (3, 3)组合及它们的等价物

$a_5a_4a_3a_2a_1a_0$	$b_3b_2b_1$	$c_3c_2c_1$	$d_3d_2d_1$	$e_3e_2e_1$	$p_3p_2p_1p_0$	$q_3q_2q_1q_0$	path
000111	543	210	000	210	4111	3000	
001011	542	310	100	310	3211	2100	
001101	541	320	110	320	3121	2010	
001110	540	321	111	321	3112	2001	
010011	532	410	200	010	2311	1200	
010101	531	420	210	020	2221	1110	
010110	530	421	211	121	2212	1101	
011001	521	430	220	030	2131	1020	
011010	520	431	221	131	2122	1011	
011100	510	432	222	232	2113	1002	
100011	432	510	300	110	1411	0300	
100101	431	520	310	220	1321	0210	
100110	430	521	311	221	1312	0201	
101001	421	530	320	330	1231	0120	
101010	420	531	321	331	1222	0111	
101100	410	532	322	332	1213	0102	
110001	321	540	330	000	1141	0030	
110010	320	541	331	111	1132	0021	
110100	310	542	332	222	1123	0012	
111000	210	543	333	333	1114	0003	

乍一看，组合的这些近亲可能使人手足无措，但它们的大多数都可由二进制表示$a_{n-1}\cdots a_1a_0$直接理解。例如，考虑“随机”的二进位串

2 $$a_{23}\cdots a_1a_0=011001001000011111101101 \qquad (13)$$

它有$s=11$个0和$t=13$个1，因此$n=24$。对偶的组合$b_s\cdots b_1$列出0的位置，即：

23 20 19 17 16 14 13 12 11 4 1

因为最左的位置是$n-1$而最右的位置是0。主要的组合$c_t\cdots c_1$列出1的位置，即：

22 21 18 15 10 9 8 7 6 5 3 2 0

对应的多重组合$d_t\cdots d_1$列出在每个1右边0的个数：

10 10 8 6 2 2 2 2 2 2 1 1 0

如果我们把另外的1想像成在左边和右边，则合成$p_t\cdots p_0$列出相连续的1之间的距离：

2 1 3 3 5 1 1 1 1 1 2 1 2 1

而非负的合成$q_t\cdots q_0$计算由1表示的“篱笆位置”之间有多少个0：

1 0 2 2 4 0 0 0 0 0 1 0 1 0

于是我们有

$$a_{n-1}\cdots a_1a_0=0^{q_t}10^{q_{t-1}}1\cdots10^{q_1}10^{q_0} \qquad (14)$$

3 表1中的通路也有一个简单的解释(参见习题2)。

词典顺序的生成。表1以词典顺序示出组合$a_{n-1}\cdots a_1 a_0$和$c_t\cdots c_1$，它也是$d_t\cdots d_1$的词典顺序。注意，对偶组合$b_s\cdots b_1$和对应的合成$p_t\cdots p_0$，$q_t\cdots q_0$然后以颠倒的词典顺序出现。

词典顺序通常提出生成组合式配置的最方便方式。确实，算法7.2.1.2L已经以$a_{n-1}\cdots a_1 a_0$的形式解决了组合的问题，因为在位串形式下的(s,t)组合和多重集合$\{s\cdot 0, t\cdot 1\}$的排列相同。当把它应用于这个特殊情况时，该通用算法可以用一些明显的方式加以流水线化。(也参见习题7.1－00，它给出7个按位操作的一个值得注意的序列，它将把任何给定的二进制数$(a_{n-1}\cdots a_1 a_0)$转换成为按词典顺序的下一个t组合，假若n不超过计算机字长的话。)

然而，且让我们专注于在其他主要形式$c_t\cdots c_2 c_1$之下生成组合的问题。这一形式更为直接地同通常需要组合的方式有关，而且当同n比较，t较小时，它也比位串更紧凑。首先我们应当记住，当t很小时，一个简单的嵌套循环将可很漂亮地完成这件事。例如，当$t=3$时，下列指令足矣:

For $c_3=2,3,\cdots,n-1$(按此顺序)做以下;
　For $c_2=1,2,\cdots,c_3-1$(按此顺序)做以下;　　(15)
　　For $c_1=0,1,\cdots,c_2-1$(按此顺序)做以下;
　　　访问组合$c_3c_2c_1$

(参见7.2.1.1-(3)的类似情况。)

另一方面，当t是变量或不那么小时，通过遵循在算法7.2.1.2L之后所讨论的一般的方法，我们可以在词典顺序下生成组合，即来找出可被增加的最右元素c_j，而后设置随继的元素$c_{j-1}\cdots c_1$成为它们最小可能的值。

算法L(词典顺序的组合)　给定$n\geqslant t\geqslant 0$，这个算法生成n个数$\{0,1,\cdots,n-1\}$的所有t组合$c_t\cdots c_2c_1$。附加的变量c_{t+1}和c_{t+2}被用作哨兵。

L1. [初始化。] 对于$1\leqslant j\leqslant t$，置$c_j\leftarrow j-1$；并置$c_{t+1}\leftarrow n$和$c_{t+2}\leftarrow 0$。

L2. [访问。] 访问组合$c_t\cdots c_2c_1$。

L3. [求j。] 置$j\leftarrow 1$，而后当$c_j+1=c_{j+1}$时置$c_j\leftarrow j-1$和$j\leftarrow j+1$；重复直到$c_j+1\neq c_{j+1}$为止。

L4. [完成了吗？] 如果$j>t$则结束本算法。

L5. [增加c_j。] 置$c_j\leftarrow c_j+1$并返回L2。 ▮

不难分析本算法的运行时间。在访问使$c_{j+1}=c_1+j$的一个组合之后，步骤L3设置$c_j\leftarrow j-1$，因而这样的组合的个数为不等式

$$n>c_t>\cdots>c_{j+1}\geqslant j \qquad (16)$$

4

的解的个数。但这个公式等价于$n-j$个对象$\{n-1,\cdots,j\}$的$(t-j)$组合，所以赋值$c_j\leftarrow j-1$恰好出现$\binom{n-j}{t-j}$次。对于$1\leqslant j\leqslant t$进行求和告诉我们，在步骤L3中的循环总共被执行了

$$\binom{n-1}{t-1}+\binom{n-2}{t-2}+\cdots+\binom{n-t}{0}=\binom{n-1}{s}+\binom{n-2}{s}+\cdots+\binom{s}{s}=\binom{n}{s+1} \qquad (17)$$

次，或者平均每次访问要做

$$\binom{n}{s+1}\bigg/\binom{n}{t}=\frac{n!}{(s+1)!(t-1)!}\bigg/\frac{n!}{s!t!}=\frac{t}{s+1} \qquad (18)$$

次，当$t\le s$时这个比例很小，因此在这样一些情况下算法L十分有效。

但当t接近于n而s很小时，量$t/(s+1)$可以令人难以置信地大。其实，当c_j已经等于$j-1$时，算法L有时不必要地设置$c_j \leftarrow j-1$。进一步的仔细分析还揭示，我们不必总是去查找在步骤L4和L5中需要的下标j，因为j的正确值通常可以从刚刚采取的动作中预测出来。例如，在我们增加c_4的值并重置$c_3c_2c_1$为它们开始的值210后，下一个组合不可避免地将增加c_3。这些发现就导致这个算法的一个调整版本：

算法T(词典顺序的组合) 这个算法和算法L类似，但是更快些。为方便起见，也假定$t<n$。

T1. [初始化。] 对于$1\le j\le t$，置$c_j \leftarrow j-1$；然后置$c_{t+1}\leftarrow n$和$c_{t+2}\leftarrow 0$，以及$j\leftarrow t$。

T2. [访问。] (这时j是使得$c_{j+1}>j$的最小下标。)访问组合$c_t\cdots c_2c_1$，然后如果$j>0$，则置$x\leftarrow j$并转到步骤T6。

T3. [是容易情况吗？] 如果$c_1+1<c_2$，则置$c_1\leftarrow c_1+1$并返回T2，否则置$j\leftarrow 2$。

T4. [求j。] 置$c_{j-1}\leftarrow j-2$且$x\leftarrow c_j+1$。如果$x=c_{j+1}$，则置$j\leftarrow j+1$，并重复这个步骤直到$x\ne c_{j+1}$为止。

T5. [完成了吗？] 如果$j>t$则结束此算法。

T6. [增加c_j。] 置$c_j\leftarrow x$，$j\leftarrow j-1$，并返回T2。 ▌

现在在步骤T2中$j=0$当且仅当$c_1>0$，所以在步骤T4中的赋值绝不多余。习题6对算法T进行了完备的分析。

注意仅在初始化步骤L1和T1中（而不在算法L和T的主要部分中）才出现参数n，因此我们可以把这个过程当作是生成一个无穷表的头$\binom{n}{t}$个组合的过程，它仅依赖于t。由于对于$n+1$个事物的t组合的表，在我们的约定之下，以n个事物的表开始，所以出现这个简化。由于这个原因，我们对于递减序列$c_t\cdots c_1$已经使用词典顺序，而不是对递增序列$c_1\cdots c_t$进行工作。

德里克·莱默(Derrick Lehmer)注意到算法L和T的另一个令人高兴的性质[埃·福·贝肯巴赫(E. F. Beckenbach)编，*Applied Combinatorial Mathematics* (1964)，
5 27-30]：

定理L 在恰好访问了

$$\binom{c_t}{t}+\cdots+\binom{c_2}{2}+\binom{c_1}{1} \qquad (19)$$

个其他的组合之后，组合$c_t \cdots c_2 c_1$被访问。

证明 对于$t \geqslant j > k$和$c_k' < c_k$，有$\binom{c_k}{k}$个组合$c_t' \cdots c_2' c_1'$且$c_j' = c_j$，即$c_t \cdots c_{k+1}$后边接着有$\{0, \cdots, c_k - 1\}$的k组合。 ▌

例如，当t=3时，对应于表1中组合$c_3 c_2 c_1$的数

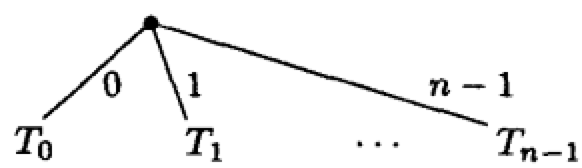
$$\binom{2}{3}+\binom{1}{2}+\binom{0}{1},\binom{3}{3}+\binom{1}{2}+\binom{0}{1},\binom{3}{3}+\binom{2}{2}+\binom{0}{1},\cdots,\binom{5}{3}+\binom{4}{2}+\binom{3}{1}$$

只不过跑遍序列0, 1, 2, ⋯, 19。定理L给了我们理解度数为t的组合数系的一个很好的方法，它把每个非负整数惟一地表示成以下形式：

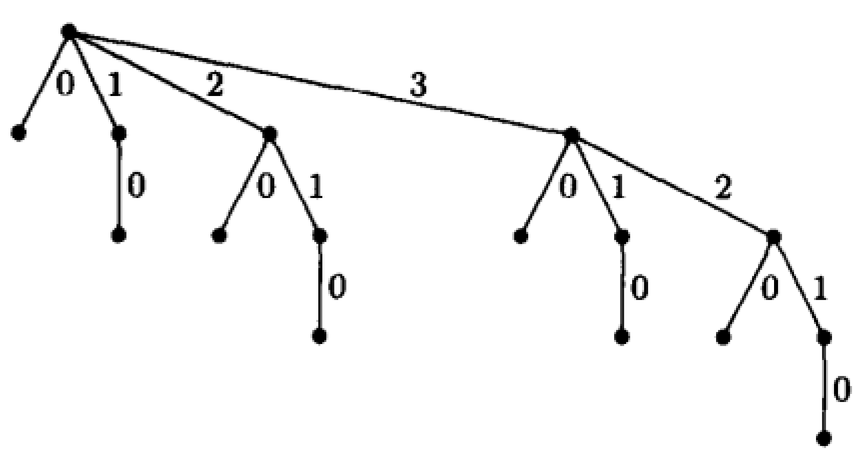
$$N = \binom{n_t}{t} + \cdots + \binom{n_2}{2} + \binom{n_1}{1}, \quad n_t > \cdots > n_2 > n_1 \geqslant 0 \qquad (20)$$

[见欧内斯托·帕斯卡(Ernesto Pascal)，*Giornale di Mathematiche* 25 (1887)，45-49。]

二项式树。对于n>0，由

$T_0 = \bullet$， $T_n =$  (21)

定义的树系T_n在若干重要的范畴中出现，并且把组合生成进一步展现出来。例如，T_4是

 (22)

而T_5则作为本书第1卷的卷首插画，以更艺术的形式呈现给大家。

注意，除开对于T_{n-1}附加的一个副本之外，T_n和T_{n-1}是类似的；因此T_n总共有2^n个节点。其次，在第t级中节点的个数是二项式系数$\binom{n}{t}$；这个事实解释了"二项式树"这个名称。确实，在从根开始到第t级的每个节点处的通路上，所遇到的标号序列定义一个组合$c_t \cdots c_1$，而且从左到右所有组合以词典顺序出现。因此算法L和T可以当作遍历二项式树T_n的第t级节点的步骤。 6

在(21)中令$n \to \infty$，就得到无穷的二项式树T_∞。这个树的根有无穷多分支，但除了在第0级的整个根之外，每个节点都是一个有限二项式子树的根。在T_∞的第t级上，所有可能的t组合都以词典顺序出现。

下面通过考虑装满一个帆布背包的所有可能的方式，来进一步熟悉二项式树。更精确地说，假设我们有n件物品，它们分别花去容量的$w_{n-1},\cdots,w_1,w_0$个单位，其中

$$w_{n-1}\geqslant\cdots\geqslant w_1\geqslant w_0 \tag{23}$$

我们要生成所有的二进制向量$a_{n-1}\cdots a_1 a_0$，使得

$$a\cdot w=a_{n-1}w_{n-1}+\cdots+a_1w_1+a_0w_0\leqslant N \tag{24}$$

其中N是一个帆布背包总的容积。等价地，我们要求$\{0, 1, \cdots, n-1\}$的所有子集C，使得$w(C)=\sum_{c\in C}w_c\leqslant N$。这样的子集将称作*能行的*（feasible）。由上边(3)的约定，我们对这些下标编上不同的号码，因为在这个问题中t是变量。

每一个能行的子集对应于T_n的一个节点，我们的目标是访问每一个能行的节点。显然，每一个能行节点的双亲都是能行的，如果有的话，它的左兄弟也是能行的。因此一个简单的树的探察过程有效地工作：

算法F（*填满一个帆布布袋*） 给定$w_{n-1},\cdots,w_1,w_0$和N，这个算法生成填满一个帆布布袋的所有能行的方式$c_1\cdots c_t$。对于$1\leqslant j<n$，我们令$\delta_j=w_j-w_{j-1}$。

F1. [初始化。] 置$t\leftarrow0$，$c_0\leftarrow n$，以及$r\leftarrow N$。

F2. [访问。] 访问组合$c_1\cdots c_t$，它使用容积的$N-r$个单位。

F3. [尝试增加w_0。]如果$c_t>0$且$r\geqslant w_0$，置$t\leftarrow t+1$，$c_t\leftarrow0$，$r\leftarrow r-w_0$，并返回F2。

F4. [尝试增加c_t。] 如果$t=0$，则结束。否则，如果$c_{t-1}>c_t+1$且$r\geqslant\delta_{c_t+1}$，则置$c_t\leftarrow c_t+1$，$r\leftarrow r+\delta_{c_t}$，并返回F2。

F5. [删去c_t。] 置$r\leftarrow r+w_{c_t}$，$t\leftarrow t-1$，并返回F4。 ▮

注意本算法含蓄地以前根顺序来访问T_n的节点，并跳过非能行的子树。在这个过程已经在其位置使用元素$c-1$考察了所有可能性之后，如果能行，就把一个元素$c>0$放进帆布布袋中。运行时间同所访问的能行组合的个数成比例。（参见习题20。）

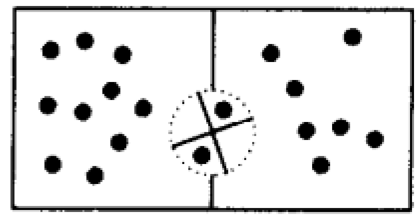
顺便指出，运筹学中的经典“背包问题”是不同的，它要求的是使得$v(C)=\sum_{c\in C}v(c)$为极大的一个能行子集C。其中对于每件物品c都赋予一个值$v(c)$。算法F不是解决这个问题的一个特别好的方法，因为它通常考虑可以被排除掉的一些情况。例如，如果C和C'都是$\{1, \cdots, n-1\}$的子集且$w(C)\leqslant w(C')\leqslant N-w_0$，$v(C)\geqslant v(C')$，算法F将考察$C\cup\{0\}$和$C'\cup\{0\}$，但后一个子集将绝不改进极大值。后边我们将考虑关于经典的背包问题的一些方法；算法F只适用于所有能行的可能性相关的那些

7 情况。

对于组合的格雷码。代替仅仅生成所有组合，我们通常更喜欢以这样一种方式来访问它们，即通过只对它的前驱做一些小的改变就得到每一个组合。

例如，我们可以看一看尼珍休斯(Nijenhuis)和威尔弗(Wilf)称之为“转动门算法”的一个算法：想像分别包含s个人和t个人的两个房间，在两者之间有一扇转动的门，每当一个人进入对面的门时，则另外一个人出来。我们能否设计一个移动

序列使得每个(s,t)组合恰好出现一次(参见右图)。



回答是肯定的，而且事实上存在有很大数目的这样的模式。例如，结果是，如果我们考察在格雷二进制编码中的有名顺序下的所有n个二进位串$a_{n-1}\cdots\ a_1\ a_0$(参见7.2.1.1节)，但只选择恰好有s个0和t个1的那些，则得到的串就形成一个转动门的码。

以下是其证明：通过7.2.1.1-(5)的递归式 $\Gamma_n=0\Gamma_{n-1},1\Gamma_{n-1}^R$，定义格雷二进制编码，因此它的$(s,t)$子序列当$st>0$时，满足递归式：

$$\Gamma_{st}=0\Gamma_{(s-1)t},1\Gamma_{s(t-1)}^R \tag{25}$$

我们也还有$\Gamma_{s0}=0^s$ 和$\Gamma_{0t}=1^t$。因此由归纳式，显然当$st>0$时，Γ_{st} 以0^s1^t开始且以10^s1^{t-1}结束。在(25)中的逗号的转换是当$t\geqslant 2$时，从$0\Gamma_{(s-1)t}$最后的元素到$1\Gamma_{s(t-1)}$最后的元素，即从$010^{s-1}1^{t-1}=010^{s-1}11^{t-2}$到$110^s1^{t-2}=110^{s-1}01^{t-2}$，因而这满足转动门的约束。情况$t=1$也校验过了，例如，$\Gamma_{33}$由以下的诸列给出

000111	011010	110001	101010	
001101	011100	110010	101100	
001110	010101	110100	100101	(26)
001011	010110	111000	100110	
011001	010011	101001	100011	

而且Γ_{23}可以在这个数组的头两列中找到。再把门转动一次就把最后一个元素变为头一个元素。[Γ_{st} 的这些性质首先是由唐道南(D. T. Tang)和刘兆宁(C. N. Liu)发现的，*IEEE Trans*. C-22 (1973)，176-180；詹·理·毕特纳(J. R. Bitner)，吉·厄尔里兹(G. Ehrlich)和爱·马·赖因戈德(E. M. Reingold)给出了一个无循环的实现，*CACM* 19 (1976)， 517-521。]

当我们把(26)中的位串$a_5\,a_4\,a_3\,a_2\,a_1\,a_0$转换成对应的下标表形式$c_3\,c_2\,c_1$时，一个引人注目的模式就出现了：

210	431	540	531	
320	432	541	532	
321	420	542	520	(27)
310	421	543	521	
430	410	530	510	

头一个分量c_3以递增顺序出现；但对于c_3的每一个固定的值，c_2的值以递减顺序出现。而对于固定的$c_3\,c_2$，c_1的值再次是递增的。一般地说，同样的如下事实为真：在转动门格雷编码Γ_{st}中，所有组合$c_t\cdots c_2\,c_1$以 8

$$(c_t,\ -c_{t-1},\ c_{t-2},\ \cdots,\ (-1)^{t-1}c_1) \tag{28}$$

的词典顺序出现。由归纳法可得出这个性质，因为对于$st>0$，当我们使用下标表记号代替二进位串记号时，(25)变成

$$\Gamma_{st}=\Gamma_{(s-1)t},\ (s+t-1)\Gamma^{R}_{s(t-1)} \qquad (29)$$

因此通过使用由威·哈·佩恩(W. H. Payne)给出的下列算法[参见*ACM Trans. Math. Software* 5 (1979), 163-172]可有效地生成这个序列:

算法R(转动门组合) 假定$n>t>1$，本算法以交替序列(28)的词典顺序生成$\{0, 1, \cdots, n-1\}$的所有t组合$c_t\cdots c_2c_1$。依赖于t是奇还是偶，步骤R3有两个版本。

R1. [初始化。] 对于$t\geqslant j\geqslant 1$，置$c_j\leftarrow j-1$和$c_{t+1}\leftarrow n$。

R2. [访问。] 访问组合$c_t\cdots c_2c_1$。

R3. [是容易情况吗？] 如果t是奇数：如果$c_1+1<c_2$，则c_1增1并返回R2，否则置$j\leftarrow 2$ 并转到R4。如果t是偶数：如果$c_1>0$，则c_1减1，并返回R2，否则置$j\leftarrow 2$ 并转到R5。

R4. [尝试减少c_j。] (这时$c_j=c_{j-1}+1$)如果$c_j\geqslant j$，则置$c_j\leftarrow c_{j-1}$，$c_{j-1}\leftarrow j-2$，并返回R2。否则j加1。

R5. [尝试增加c_j。] (这时$c_{j-1}=j-2$)如果$c_j+1<c_{j+1}$，则置$c_{j-1}\leftarrow c_j$，$c_j\leftarrow c_j+1$，并返回R2。否则j加1，而且如果$j\leqslant t$，则转到R4。 ▮

习题21~25进一步剖析了这个有趣序列的性质。它们之一是定理L的一个漂亮的搭配：在恰好访问了

$$N=\binom{c_t+1}{t}-\binom{c_{t-1}+1}{t-1}+\cdots+(-1)^t\binom{c_2+1}{2}-(-1)^t\binom{c_1+1}{1}-[t\text{为奇数}] \qquad (30)$$

个其他组合之后，由算法R访问组合$c_tc_{t-1}\cdots c_2c_1$。我们可以把这称为在度数t的“交替组合数系”中N的表示。例如，一个结果是，每个正整数都有形如$N=\binom{a}{3}-\binom{b}{2}+\binom{c}{1}$的表示，其中$a>b>c>0$。算法R告诉我们在这个数系中如何把1加到$N$中。

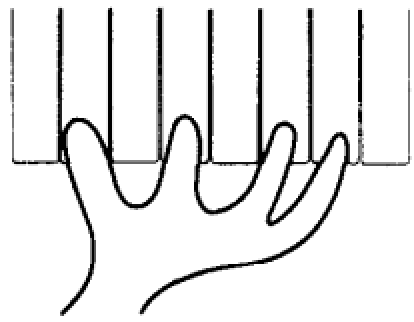
尽管(26)和(27)的串不是词典顺序的，但它们是叫做广义词典顺序(genlex order)的更一般概念的例子，这个名字是由蒂莫西·沃尔什(Timothy Walsh)杜撰的。对一个串序列$\alpha_1, \cdots, \alpha_N$，当具有共同前缀的所有串连续地出现时，则称它处于广义词典顺序之下。例如，在(27)中所有以53开始的3组合在一起出现。

广义词典顺序意味着，如同在6.3节的图31中那样，串可以被安排到一个检索结构中，但每个节点的子女可以有任意的顺序。当以任何顺序遍历一个检索结构，使得每个节点正好在它的后裔之前或之后被访问时，所有带有一个共同前缀的节点（即一个子检索结构的所有节点）连续地出现。这个原理使广义词典顺序十分
9 方便，因为它对应于递归生成方案。我们已经见过的、用于生成n元组的许多算法因此以广义词典顺序的某种版本产生它们的结果；类似地，“平易改动”方法(算法7.2.1.2P)在对应于反演表的广义词典顺序下访问排列。

算法R的转动门方法是一个例行程序，在每步中仅改变组合的一个元素的广义词典顺序。但它不是完全令人满意的，因为为了保持$c_t>\cdots>c_2>c_1$的条件，它经常

必须同时改变两个下标c_j。例如，算法R改变210成为320，而且(27)包含9个这样的“交叉”改动。

这个缺陷的来源可以追溯到我们对(25)满足转动门性质的证明。我们发现，当$t \geqslant 2$时，串$010^{s-1}11^{t-2}$后边跟着$110^{s-1}01^{t-2}$。因此当像11000这样的子串被变成01001或反过来时，递归构造Γ_{st}涉及形如$110^a0 \leftrightarrow 010^a1$这样的转换。



对于组合的一个格雷通路，如果在每一步中它只改变下标c_j之一，则说它是*同族的*。对于$a \geqslant 1$，当我们从一个串通到另一个串时，一个同族的方案是通过在这些串内仅有形如$10^a \leftrightarrow 0^a1$的转换而以位串形式来表征的。例如，对于一个同族方案，通过一次只移动一个手指头，我们能够在一个n个字符的键盘上演奏所有t音的弦。

对(25)稍作修改，可以产生令人愉快的同族的(s, t)组合的广义词典顺序方案。基本思想是构造以0^s1^t开始和以1^t0^s结尾的一个序列，以下的递归立即提了出来：令$K_{s0}=0^s$，$K_{0t}=1^t$，$K_{s(-1)}=\phi$，以及对于$st>0$

$$K_{st} = 0K_{(s-1)t},\ 10K^R_{(s-1)(t-1)},\ 11K_{s(t-2)} \tag{31}$$

在这个序列的逗号处，我们有由$101^{t-1}0^{s-1}$紧随其后的01^t0^{s-1}，和由110^s1^{t-2}紧随其后的10^s1^{t-1}；这两个转换都是同族的，尽管第二个要求1跳越过s个0。对于$s=t=3$的组合K_{33}，在位串形式下为

000111	010101	101100	100011	
001011	010011	101001	110001	
001101	011001	101010	110010	(32)
001110	011010	100110	110100	
010110	011100	100101	111000	

而对应的“手指头模式”为

210	420	532	510	
310	410	530	540	
320	430	531	541	(33)
321	431	521	542	
421	432	520	543	10

当把通常组合$c_t\cdots c_1$的一个同族方案转换成带有重复$d_t\cdots d_1$的组合的对应方案(6)时，它保留这样一个性质，即在每个步骤中仅有一个下标d_j改变。而且当把它转换成合成$p_t\cdots p_0$或$q_t\cdots q_0$的方案(9)或(11)时，当c_j改变时，只有两个(相邻的)部分改变。

近乎完美的方案。但我们甚至可以做得更好！所有(s, t)组合都可以通过或者是$01 \leftrightarrow 10$或者是$001 \leftrightarrow 100$的一个强同族转换序列生成。换言之，我们可以坚持，每个步骤引起一个下标c_j改变至多为2。我们把这样的生成方案称为*近乎完美的*。

强迫这样强的条件实际上使得相当容易发现近乎完美的方案，因为可以利用的

选择就相对少了。确实，如果我们把自己限制到作为n个二进位的串的近乎完美的广义词典顺序方法，托·阿·詹金斯(T. A. Jenkyns)和戴·麦卡锡(D. McCarthy)发现，所有这样的方法都可容易地表征[*Ars Combinatoria* 40 (1995)，153-159]。

定理N 如果$st>0$，则恰有$2s$个近乎完美的方法来把所有(s, t)组合成一个广义词典顺序。事实上，当$1\leqslant a\leqslant s$时，恰有一个这样的表N_{sta}，它以$1^t\,0^s$开始并以$0^a\,1^t\,0^{s-a}$结束，其他s个可能性是颠倒表N^R_{sta}。

证明 当$s=t=1$时，这个结果成立。否则，我们对$s+t$使用归纳法。表N_{sta}如果存在的话，必定有对于某个近乎完美的广义词典顺序的表$X_{s(t-1)}$和$Y_{(s-1)t}$的形式$1X_{s(t-1)}$, $0Y_{(s-1)t}$。如果$t=1$，$X_{s(t-1)}$是单个串0^s；因此如果$a>1$，$Y_{(s-1)t}$必定是$N_{(s-1)1(a-1)}$，而且对于$a=1$它必定是$N^R_{(s-1)11}$。另一方面，如果$t>1$，近乎完美的条件意味着$X_{s(t-1)}$的最后的串不能以1开始；因此对于某个b，$X_{s(t-1)}=N_{s(t-1)b}$。如果$a>1$，$Y_{(s-1)t}$必定是$N_{(s-1)t(a-1)}$，因此b必定是1。类似地，如果$s=1$，则b必定是1。否则我们有$a=1<s$，而且对于某个c，这迫使$Y_{(s-1)t}=N^R_{(s-1)tc}$。仅当$c=1$和$b=2$时，从$10^b\,1^{t-1}0^{s-b}$到$0^{c+1}1^t\,0^{s-1-c}$的转换才是近乎完美的。 ∎

当$st>0$时，定理N的证明产生下列的递归公式：

$$N_{sta}=\begin{cases}1N_{s(t-1)1}, 0N_{(s-1)t(a-1)}, & \text{如果}1<a\leqslant s\\ 1N_{s(t-1)2}, 0N^R_{(s-1)t1}, & \text{如果}1=a<s\\ 1N_{1(t-1)1}, 01^t, & \text{如果}1=a=s\end{cases} \tag{34}$$

而且当然$N_{s0a}=0^s$。

让我们置$A_{st}=N_{st1}$和$B_{st}=N_{st2}$，由菲利普·约·蔡斯(Phillip J. Chase)于1976年发现的这些近乎完美表，分别有移动最左的1的块到右边一两个位置的纯效果，而且它们满足下列相互递归的式子：

$$A_{st}=1B_{s(t-1)}, 0A^R_{(s-1)t}\ ;\quad B_{st}=1A_{s(t-1)}, 0A_{(s-1)t} \tag{35}$$

"为向前一步，则向前两步，而后后退一步；为向前两步，则先向前一步，而后再
11 向前另一步。"如果当s或t为负时，我们定义A_{st}和B_{st}为ϕ，除非$A_{00}=B_{00}=\epsilon$（空串），则对于s和t的所有整数值，这些等式都成立。于是A_{st}实际上取$\min(s, 1)$的向前的步，而B_{st}实际上取$\min(s, 2)$。例如，表2示出对于$s=t=3$的相关表，并且使用一个等价的下标表的形式$c_3c_2c_1$，以代替位串$a_5a_4a_3a_2a_1a_0$。

表2 对于(3, 3)组合的蔡斯序列

$A_{33}=\widehat{C}^R_{33}$			
543	531	321	420
541	530	320	421
540	510	310	431
542	520	210	430
532	521	410	432

$B_{33}=C_{33}$			
543	520	432	410
542	510	430	210
540	530	431	310
541	531	421	320
521	532	420	321

蔡斯注意到，如果我们定义

$$C_{st}=\begin{cases}A_{st}\text{，如果}s+t\text{为奇数；}\\B_{st}\text{，如果}s+t\text{为偶数；}\end{cases}\qquad \hat{C}_{st}=\begin{cases}A_{st}^{R}\text{，如果}s+t\text{为偶数}\\B_{st}^{R}\text{，如果}s+t\text{为奇数}\end{cases}\tag{36}$$

这些序列的计算机实现会变得更简单些[参见*Congressus Numerantium* 69 (1989), 215-242]。于是我们有

$$C_{st}=\begin{cases}1C_{s(t-1)},0\hat{C}_{(s-1)t}\text{，如果}s+t\text{为奇数}\\1C_{s(t-1)},0C_{(s-1)t}\text{，如果}s+t\text{为偶数}\end{cases}\tag{37}$$

$$\hat{C}_{st}=\begin{cases}0C_{(s-1)t},1\hat{C}_{s(t-1)}\text{，如果}s+t\text{为偶数}\\0\hat{C}_{(s-1)t},1\hat{C}_{s(t-1)}\text{，如果}s+t\text{为奇数}\end{cases}\tag{38}$$

当二进位a_j已准备好来改变时，通过测试j是奇数还是偶数，我们可以得知我们在递归式的何处。

确实，基于可应用于任何广义词典顺序的方案的一般思想，通过相当简单的算法，可以生成序列C_{st}。我们说二进位a_j在一个广义词典顺序算法中是活动的——如果设想在它左边的任何东西被改变之前它改变的话。(换言之，在对应的检索结构中一个活动二进位的节点不是它的双亲的最右子女。)假设我们有一个辅助的表$w_n\cdots w_1w_0$，其中$w_j=1$当且仅当a_j是活动的或者$j<r$，其中r是使得$a_r\neq a_0$的最小下标；我们也令$w_n=1$。于是下列方法将求出$a_{n-1}\cdots a_1a_0$的后继者：

> 置$j\leftarrow r$。如果$w_j=0$，则置$w_j\leftarrow1$，$j\leftarrow j+1$并且重复直到$w_j=1$为止。
> 如果$j=n$则结束，否则置$w_j\leftarrow0$。把a_j变为$1-a_j$，并对$a_{j-1}\cdots a_0$和r (39)
> 作任何其他的、适用于正被使用的特殊的广义词典顺序的变化。

这个方法的美妙之处源于这样一个事实，即循环保证有效。我们可以证明，平均说来，操作$j\leftarrow j+1$在每一生成步中将被执行少于一次(参见习题36)。 12

通过分析当在(37)和(38)中的二进位变化时出现的转换，我们可以容易地列出剩余的细节：

算法C(*蔡斯序列*) 本算法以蔡斯序列C_{st}的近乎完美顺序，访问所有(s,t)组合$a_{n-1}\cdots a_1a_0$，其中$n=s+t$。

- C1. [初始化。] 对于$0\leqslant j<s$，置$a_j\leftarrow0$，对于$s\leqslant j<n$，置$a_j\leftarrow1$，而且对于$0\leqslant j<n$，置$w_j\leftarrow1$。如果$s>0$，则置$r\leftarrow s$；否则置$r\leftarrow t$。
- C2. [访问。] 访问组合$a_{n-1}\cdots a_1a_0$。
- C3. [求j和转移。] 置$j\leftarrow r$。如果$w_j=0$，则置$w_j\leftarrow1$，$j\leftarrow j+1$，并重复直到$w_j=1$为止。如果$j=n$则结束；否则置$w_j\leftarrow0$，并作四路转移：如果j为奇数且$a_j\neq0$则转到C4；如果j为偶数且$a_j\neq0$则转到C5；如果j为偶数且$a_j=0$则转到C6；如果j为奇数且$a_j=0$则转到C7。
- C4. [向右移1位。]置$a_{j-1}\leftarrow1$，$a_j\leftarrow0$。如果$r=j>1$，则置$r\leftarrow j-1$；否则如果$r=j-1$，则置$r\leftarrow j$。返回到C2。
- C5. [向右移2位。]如果$a_{j-2}\neq0$，转到C4。否则置$a_{j-2}\leftarrow1$，$a_j\leftarrow0$。如果$r=j$，

则置$r \leftarrow \max(j-2, 1)$；否则如果$r=j-2$，则置$r \leftarrow j-1$。返回到C2。

C6. [向左移1位。] 置$a_j \leftarrow 1$，$a_{j-1} \leftarrow 0$。如果$r=j>1$，则置$r \leftarrow j-1$；否则如果$r=j-1$，则置$r \leftarrow j$。返回到C2。

C7. [向左移2位。] 如果$a_{j-1} \neq 0$，则转到C6。否则置$a_j \leftarrow 1$，$a_{j-2} \leftarrow 0$。如果$r=j-2$，则置$r \leftarrow j$；否则如果$r=j-1$，则置$r \leftarrow j-2$。返回到C2。 ▮

***对蔡斯序列的分析。**算法C的魔术般的性质要求作进一步的剖析，而且更仔细地考察证明是十分有教益的。给定一个二进位串$a_{n-1} \cdots a_1 a_0$，我们定义$a_n=1$，$u_n = n \bmod 2$且对于$n>j \geqslant 0$，

$$u_j = (1-u_{j+1})a_{j+1},\ v_j = (u_j + j) \bmod 2,\ \ w_j = (v_j + a_j) \bmod 2 \tag{40}$$

例如，我们可以有$n=26$，且

$$\begin{aligned} a_{25} \cdots a_1 a_0 &= 11001001000011111101101010 \\ u_{25} \cdots u_1 u_0 &= 10100100100001010100100101 \\ v_{25} \cdots v_1 v_0 &= 00001110001011111110001111 \\ w_{25} \cdots w_1 w_0 &= 11000111001000000011100101 \end{aligned} \tag{41}$$

通过这些定义我们可以用归纳法证明$v_j=0$，当且仅当二进位a_j在生成$a_{n-1} \cdots a_1 a_0$的递归式(37)和(38)中是由C“控制”的而不是由$\hat{C}$来控制，除非当a_j是在右端中最后的0或1的运程的部分时例外。因此，对于$r \leqslant j<n$，w_j同由算法C在访问$a_{n-1} \cdots a_1 a_0$的时刻所计算的值相一致。这些公式可用来精确地确定在蔡斯序列中一个给定的组合在何处出现(参见习题39)。

如果我们要对下标表$c_t \cdots c_2 c_1$进行工作而不是对二进位串$a_{n-1} \cdots a_1 a_0$进行工作，

13 则对记号稍作修改是很方便的，而且当$s+t=n$时把$C_t(n)$写作C_{st}并把$\hat{C}_t(n)$写作$\hat{C}_{st}$。于是$C_0(n)=\hat{C}_0(n)=\varepsilon$，而且对于$t \geqslant 0$，递归式取下列形式：

$$C_{t+1}(n+1) = \begin{cases} nC_t(n), \hat{C}_{t+1}(n)\ ，\ 如果n为偶数 \\ nC_t(n), C_{t+1}(n)\ ，\ 如果n为奇数 \end{cases} \tag{42}$$

$$\hat{C}_{t+1}(n+1) = \begin{cases} C_{t+1}(n), n\hat{C}_t(n)\ ，\ 如果n为奇数 \\ \hat{C}_{t+1}(n), n\hat{C}_t(n)\ ，\ 如果n为偶数 \end{cases} \tag{43}$$

例如，这些新的等式可加以扩展以告诉我们：

$$\begin{aligned} C_{t+1}(9) &= 8C_t(8), 6C_t(6), 4C_t(4), \cdots, 3\hat{C}_t(3), 5\hat{C}_t(5), 7\hat{C}_t(7) \\ C_{t+1}(8) &= 7C_t(7), 6C_t(6), 4C_t(4), \cdots, 3\hat{C}_t(3), 5\hat{C}_t(5) \\ \hat{C}_{t+1}(9) &= \qquad 6C_t(6), 4C_t(4), \cdots, 3\hat{C}_t(3), 5\hat{C}_t(5), 7\hat{C}_t(7), 8\hat{C}_t(8) \\ \hat{C}_{t+1}(8) &= \qquad 6C_t(6), 4C_t(4), \cdots, 3\hat{C}_t(3), 5\hat{C}_t(5), 7\hat{C}_t(7) \end{aligned} \tag{44}$$

注意，在所有4个序列中相同的模式预先占支配地位，中间的“…”的意义依赖于t的值：在$n<t$处我们干脆省却$nC_t(n)$和$n\hat{C}_t(n)$的所有项。

除开对于开头和末尾处的边际效应外，(44)中的所有扩展都是基于无穷级数

$$\cdots, 10, 8, 6, 4, 2, 0, 1, 3, 5, 7, 9, \cdots \tag{45}$$

这是把非负整数安排成双重无穷序列的一个自然的方式。对给定任何整数$t>0$，如果我们省去$<t$的(45)的所有项，则剩下的项保留这样一个性质，即相邻的元素相异1或2。对于这个序列，理查德·斯坦利(Richard Stanley)已经提议使用“偶减奇顺序”(endo-order)。因为我们可以通过想像“偶数递减，奇数……”来记住它。(注意，如果我们只保留小于N的项并且相对于N求补，则偶减奇顺序就变成管风琴顺序；参见习题6.1-18。)

我们可以直接对(42)和(43)的递归式编程序，但使用(44)来展开它是有趣的，这就得到类似于算法C的一个迭代算法。这一结果仅需要$O(t)$个存储单元，而且当同n相比较t是相对小的时，它特别有效。习题45包含有其细节。

***近乎完美多重集合的排列**。蔡斯序列以一种自然的方式导致将以近乎完美的方式生成任何所希多重集合$\{s_0\cdot 0,\ s_1\cdot 1,\ \cdots,\ s_d\cdot d\}$的排列。意思是：

i) 每一个转换是$a_{j+1}a_j\leftrightarrow a_j a_{j+1}$或$a_{j+1}a_j a_{j-1}\leftrightarrow a_{j-1}a_j a_{j+1}$。

ii) 第二类转换有$a_j=\min(a_{j-1}, a_{j+1})$。

算法C告诉我们，当$d=1$时如何做，而通过以下递归构造我们可以把它扩充成更大的d值[*CACM* **13** (1970)，368-369, 376]。假设

$$\alpha_0, \alpha_1, \cdots, \alpha_{N-1}$$ 14

是$\{s_1\cdot 1,\ \cdots,\ s_d\cdot d\}$的排列的近乎完美表，则通过$s=s_0$和$t=s_1+\cdots+s_d$，算法C告诉我们如何生成一个表

$$\Lambda_j=\alpha_j 0^s,\ \ldots, 0^a\alpha_j 0^{s-a} \tag{46}$$

其中所有的转换是$0x\leftrightarrow x0$或$00x\leftrightarrow x00$；依赖于s和t，最后的条目有$a=1$或2个前导零。因此序列

$$\Lambda_0,\ \Lambda_1^R, \Lambda_2, \cdots, (\Lambda_{N-1} \text{ 或 } \Lambda_{N-1}^R) \tag{47}$$

的所有转换是近乎完美的；而这个表显然包含所有排列。

例如，以这个方法生成的$\{0, 0, 0, 1, 1, 2\}$的排列为：

211000, 210100, 210001, 210010, 200110, 200101, 200011, 201001, 201010, 201100,
021100, 021001, 021010, 020110, 020101, 020011, 000211, 002011, 002101, 002110,
001120, 001102, 001012, 000112, 010012, 010102, 010120, 011020, 011002, 011200,
101200, 101020, 101002, 100012, 100102, 100120, 110020, 110002, 110200, 112000,
121000, 120100, 120001, 120010, 100210, 100201, 100021, 102001, 102010, 102100,
012100, 012001, 012010, 010210, 010201, 010021, 000121, 001021, 001201, 001210

***完美方案**。为什么我们要设置像C_{st}这样一个近乎完美的生成程序，而不是坚持所有转换都有最简可能的形式$01\leftrightarrow 10$？

一个原因是完美方案并不总是存在。例如，我们在7.2.1.2-(2)中发现，没有办法生成带有相邻交换的$\{1, 1, 2, 2\}$的所有6个排列；因此，也就没有对于(2, 2)组合

的完美方案。事实上，我们实现完美的机会仅仅是四分之一。

定理P 通过相邻交换01↔10的所有(s, t)组合$a_{s+t-1}\cdots a_1 a_0$的生成是可能的，当且仅当$s\le 1$或$t\le 1$或st为奇数。

证明 考虑多重集合$\{s\cdot 0, t\cdot 1\}$的所有排列。在习题5.1.2-16中我们得知，有k个反演的这样排列的个数m_k是在z多项式中z^k的系数

$$\binom{s+t}{t}_z=\prod_{k=s+1}^{s+t}(1+z+\cdots+z^{k-1})\Big/\prod_{k=1}^{t}(1+z+\cdots+z^{k-1}) \qquad (48)$$

每一个相邻交换把反演数改变± 1，因此仅当所有排列中近乎一半有奇数个反演时，一个完美的生成方案才是可能的。更精确地说$\binom{s+t}{t}_{-1}=m_0-m_1+m_2-\cdots$的值必须为0或$\pm 1$。但习题49表明

$$\binom{s+t}{t}_{-1}=\binom{\lfloor (s+t)/2\rfloor}{\lfloor t/2\rfloor}[st\text{ 为偶数}] \qquad (49)$$

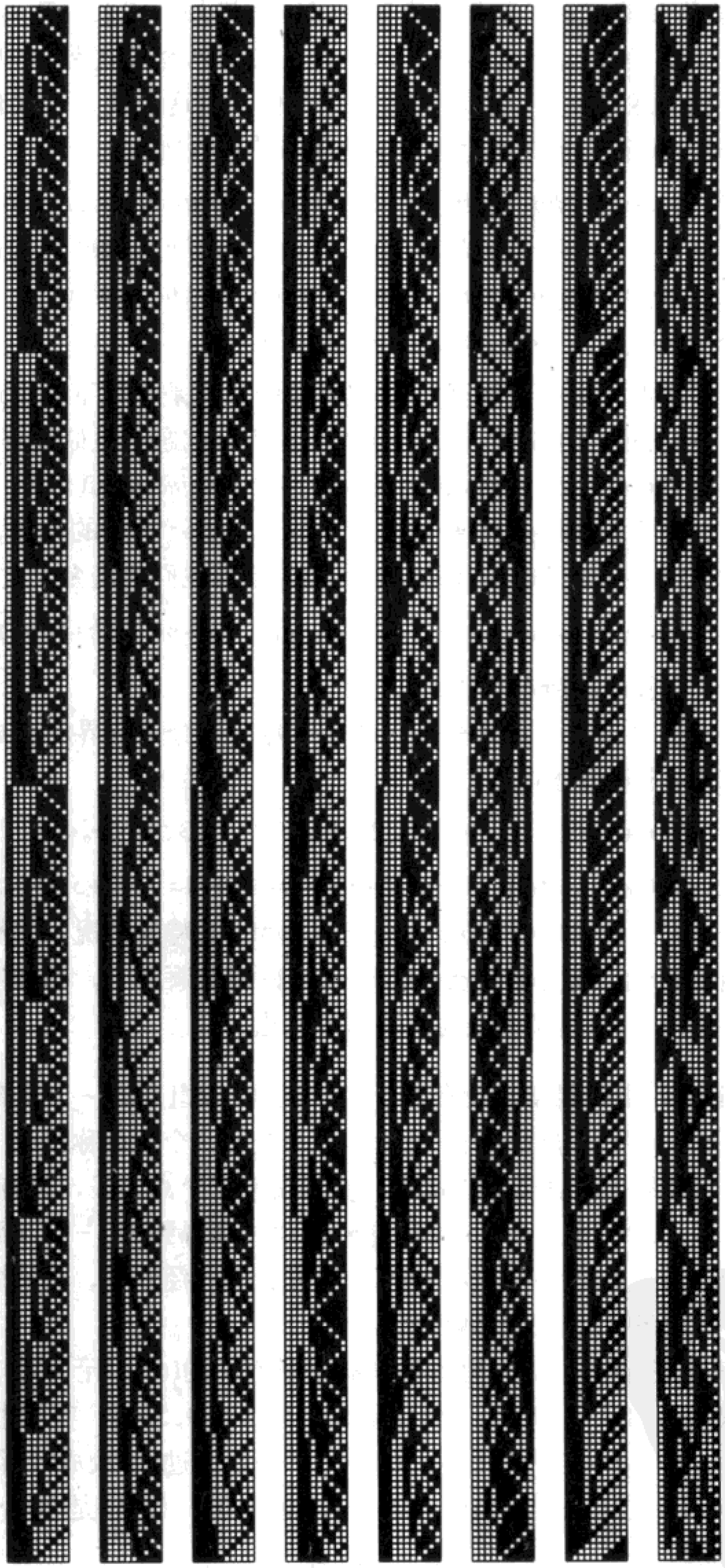
而除非$s\le 1$或$t\le 1$或st为奇数，否则这个量超过1。

反之，对于$s\le 1$或$t\le 1$，完美方案是容易的，而且每当st为奇数时，它们也是可能的。对于$s=t=3$出现头一个非平凡的情况，此时有4个实际上不同的解；这些解当中最为对称的是

210—310—410—510—520—521—531—532—432—431—
15 421—321—320—420—430—530—540—541—542—543 (50)

(参见习题51)。对于任意的奇数s和t，好多作者已经构造出在相关图中的哈密顿通路；例如，埃迪斯(Eades)、希克基(Hickey)和里德(Read)[*JACM* 31 (1984), 19-29]通过递归共行程序做了程序设计方面的一个有趣习题。然而，不幸的是，在已知构造中没有一个是充分简单地在一个短的空间中进行描述，或者以合理的有效性实现。因此完美组合生成程序还未被证明是具有实际重要性的。 ▌

于是，总之，我们已经看到，对于(s, t)组合的研究导致了许多迷人的模式，其中有些模式有很重要的实用性，而有些则仅仅是优雅和/或美丽而已。图26示出当出现$\binom{10}{5}=252$种组合时，在$s=t=5$的情况下主要的选项。词典顺序排序(算法L)、转动门格雷码(算法R)、(31)的同族方案K_{55}，以及蔡斯近乎完美的方案(算法C)，被示于该图的a)、b)、c)和d)部分。部分e)示出尽可能接近于完美的近乎完美方案，同时又还是在c数组的广义词典顺序下(参见习题34)。至于部分f)是埃迪斯、希克基和里德的完美方案。最后图26g和图26h是通过转动$a_j a_{j-1}\cdots a_0\leftarrow a_{j-1}\cdots a_0 a_j$或通过交换$a_j\leftrightarrow a_0$而进行的表。它们类似于算法7.2.1.2C和7.2.1.2E(参见习题55和习题56)。



a) b) c) d) e) f) g) h)

图26 (5, 5)组合的例子: a) 词典顺序; b) 转动门; c) 同族; d) 近乎完美; e) 更近乎完美; f) 完美; g) 后缀转动; h) 右交换

***一个多重集合的组合**。如果多重集合可以有排列，它们也能有组合。例如，考虑多重集合$\{b, b, b, b, g, g, g, r, r, r, w, w\}$，表示包含四个蓝色的球、三个绿色的球、三个红色的球以及两个白色的球的一个栈。由这个栈，可以有37种方法选择5个球；在词典顺序下(但在每个组合中为递减的)，它们是

$$\begin{aligned}&gbbbb, ggbbb, gggbb, rbbbb, rgbbb, rggbb, rgggb, rrbbb, rrgbb, rrggb, rrggg,\\&rrrbb, rrrgb, rrrgg, wbbbb, wgbbb, wggbb, wgggb, wrbbb, wrgbb, wrggb,\\&wrggg, wrrbb, wrrgb, wrrgg, wrrrb, wrrrg, wwbbb, wwgbb, wwggb,\\&wwggg, wwrbb, wwrgb, wwrgg, wwrrb, wwrrg, wwrrr\end{aligned} \tag{51}$$

这个事实可能看起来是无聊至极和/或神秘的，但是在以下的定理W中我们将看到，多重集合组合的词典顺序生成对于重要的组合问题产生最优解。

詹姆斯·贝努利(James Bernoulli)在他的[*Ars Conjectandi*(1713)，119-123]中发现，通过观察在乘积$(1+z+z^2)(1+z+z^2+z^3)^2\ (1+z+z^2+z^3+z^4)$中$z^5$的系数，我们能枚举这样的组合。其实，他的发现很容易理解，因为如果我们乘出多项式

$$(1+w+ww)(1+r+rr+rrr)(1+g+gg+ggg)(1+b+bb+bbb+bbbb)$$

我们就得到从这个栈中的所有可能的选择。

多重集合组合也等价于*有界合成*，即其中每个部分是有界的合成。例如，在(51)中列出的37种多重组合对应于

16~17 $$5=r_3+r_2+r_1+r_0,\ 0\leqslant r_3\leqslant 2,\ 0\leqslant r_2,\ r_1\leqslant 3,\ 0\leqslant r_0\leqslant 4$$

的37个解。即5=0+0+1+4=0+0+2+3=0+0+3+2=0+1+0+4=…=2+3+0+0。

有界合成，进而是*偶然性表*(contingency table)的特殊情况。偶然性表在统计学中有很大的重要性。而且通过类格雷码以及在词典顺序下，都可生成所有这些组合配置。习题60~63剖析了涉及的某些基本思想。

***阴影**。组合的集合经常出现在数学中。例如，2组合的一个集合（即对偶的集合）实际上是一个图，而且对于一般的t，t组合的一个集合称作统一的超图。如果一个凸的多边体的顶点稍微地被扰乱，使得没有三个点共线，没有四个点共面，而且一般地没有$t+1$个点位于$(t-1)$维的超平面中，则得到的$(t-1)$维面是其顶点在计算机应用中有很大重要性的“单纯形”。研究者们已经知道，这种组合的集合有着重要的同词典顺序生成相关联的性质。

如果α是任何t组合$c_t\cdots c_2c_1$，它的*阴影*$\partial\alpha$是所有它的$(t-1)$元子集$c_{t-1}\cdots c_2c_1$，…，$c_t\cdots c_3c_1$，$c_t\cdots c_3c_2$的集合。例如，$\partial 5310=\{310, 510, 530, 531\}$，我们也可把一个$t$组合表示为一个二进位串$a_{n-1}\cdots a_1a_0$，在该情况下$\partial\alpha$是通过把1改成0所得到的所有串的集合：$\partial 101011=\{001011, 100011, 101001, 101010\}$。如果$A$是$t$组合的任何集合，我们定义它的阴影

$$\partial A=\cup\{\partial\alpha \mid \alpha\in A\} \tag{52}$$

为在它的成员的阴影中所有$(t-1)$组合的集合。例如，$\partial\partial 5310=\{10, 30, 31, 50, 51, 53\}$。

这些定义也适用于带有重复的组合，即多重组合$\partial 5330=\{330, 530, 533\}$，以及$\partial\partial 5330=\{30, 33, 50, 53\}$。一般地说，当$A$是$t$元多重集合的一个集合时，$\partial A$是$(t-1)$元多重集合的集合。然而，注意，$\partial A$本身决无重复的元素。

相对于一个全集U的上阴影$\varrho\alpha$可类似地定义，但它从t组合转向$(t+1)$组合：

$$\varrho\alpha=\{\beta\subseteq U \mid \alpha\in\partial\beta\}，对于\alpha\in U \tag{53}$$

$$\varrho A=\cup\{\varrho\alpha \mid \alpha\in A\}，对于 A\subseteq U \tag{54}$$

例如，如果$U=\{0, 1, 2, 3, 4, 5, 6\}$，我们有$\varrho 5310=\{53210, 54310, 65310\}$；另一方面，如果$U=\{\infty\cdot 0, \infty\cdot 1, \cdots, \infty\cdot 6\}$，我们有$\varrho 5310=\{53100, 53110, 53210, 53310, 54310, 55310, 65310\}$。

以下的一些基本定理，它们在数学和计算机科学的各个分支中有许多应用，它们告诉我们一个集合的阴影可以有多小。

定理K 如果A是包含在$U=\{0, 1, \cdots, n-1\}$中的N个t组合的集合，则

$$|\partial A|\geqslant|\partial P_{Nt}| \text{ 且 } |\varrho A|\geqslant|\varrho Q_{Nnt}| \tag{55}$$

其中P_{Nt}表示由算法L生成的头N个组合，即N个满足(3)的词典顺序下最小的组合$c_t\cdots c_2c_1$，而且Q_{Nnt}表示N个词典顺序下最大的。 ▮ 18

定理M 如果A是包含在多重集合$U=\{\infty\cdot 0, \infty\cdot 1, \cdots, \infty\cdot s\}$中$N$个$t$多重组合的集合，则

$$|\partial A|\geqslant|\partial \hat{P}_{Nt}| \quad 且 \quad |\varrho A|\geqslant|\varrho \hat{Q}_{Nst}| \tag{56}$$

其中$\hat{P}_{Nt}$表示N个词典顺序下满足(6)的最小多重组合$d_t\cdots d_2d_1$，而$\hat{Q}_{Nst}$表示N个在词典顺序下最大的。 ▮

这两个定理是我们后面将证明的一个更强结果的推论。定理K一般称作克鲁斯卡尔－卡托纳定理，因为它是由约·贝·克鲁斯卡尔发现的[由理·贝尔曼(R. Bellman)编辑的*Math. Optimization Techniques* (1963)，251-278]，并由古·卡托纳(G. Katona)重新发现[*Theory of Graphs*(图论)，Tihany 1966，由埃尔多斯(Erdös)和卡托纳编辑的(Academic Press，1968)，187-207]；马·鲍·舒曾伯格(M. P. Schützenberger)在一本不大出名的出版物上已经指出它，并带有不完整的证明[*RLE Quarterly Progress Report* **55** (1959)，117-118]。定理M可追溯到许多年前的弗·索·麦考莱(F. S. Macaulay)[*Proc. London Math. Soc.* (2) **26** (1927)，531-555]。

在证明(55)和(56)之前，让我们更仔细地观察这些公式意味着什么。从定理L我们知道，由算法L所访问的所有t组合中的头N个是居于$n_t\cdots n_2n_1$之前的那些，其中

$$N=\binom{n_t}{t}+\cdots+\binom{n_2}{2}+\binom{n_1}{1}，\quad n_t>\cdots>n_2>n_1\geqslant 0$$

是N的度数t的组合表示。有时这个表示有少于t个非0项，因为n_j可能等于$j-1$；且让我们去掉这些零，而且写

$$N=\binom{n_t}{t}+\binom{n_{t-1}}{t-1}+\cdots+\binom{n_v}{v}, \quad n_t>n_{t-1}>\cdots>n_v\geqslant v\geqslant 1 \tag{57}$$

现在头$\binom{n_t}{t}$个组合$c_t\cdots c_1$是$\{0, \cdots, n_t-1\}$的t组合；其次的$\binom{n_{t-1}}{t-1}$是其中$c_t=n_t$而且$c_{t-1}\cdots c_1$是$\{0, \cdots, n_{t-1}-1\}$的一个$(t-1)$组合；等等。例如，如果$t=5$和$N=\binom{9}{5}+\binom{7}{4}+\binom{4}{3}$，则头$N$个组合是

$$P_{N5}=\{43210, \cdots, 87654\}\cup\{93210, \cdots, 96543\}\cup\{97210, \cdots, 97321\} \tag{58}$$

幸而，这个集合P_{N5}的阴影容易理解，它是:

$$\partial P_{N5}=\{3210, \cdots, 8765\}\cup\{9210, \cdots, 9654\}\cup\{9710, \cdots, 9732\} \tag{59}$$

即当$t=4$时在词典顺序下的头$\binom{9}{4}+\binom{7}{3}+\binom{4}{2}$组合。

换言之，如果当N有惟一表示（57）时，我们通过公式

$$\kappa_t N=\binom{n_t}{t-1}+\binom{n_{t-1}}{t-2}+\cdots+\binom{n_v}{v-1} \tag{60}$$

来定义克鲁斯卡尔函数，我们有

19 $$\partial P_{Nt}=P_{(\kappa_t N)(t-1)} \tag{61}$$

例如，定理K告诉我们，具有100万个边的一个图至多可含

$$\binom{1414}{3}+\binom{1009}{2}=470\ 700\ 300$$

个三角形，即至多有顶点$\{u, v, w\}$的470 700 300个集合且u—v—w—u。原因是由习题17，$1000000=\binom{1414}{2}+\binom{1009}{1}$，而且边$P_{(1000000)2}$确实支持$\binom{1414}{3}+\binom{1009}{2}$个三角形；但如果有更多，则这个图将必要地至少有$\kappa_3 470700301=\binom{1414}{2}+\binom{1009}{1}+\binom{1}{0}=1000001$个边在它们的阴影内。

克鲁斯卡尔定义伴随函数

$$\lambda_t N=\binom{n_t}{t+1}+\binom{n_{t-1}}{t}+\cdots+\binom{n_v}{v+1} \tag{62}$$

来处理像这样的问题。κ和λ函数通过在习题72中证明的一个有趣定律来相关联:

$$M+N=\binom{s+t}{t}\text{ 意味着如果}st>0\text{，则 }\kappa_s M+\lambda_t N=\binom{s+t}{t+1} \tag{63}$$

转到定理M，$\partial\hat{P}_{Nt}$和$\varrho\,\hat{Q}_{Nst}$的大小结果是

$$|\partial \hat{P}_{Nt}| = \mu_t N \text{ 和 } |\varrho\, \hat{Q}_{Nst}| = N + \kappa_s N \tag{64}$$

(参见习题81)，当N有组合表示(57)时，其中函数μ_t满足

$$\mu_t N = \binom{n_t - 1}{t-1} + \binom{n_{t-1} - 1}{t-2} + \cdots + \binom{n_v - 1}{v-1} \tag{65}$$

表3示出对于t和N的小的值，这些函数$\kappa_t N$、$\lambda_t N$以及$\mu_t N$的特性。当t和N很大时，借助于由高木贞治(Teiji Takagi)在1903年提出的一个著名函数$\tau(x)$，可以对它们很好地求近似值，参见图27和习题82~85。

表3　克鲁斯卡尔－麦考莱函数κ、λ和μ举例

$N =$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$\kappa_1 N =$	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$\kappa_2 N =$	0	2	3	3	4	4	4	5	5	5	5	6	6	6	6	6	7	7	7	7	7
$\kappa_3 N =$	0	3	5	6	6	8	9	9	10	10	10	12	13	13	14	14	14	15	15	15	15
$\kappa_4 N =$	0	4	7	9	10	10	13	15	16	16	18	19	19	20	20	20	23	25	26	26	28
$\kappa_5 N =$	0	5	9	12	14	15	15	19	22	24	25	25	28	30	31	31	33	34	34	35	35
$\lambda_1 N =$	0	0	1	3	6	10	15	21	28	36	45	55	66	78	91	105	120	136	153	171	190
$\lambda_2 N =$	0	0	0	1	1	2	4	4	5	7	10	10	11	13	16	20	20	21	23	26	30
$\lambda_3 N =$	0	0	0	0	1	1	1	2	2	3	5	5	5	6	6	7	9	9	10	12	15
$\lambda_4 N =$	0	0	0	0	0	1	1	1	1	2	2	2	3	3	4	6	6	6	6	7	7
$\lambda_5 N =$	0	0	0	0	0	0	1	1	1	1	1	2	2	2	2	3	3	3	4	4	5
$\mu_1 N =$	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$\mu_2 N =$	0	1	2	2	3	3	3	4	4	4	4	5	5	5	5	5	6	6	6	6	6
$\mu_3 N =$	0	1	2	3	3	4	5	5	6	6	6	7	8	8	9	9	9	10	10	10	10
$\mu_4 N =$	0	1	2	3	4	4	5	6	7	7	8	9	9	10	10	10	11	12	13	13	14
$\mu_5 N =$	0	1	2	3	4	5	5	6	7	8	9	9	10	11	12	12	13	14	14	15	15

定理K和M是由王大伦和王(杨)平[*SIAM J.Applied Math*. 33 (1997)，55-59]发现的一个更一般得多的离散几何定理的推论，我们现在来研究它。考虑其元素为整数向量$x=(x_1, \cdots, x_n)$的离散n维圆环体$T(m_1, \cdots, m_n)$，且有$0 \leqslant x_1 < m_1, \cdots, 0 \leqslant x_n < m_n$。我们像在等式4.3.2- (2)和4.3.2- (3)中那样定义这样两个向量的和与差。

$$x+y=((x_1+y_1) \bmod m_1, \cdots, (x_n+y_n) \bmod m_n) \tag{66}$$

$$x-y=((x_1-y_1) \bmod m_1, \cdots, (x_n-y_n) \bmod m_n) \tag{67}$$

通过说$x \preceq y$当且仅当

$$\nu x < \nu y \text{或}(\nu x = \nu y \text{和按词典顺序} x \geqslant y) \tag{68}$$

我们也在这样的向量上定义所谓交叉次序；这里和通常一样，$\nu(x_1, \cdots, x_n)=x_1+\cdots+x_n$。例如，当$m_1=m_2=2$且$m_3=3$时，在递增的交叉次序下的12个向量$x_1x_2x_3$是

$$000, 100, 010, 001, 110, 101, 011, 002, 111, 102, 012, 112 \tag{69}$$

为方便起见，这里省略了括弧和逗号。在$T(m_1, \cdots, m_n)$中，一个向量的补为

$$\bar{x} = (m_1-1-x_1, \cdots, m_n-1-x_n) \tag{70}$$

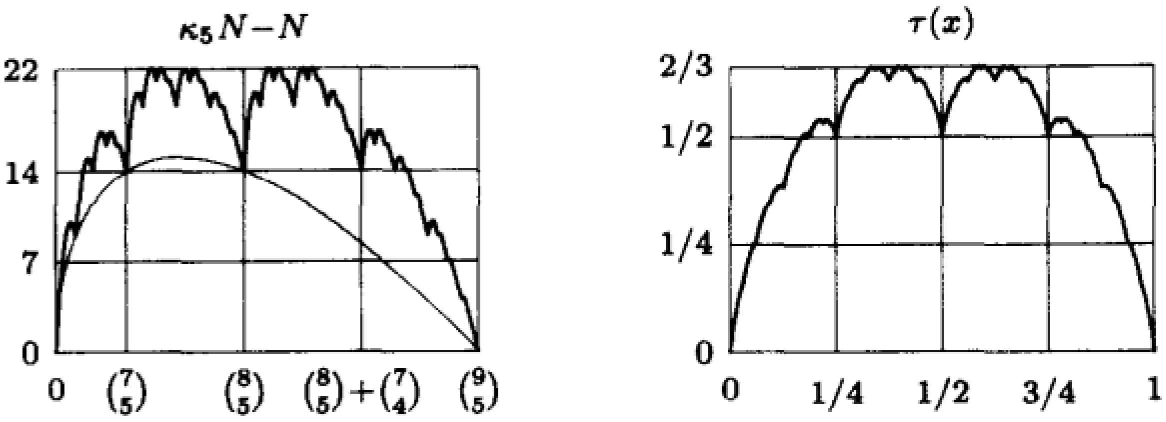


图27 以高木贞治函数近似一个克鲁斯卡尔函数

(右图中的圆滑曲线是习题80的下界 $\underline{\kappa}_5N-N$)

注意$x\preceq y$成立当且仅当 $\bar{x}\succeq\bar{y}$ 。因此如果rank(x)表示在交叉次序下居于x之前的向量个数，则有

$$\text{rank}(x)+\text{rank}(\bar{x})=T-1,\ \text{其中}T=m_1\cdots m_n \tag{71}$$

我们将发现，把向量称作“点”和以递增的交叉次序来命名点e_0, e_1, …, e_{T-1}是方便的。于是在(69)中我们有e_7= 002，以及一般地 $\bar{e}_r=e_{T-1-r}$ 。注意

20~21 $$e_1=100\cdots00,\ e_2=010\cdots00,\ \cdots,\ e_n=000\cdots01 \tag{72}$$

这些是所谓的单位向量。由最小N个点组成的集合

$$S_N=\{e_0,e_1,\cdots,e_{N-1}\} \tag{73}$$

称为一个标准集合，而且在特殊情况$N=n+1$之下我们写

$$E=\{e_0,e_1,\cdots,e_n\}=\{000\cdots00,100\cdots00,010\cdots00,\cdots,000\cdots01\} \tag{74}$$

点X的任何集合有一个由以下规则定义的散布X^+、核X°以及对偶X^-:

$$X^+=\{x\in S_T\mid x\in X\text{ 或 }x-e_1\in X\text{ 或 }\cdots\text{或 }x-e_n\in X\} \tag{75}$$

$$X^\circ=\{x\in S_T\mid x\in X\text{ 且 }x+e_1\in X\text{ 且 }\cdots\text{且 }x+e_n\in X\} \tag{76}$$

$$X^-=\{x\in S_T\mid \bar{x}\notin X\} \tag{77}$$

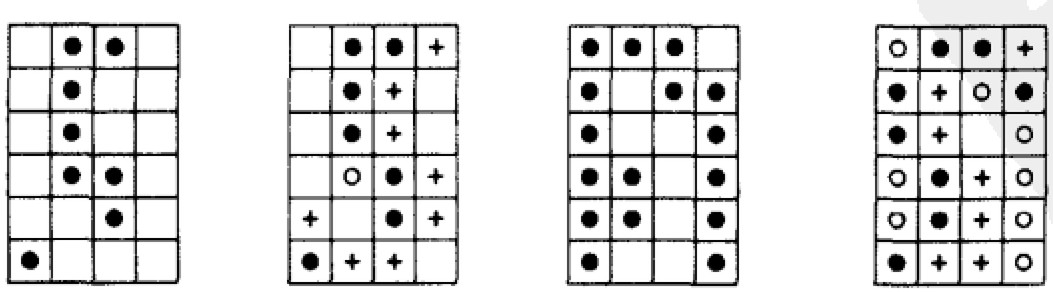
我们也可代数地定义X的散布，并写

$$X^+=X+E \tag{78}$$

其中$X+Y$表示$\{x+y|x\in X\text{且}y\in Y\}$。显然

$$X^+\subseteq Y\text{ 当且仅当 }x\subseteq Y^\circ \tag{79}$$

通过多少有些随机的圆环体状的安排X={00, 12, 13, 14, 15, 21, 22, 25}，可以图示在二维情况下m_1=4，m_2=6的这些记号的情况，从图形上看，我们有



(80)

X　　X°和X^+　　X^-　　$X^{-\circ}$和X^{-+}

这里在头两个框图中的X由标记为•和∘的点组成。X°仅由∘组成，而X^{+}由诸+和诸•和诸∘组成。注意，如果我们转动$X^{-\circ}$和X^{-+}的框图180°，就得到对于X°和X^{+}的框图。但有(•, ∘, +,)分别地变成(+, , •, ∘)；而且事实上一般地

$$X^{\circ}=X^{-+-},\ X^{+}=X^{-\circ-} \tag{81}$$

成立（见习题86）。

现在我们已准备好来论述王大伦和王（杨）平的定理。

定理W 令X是离散圆环体$T(m_1, \cdots, m_n)$中N个点的任意集合，这里$m_1 \leqslant \cdots \leqslant m_n$。则$|X^{+}| \geqslant \left|S_N^{+}\right|$且$|X^{\circ}| \leqslant \left|S_N^{\circ}\right|$。

换言之，标准集合S_N在所有N个点的集合中有最小的散布和最大的核心。我们将证明这一结果，通过首先由弗 · 约 · 威 · 惠普尔(F. W. J. Whipple)证明定理M时所用的以下一般方法[*Proc. London Math. Soc.* (2) 28 (1928)，431-437]。头一步是来证明标准集合的散布和核心是标准的。 22

引理S 有函数α和β使得 $S_N^{+} = S_{\alpha N}$ 和 $S_N^{\circ} = S_{\beta N}$ 。

证明 我们可以假设$N>0$，令r是使$e_r \in S_N^{+}$的极大值，并令$\alpha N=r+1$；我们必须证明对于$0 \leqslant q<r$， $e_q \in S_N^{+}$ 。假设$e_q=x=(x_1, \cdots, x_n)$和$e_r=y=(y_1, \cdots, y_n)$，并令k是使$x_k>0$的最大下标。由于$y \in S_N^{+}$，因此有下标j使得$y-e_j \in S_N$。$x-e_k \preceq y-e_j$即可，而习题88就做这件事。

从(81)，对于$\beta N=T-\alpha(T-N)$，第二部分就得出了，因为 $S_N^{-} = S_{T-N}$ 。 ▮

当$n=1$时，定理W是显然的，所以通过归纳法我们假设在$n-1$维中它已被证明了。下一步是把给定集合X压缩到第k坐标的位置，对于$0 \leqslant a<m_k$，这通过把X分划成不相交集合

$$X_k(a)=\{x \in X \mid x_k=a\} \tag{82}$$

来做，而且以

$$X_k'(a)=\{(s_1,\cdots,s_{k-1}, a, s_k,\cdots,s_{n-1}) \mid (s_1,\cdots,s_{n-1}) \in S_{|X_k(a)|}\} \tag{83}$$

来代替每一个$X_k(a)$，这是一个具有相同元素数的集合。(83)中的集合S在$(n-1)$维的圆环体$T(m_1, \cdots, m_{k-1}, m_{k+1}, \cdots, m_n)$中是标准的。注意，我们有$(x_1, \cdots, x_{k-1}, a, x_{k+1}, \cdots, x_n) \preceq (y_1, \cdots, y_{k-1}, a, y_{k+1}, \cdots, y_n)$，当且仅当$(x_1, \cdots, x_{k-1}, x_{k+1}, \cdots, x_n) \preceq (y_1, \cdots, y_{k-1}, y_{k+1}, \cdots, y_n)$；因此，$X_k'(a)=X_k(a)$当且仅当使得$(x_1, \cdots, x_{k-1}, a, x_{k+1}, \cdots, x_n) \in X$的$(n-1)$维点$(x_1, \cdots, x_{k-1}, x_{k+1}, \cdots, x_n)$，当把它投影到$(n-1)$维圆环体时，是尽可能小的。我们令

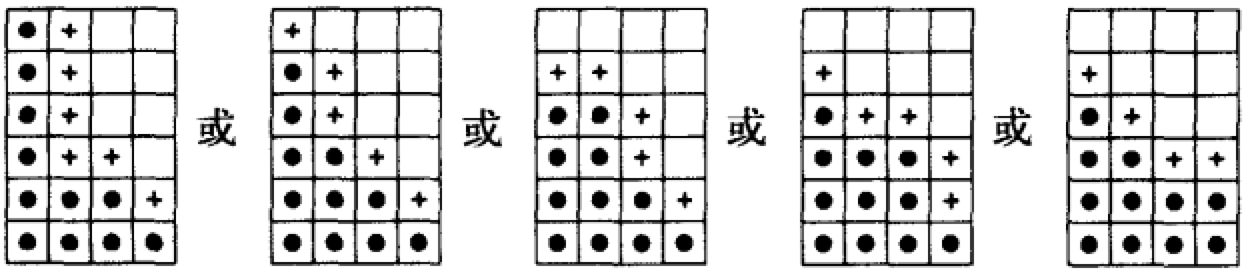
$$C_k X = X_k'(0) \cup X_k'(1) \cup \cdots \cup X_k'(m_k-1) \tag{84}$$

是在位置k中X的压缩。习题90证明了以下基本事实，即压缩不增加散布的大小：对于$1 \leqslant k \leqslant n$

$$|X^+| \geqslant |(C_k X)^+| \tag{85}$$

其次，如果压缩改变X，它以较低阶的其他元素来代替某个元素。因此我们仅对于完全被压缩（并且对于所有k有$X=C_kX$）的集合X需要证明定理W。

例如，考虑$n=2$的情况。在二维中的一个完全被压缩的集合的所有点向它们的行左边和向它们的列下部移动，如同在11个点的集合中



这些框图中最右的是标准的，而且有最小的散布。习题91完成在二维中定理W的
23 证明。

当$n>2$时，假设$x=(x_1, \cdots, x_n)\in X$且$x_j>0$。条件$C_kX=X$意味着，如果$0\leqslant i<j$且$i\neq k\neq j$，则我们有$x+e_i-e_j\in X$。对于$k$的三个值应用这一事实告诉我们，每当$0\leqslant i<j$时$x+e_i-e_j\in X$。结果，对于$0<a<m$，

$$X_n(a)+E_n(0)\subseteq X_n(a-1)+e_n \tag{86}$$

其中$m=m_n$且$E_n(0)$是集合$\{e_0, \cdots, e_{n-1}\}$的一个很好的缩写。

令$X_n(a)$有N_a个元素，使得$N=|X|=N_0+N_1+\cdots+N_{m-1}$，而且令$Y=X^+$。于是

$$Y_n(a)=(X_n((a-1)\bmod m)+e_n)\cup(X_n(a)+E_n(0))$$

在$n-1$维中是标准的，而且（86）告诉我们

$$N_{m-1}\leqslant \beta N_{m-2}\leqslant N_{m-2}\leqslant\cdots\leqslant N_1\leqslant\beta N_0\leqslant N_0\leqslant\alpha N_0$$

其中α和β指的是坐标1到$n-1$。因此

$$\begin{aligned}|Y|&=|Y_n(0)|+|Y_n(1)|+|Y_n(2)|+\cdots+|Y_n(m-1)|\\&=\alpha N_0+N_0+N_1+\cdots+N_{m-2}=\alpha N_0+N-N_{m-1}\end{aligned}$$

现在定理W的证明有一个漂亮的结论。令$Z=S_N$，并假设$|Z_n(a)|=M_a$，我们要来证明$|X^+|\geqslant|Z^+|$，即

$$\alpha N_0+N-N_{m-1}\geqslant\alpha M_0+N-M_{m-1} \tag{87}$$

因为上一段的论证既可应用到Z也可应用到X。通过证明$N_{m-1}\leqslant M_{m-1}$和$N_0\geqslant M_0$，我们将证明(87)。

使用$(n-1)$维的α和β函数，我们定义

$$N'_{m-1}=N_{m-1}, N'_{m-2}=\alpha N'_{m-1}, \cdots, N'_1=\alpha N'_2, N'_0=\alpha N'_1 \tag{88}$$

$$N''_0=N_0, N''_1=\beta N''_0, N''_2=\beta N''_1, \cdots, N''_{m-1}=\beta N''_{m-2} \tag{89}$$

则对于$0\leqslant a<m$，我们有$N'_a\leqslant N_a\leqslant N''_a$，而且由此得出

$$N'=N'_0+N'_1+\cdots+N'_{m-1}\leqslant N\leqslant N''=N''_0+N''_1+\cdots+N''_{m-1} \tag{90}$$

习题92证明，标准集合$Z'=S_{N'}$恰有N'_a个元素且对于每个a，第n个坐标等于a；而且由α和β之间的对偶性，标准集合$Z''=S_{N''}$类似地恰有N''_a个元素且有第n个坐标a，因此最后

$$M_{m-1}=|Z_n(m-1)|\geqslant |Z'_n(m-1)|=N_{m-1}$$
$$M_0=|Z_n(0)|\leqslant |Z''_n(0)|=N_0$$

因为由(90)，$Z'\subseteq Z\subseteq Z''$。由(81)我们还有$|X^{\circ}|\leqslant|Z^{\circ}|$。 ▌

现在我们已经做好了证明定理K和M的准备，事实上它们是适用于任何多重集合的克莱门兹(Clements)和林德斯特洛姆(Lindström)更一般得多的定理的特殊情况[*J. Combinatorial Theory* 7 (1969)， 230-238]: 24

推论C 如果A是包含在多重集合$U=\{s_0\cdot 0,\ s_1\cdot 1,\cdots,s_d\cdot d\}$中的$N$个$t$多重组合的一个集合，其中$s_0\geqslant s_1\geqslant\cdots\geqslant s_d$，则

$$|\partial A|\geqslant|\partial P_{Nt}|\quad 和\quad |\varrho A|\geqslant|\varrho Q_{Nt}| \tag{91}$$

其中P_{Nt}表示U的N个词典顺序下最小的多重组合$d_t\cdots d_2d_1$，而且Q_{Nt}表示N个词典顺序下最大的。

证明 U的多重组合可以表示为圆环体$T(m_1,\cdots,m_n)$的点$x_1\cdots x_n$，其中$n=d+1$而$m_j=s_{n-j}+1$。我们令x_j为$n-j$的出现次数，这个对应维持词典顺序。例如，如果$U=\{0,0,0,1,1,2,3\}$，则它的3多重组合在词典顺序下为

$$000,100,110,200,210,211,300,310,311,320,321 \tag{92}$$

而且对应点$x_1x_2x_3x_4$为

$$0003,0012,0021,0102,0111,0120,1002,1011,1020,1101,1110 \tag{93}$$

令T_w是有权$x_1+\cdots+x_n=w$的圆环体的点，则t多重组合的每一个允许的集合A是T_t的一个子集。其次，而且这是要点，$T_0\cup T_1\cup\cdots\cup T_{t-1}\cup A$的散布是

$$\begin{aligned}(T_0\cup T_1\cup\cdots\cup T_{t-1}\cup A)^+&=T_0^+\cup T_1^+\cup\cdots\cup T_{t-1}^+\cup A^+\\&=T_0\cup T_1\cup\cdots\cup T_t\cup\varrho A\end{aligned} \tag{94}$$

于是上阴影ϱA只不过是$(T_0\cup T_1\cup\cdots\cup T_{t-1}\cup A)+\cap T_{t+1}$，因而定理W实质上告诉我们$|A|=N$意味着$|\varrho A|\geqslant|\varrho(S_{M+N}\cap T_t)|$，其中$M=|T_0\cup\cdots\cup T_{t-1}|$。因此由交叉次序的定义，$S_{M+N}\cap T_t$由词典顺序下最大的$N$个$t$多重组合（即$Q_{Nt}$）组成。

由取补现在就得出$|\partial A|\geqslant|\partial P_{Nt}|$的证明(参见习题94)。 ▌

习 题

1. [*M23*] 试说明为什么戈罗姆布的规则(8)使所有集合$\{c_1,\cdots,c_t\}\subseteq\{0,\cdots,n-1\}$惟一地对应于多重集合$\{e_1,\cdots,e_t\}\subseteq\{\infty\cdot 0,\cdots,\infty\cdot n-t\}$。

2. [*16*] 在11×13的栅格中什么通路对应于二进位串(13)?

▶ 3. [*21*] (罗·罗·菲尼彻尔(R. R. Fenichel)，1968。)试证明，s分成$t+1$个非负部分的合

成$q_t+\cdots+q_1+q_0$可通过一个简单的无循环算法在词典顺序下生成。

4. [*16*] 试证明，s分成$t+1$个非负部分的每一个合成$q_t\cdots q_0$对应于t分为$s+1$个非负部分的一个合成$r_s\cdots r_0$。在这个对应之下，什么合成对应于10224000001010？

▶ 5. [*20*] 什么是生成下列不等式组所有整数解的一个好方法？

a) $n>x_t \geqslant x_{t-1}>x_{t-2} \geqslant x_{t-3}>\cdots>x_1 \geqslant 0$，当$t$为奇数时。

b) $n>>x_t>>x_{t-1}>>\cdots>>x_2>>x_1>>0$，其中$a>>b$意味着$a \geqslant b+2$。

25 6. [*M22*] 算法T中的每一步骤被执行的频率如何？

7. [*22*] 试设计一个算法，它以递减的词典顺序(参见习题(5)和表1)跑遍“对偶”的组合$b_s\cdots b_2b_1$，和算法T类似，你的算法应当避免多余的赋值和不必要的查找。

8. [*M23*] 试设计一个算法，它以二进位串形式按词典顺序生成所有(s,t)组合$a_{n-1}\cdots a_1a_0$。假设$st>0$，总共的运行时间应是$O\left(\binom{n}{t}\right)$。

9. [*M26*] 当以词典顺序列出所有(s,t)组合$a_{n-1}\cdots a_1a_0$时，令$2A_{st}$是在相邻的串之间位变化的总数。例如，$A_{33}=25$，因为在表1的20个串之间分别有

$$2+2+2+4+2+2+4+2+2+6+2+2+4+2+2+4+2+2+2=50$$

个位的变化。

a) 试证明当$st>0$时$A_{st}=\min(s,t)+A_{(s-1)t}+A_{s(t-1)}$，当$st=0$时$A_{st}=0$。

b) 试证明$A_{st}<2\binom{s+t}{t}$。

▶ 10. [*21*] 垒球的“世界联赛”传统上是这样一个竞赛，其中美国联盟冠军(A)和国家联盟冠军(N)对垒，直到它们中的一个队战胜对方四次。列出所有可能的结局AAAA，AAANA，AAANNA，…，NNNN的一个好方法是什么？什么是对这些结局赋予连续整数的一个简单方法？

11. [*19*] 在20世纪初，习题10中的哪个结局最经常出现？哪个结局从未出现过？[提示：世界联赛的比分可在因特网中很容易地找出。]

12. [*HM32*] 在加法modulo 2之下封闭的n个二进位向量的一个集合V说是一个二进制向量空间。

a) 试证明对于某个整数t，每一个这样的V包含2^t个元素，而且可以表示为集合$\{x_1\alpha_1\oplus\cdots\oplus x_t\alpha_t \mid 0\leqslant x_1,\cdots,x_t\leqslant 1\}$，其中向量$\alpha_1,\cdots,\alpha_t$形成一个具有如下性质的典范基底(canonical basis)：有$\{0,1,\cdots,n-1\}$的t组合$c_t\cdots c_2c_1$，使得如果α_k是二进制向量$a_{k(n-1)}\cdots a_{k1}a_{k0}$，我们有，对于$1\leqslant j,\ k\leqslant t$，$a_{kc_j}=[j=k]$；对于$0\leqslant l<c_k$，$1\leqslant k\leqslant t$，$a_{kl}=0$。

例如，对于$n=9$，$t=4$以及$c_4c_3c_2c_1=7641$的典型基底，有一般形式如下：

$$\alpha_1=*00*0**10$$
$$\alpha_2=*00*10000$$
$$\alpha_3=*01000000$$
$$\alpha_4=*10000000$$

共有2^8种方法通过以0和/或1来代替8个星号，而且这些的每一个都定义一个典范基底。我们称t为V的维数。

b) 对于n个二进位向量，有多少可能的t维空间？

c) 试设计一个算法来生成维数为t的所有典范基底$(\alpha_1, \cdots, \alpha_t)$。提示：令相关联的组合$c_t \cdots c_1$像在算法L中那样按词典顺序增加。

d) 当n=9，t=4时，通过你的算法访问的第1 000 000个基底是什么？

13. [*25*] 长度n、权数t和能源r的一维艾辛(Ising)配置为这样的二进制串$a_{n-1} \cdots a_0$，即$\sum_{j=0}^{n-1} a_j = t$ 和 $\sum_{j=1}^{n-1} b_j = r$ ，其中$b_j = a_j \oplus a_{j-1}$。例如，$a_{12} \cdots a_0$=1100100100011有权数6和 26
能源6，因为$b_{12} \cdots b_1$=010110110010。

给定n, t和r，试设计一个算法来生成所有这样的配置。

14. [*26*] 当以词典顺序生成(s, t)组合的二进制串$a_{n-1} \cdots a_1 a_0$时，我们有时需要改变2 $\min(s, t)$个二进位来从一个组合获得下一个。例如，表1中011100之后是100011，因此我们显然不能希望通过一个无循环算法来生成所有组合，除非我们以某个其他顺序来访问它们。

然而，试证明，如果以如下的双重链接表来表示每个组合，有一个在$O(1)$步内计算一个给定组合的词典顺序的后继的方法：对于$0 \le j \le n$，有数组$l[0], \cdots, l[n]$和$r[0], \cdots, r[n]$使得$l[r[j]]=j$。如果$x_0 = l[0]$且对于$0<j<n$，$x_j = l[x_{j-1}]$，则对于$0 \le j < n$，$a_j = [x_j > s]$。

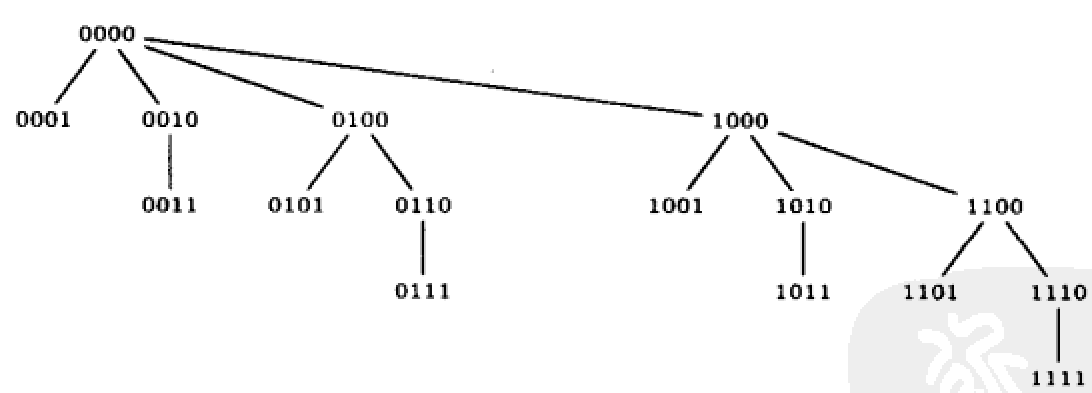
15. [*M22*] 使用以下事实，即对偶组合$b_s \cdots b_2 b_1$以颠倒的词典顺序出现，来证明和$\binom{b_s}{s}+\cdots+\binom{b_2}{2}+\binom{b_1}{1}$ 同和$\binom{c_t}{t}+\cdots+\binom{c_2}{2}+\binom{c_1}{1}$ 有一个简单的关系。

16. [*M21*] 当t是(a)2，(b)3，(c)4，(d)5，(e)1000000时，由算法L生成的第100万个组合是什么？

17. [*HM25*] 给定N和t，计算组合表示(20)的一个好方法是什么？

▶ 18. [*20*] 当像在习题2.3.2-5中那样，以“右儿子”和“左兄弟”指针来表示二项式树T_n时，我们得到什么二进制树？

19.[*21*] 代替像在(22)中所示的那样来对二项式树T_4的分支进行标号，我们可以用它对应组合的二进位串来对每一个节点进行标号。



如果用这种方式已把T_∞标了号，就去掉前导零，前根顺序和二进制记号的通常的递增顺序是一样的；所以第100万个节点结果是11110100001000111111。但在后根顺序下T_∞的第100万个节点是什么？

20. [*M20*] 试求生成函数g和h，使得算法F恰好求$[z^N]g(z)$个能行的组合并且恰好置$t \leftarrow t+1$共$[z^N]h(z)$次。

21. [*M22*] 试证明交替组合律(30)。

22. [*M23*] 当t是(a)2，(b)3，(c)4，(d)5，(e)1000000时，由算法R所访问的第100万个转动门组合是什么？

23. [*M23*] 假设在步骤R1中通过置$j \leftarrow t+1$我们增广算法R，而且如果R3直接转到R2则
27 置$j \leftarrow 1$。试求j的概率分布以及它的平均值。关于算法的运行时间这意味着什么？

▶ 24. [*M25*] (威·哈·佩恩，1974。)继续上一道题，令j_k是由算法R在进行第k次访问时j的值，试证明$|j_{k+1}-j_k| \le 2$，并说明如何通过利用这个性质使这个算法无循环。

25. [*M35*] 令$c_t \cdots c_2 c_1$和$c'_t \cdots c'_2 c'_1$是通过转动门方法，即算法R生成的第N次和第N'次组合。如果集合$C=\{c_t, \cdots, c_2, c_1\}$有$m>0$个不在$C'=\{c'_t, \cdots, c'_2, c'_1\}$中的元素。试证明$|N-N'| > \sum_{k=1}^{m-1}\binom{2k}{k-1}$。

26. [*26*] 如果我们仅抽取这样的n元组$a_{n-1} \cdots a_1 a_0$，使得(a)$a_{n-1}+\cdots+a_1+a_0=t$，(b)$\{a_{n-1}, \cdots, a_1, a_0\}=\{r \cdot 0, s \cdot 1, t \cdot 2\}$，试问三进制反射格雷码是否有类似于转动门格雷码$\Gamma_n$的性质？

▶ 27. [*25*] 仅用类似格雷码的转换$0 \leftrightarrow 1$和$01 \leftrightarrow 10$，试证明有一个简单的方法来生成$\{0, 1, \cdots n-1\}$的至多t个元素的所有组合。(换言之，每一步应是或插入一个元素，或删去一个元素，或移动一个元素± 1次。)例如，当$n=4$和$t=2$时，

$$0000, 0001, 0011, 0010, 0110, 0101, 0100, 1100, 1010, 1001, 1000$$

是一个这样的序列。提示：考虑中国环。

28. [*M21*] 真或假：在位串形式下的(s, t)组合$a_{n-1} \cdots a_1 a_0$的一个表是在广义词典顺序下，当且仅当对应的下标形式表$b_s \cdots b_2 b_1$(对于0而言)和$c_t \cdots c_2 c_1$(对于1而言)都在广义词典顺序下。

▶ 29. [*M28*] (菲·约·蔡斯。)给定符号+，−和0的一个串，说一个R块是以0居前和后边不跟有−的形如$-^{k+1}$的一个子串；一个L块是后边跟着一个0的形如$+-^k$的子串。在两种情况下，都有$k \ge 0$。例如，在格雷码形式下，串 +00++−+++−000− 有两个L块一个R块。注意这些块不能重叠。

每当至少存在一个块时，我们形成这样一个串的后继如下：如果最右块是一个R块，则以$-+^k0$来代替最右的$0-^{k+1}$，否则以$0+^{k+1}$来代替最右的$+-^k0$。如果有的话，还对头一个符号取负，它出现在已被改动的块的右边。例如：

$$-+00++- \to -0+0-+- \to -0+-0-- \to -0+--+0 \to -0+--0+ \to -00+++-$$

这里记号$\alpha \to \beta$意味着β是α的后继。

a) 什么样的串无块(因此无后继)？

b) 能否有一个串的循环使$\alpha_0 \to \alpha_1 \to \cdots \to \alpha_{k-1} \to \alpha_0$？

c) 证明如果$\alpha \to \beta$，则$-\beta \to -\alpha$。这里“−”意味着“对所有符号求负”(因此每一个串至多有一个前驱)。

d) 试证明如果$\alpha_0 \to \alpha_1 \to \cdots \to \alpha_k$且$k>0$，则串$\alpha_0$和$\alpha_k$不能在相同位置中拥有它们的0。(因此如果$\alpha_0$有$s$个符号和$t$个零，则$k$必定小于$\binom{s+t}{t}$)。

e) 证明有s个符号和t个零的每个串α恰属于一个链$\alpha_0 \to \alpha_1 \to \cdots \to \alpha_{\binom{s+t}{t}-1}$。

30. [*M32*] 上一道题通过映射$+ \mapsto 0$，$- \mapsto 0$以及$0 \mapsto 1$，定义了2^s种方法来生成s个0和t个1的所有组合。试证明这些方法中的每一个都是同族的广义词典顺序，可由一个适当的递归式定义。蔡斯序列(37)是否是这个一般构造的一个特殊情况？

31. [*M23*] 在(a)位串形式$a_{n-1} \cdots a_1 a_0$和(b)下标形式$c_t \cdots c_2 c_1$中，有多少个广义词典顺序的
28 (s, t)组合表？

▶ 32. [*M32*] 有多少(s,t)组合串$a_{n-1}\cdots a_1a_0$的广义词典顺序表(a)有转动门性质？(b)是同族的？

33. [*HM33*] 在习题31(b)中的广义词典顺序表中有多少是近乎完美的？

34. [*M32*] 继续习题33，试说明，当s和t都不太大时，在“不完美的”转换$c_j \leftarrow c_j \pm 2$的个数为极小的意义下，说明如何来求尽可能近乎完美的方案？

35. [*M26*] 蔡斯序列C_{st}中有多少步使用不完美的转换？

▶ 36. [*M21*] 给定在位串形式下的组合的任何广义词典顺序下的一个方案，试证明方法(39)在生成所有(s,t)组合$a_{n-1}\cdots a_1a_0$时，恰好执行了$\binom{s+t}{t}-1$次$j \leftarrow j+1$的操作。

▶ 37. [27] 在(a)词典顺序，(b)算法R的转动门次序，以及(c)(31)的同族顺序中，当一般的广义词典顺序方法被用来产生(s,t)组合$a_{n-1}\cdots a_1a_0$时，算法的结果是什么？

38. [26] 对于颠倒的序列C_{st}^R，试设计如同算法C的一个广义词典顺序的算法。

39. [*M21*] 当$s=12$和$t=14$时，在蔡斯序列C_{st}中，居于位串110010010000 11111101101010前的组合有多少？

40. [*M22*] 当$s=12$和$t=14$时，在蔡斯序列C_{st}中的第100万个组合是什么？

41. [*M27*] 试证明有一个非负整数的排列$c(0), c(1), c(2), \cdots$，使得蔡斯序列C_{st}的元素，可通过对于$0 \leqslant k<2^{s+t}$和有权$\nu(c(k))=s$的元素$c(k)$最低有效的$s+t$个二进位取补得到。(作为结果，因此序列$\bar{c}(0),\cdots,\bar{c}(2^n-1)$包含使得$s+t=n$的所有$C_{st}$，就如同格雷二进制码$g(0),\cdots,g(2^n-1)$包含所有转动门序列$\Gamma_{st}$一样。)试说明如何从二进表示$k=(\cdots b_2b_1b_0)_2$来计算二进表示$c(k)=(\cdots a_2a_1a_0)_2$?

42. [*HM*34] 试用形如$\sum_{s,t}g_{st}w^sz^t$的生成函数来分析算法C的每一步。

43. [20] 证明或否定：如果$s(x)$和$p(x)$分别表示在“偶数递减奇”顺序下的x的后继和前驱，则$s(x+1)=p(x)+1$。

▶ 44. [*M21*] 令$C_t(n)-1$表示通过勾销掉具有$c_1=0$的所有组合，然后在剩下的组合中以$(c_t-1)\cdots(c_1-1)$来代替$c_t\cdots c_1$而由$C_t(n)$得到的序列。试证明$C_t(n)-1$是近乎完美的。

45. [32] 利用“偶数递减奇”和在(44)中所勾勒的扩展，来生成使用一个非递归过程的蔡斯序列$C_t(n)$的组合$c_t\cdots c_2c_1$。

▶ 46. [33] 对于蔡斯序列C_{st}的对偶组合$b_s\cdots b_2b_1$，即对于在$a_{n-1}\cdots a_1a_0$中的诸零的位置，试构造一个非递归的算法。

47. [26] 试实现(46)和(47)的近乎完美的多重集合的排列方法。

48. [*M21*] 假设$\alpha_0,\alpha_1,\cdots,\alpha_{N-1}$是多重集合$\{s_1\cdot 1,\cdots,s_d\cdot d\}$的排列的任何表，其中$\alpha_k$和$\alpha_{k+1}$通过交换两个元素而不同。令$\beta_0,\cdots,\beta_{M-1}$是$(s,t)$组合的任何转动门的表，其中$s=s_0$，$t=s_1+\cdots+s_d$，且$M=\binom{s+t}{t}$。则令$\Lambda_j$为以$\alpha_j\uparrow\beta_0$开始，并且应用转动门交换得到的$M$个元素的表；这里$\alpha\uparrow\beta$表示在保持左右次序下，通过以$\alpha$的元素代替$\beta$中的1得到的串。例如，如果$\beta_0,\cdots,\beta_{M-1}$是0110，0101，1100，1001，0011，1010，而且如果$\alpha_j=12$，则Λ_j是0120，0102，1200，1002，0012，1020。(转动门表无须是同族的。) 29

试证明表(47)含$\{s_0\cdot 0, s_1\cdot 1,\cdots,s_d\cdot d\}$的所有排列，而且相邻的排列彼此通过两个元素的交换而不同。

49. [*HM23*] 如果q是1的第m个本原根，例如$e^{2\pi i/m}$，试证明：

$$\binom{n}{k}_q=\binom{\lfloor n/m\rfloor}{\lfloor k/m\rfloor}\binom{n \bmod m}{k \bmod m}_q$$

▶ 50. [*HM25*] 把上题的公式推广到q多项式系数

$$\binom{n_1+\cdots+n_t}{n_1,\cdots,n_t}_q$$

51. [*25*] 试找出下图中的所有哈密顿通路：图的顶点是通过相邻转置而相关的{0, 0, 0, 1, 1, 1}的排列。在0和1交换和/或左右反射的操作下哪些通路等价？

52. [*M37*] 推广定理P，试求“多重集合$\{s_0\cdot 0,\cdots,s_d\cdot d\}$的所有排列可通过相邻转置$a_j a_{j-1}\leftrightarrow a_{j-1}a_j$生成”的必要和充分条件。

53. [*M46*] (德·亨·莱默，1965。)假设$\{s_0\cdot 0,\cdots,s_d\cdot d\}$的$N$个排列不可能由一个完美方案生成，因为它们中的$(N+x)/2$有偶数个反演，其中$x\geqslant 2$。对于$1\leqslant k<N+x-1$，通过$N+x-2$个相邻交换$a_{\delta_k}\leftrightarrow a_{\delta_k-1}$的一个序列，是否有可能来把它们全部生成出来？其中$x-1$个情况是通过把我们带回到我们刚刚见到的排列的$\delta_k=\delta_{k-1}$的“鼓励品”。例如，对于{0, 0, 1, 1, 2, 2}的90个排列的一个适当序列$\delta_1\cdots\delta_{94}$，其中$x=\binom{2+2+2}{2,2,2}_{-1}=6$，如果我们以$a_5a_4a_3a_2a_1a_0$=221100开始，则它是234535432523451α42α^R51α42α^R51α4，其中α=45352542345355。

54. [*M40*] 除了相邻交换$a_j\leftrightarrow a_{j-1}$外，如果我们允许绕末端的交换$a_{n-1}\leftrightarrow a_0$，对于什么样的$s$和$t$值，能够生成所有的$(s,t)$组合？

▶ 55. [*33*] (弗朗克·拉斯基，2004。)(a)证明通过逐次的转动$a_j a_{j-1}\cdots a_0\leftarrow a_{j-1}\cdots a_0 a_j$，可以有效地生成$(s,t)$的所有组合$a_{s+t-1}\cdots a_1 a_0$。(b)当$s+t<64$时，什么MMIX指令将使$(a_{s+t-1}\cdots a_1 a_0)_2$成为它的后继？

56. [*M49*] (巴克(Buck)和威德曼(Wiedemann)，1984。)通过重复地把a_0同某个其他元素相交换，能否生成所有(t,t)组合$a_{2t-1}\cdots a_1 a_0$?

▶ 57. [*22*] (弗朗克·拉斯基。)一次只改变一个手指头，一个钢琴演奏者能否跑遍所有可能的4音符，它至多跨过一个八音度？这是生成所有这样的组合$c_t\cdots c_1$，使得$n>c_t>\cdots>c_1\geqslant 0$和$c_t-c_1<m$的问题，其中$t=4$且(a)如果我们仅考虑一个钢琴的白音符，则$m=8$，$n=52$，(b)如果我们也考虑黑音符，则$m=13$，$n=88$。

58. [*20*] 附加以下条件，考虑习题57的钢琴演奏者问题：弦不涉及相邻的音符。(换言之，对于$t>j\geqslant 1$，$c_{j+1}>c_j+1$。这样的弦趋向于更和谐。)

59. [*M25*] 对于4音符钢琴演奏者问题，是否有一个完美的解？其中每步移动一个手指头到一个相邻的键上。

60. [*23*] 试设计一个算法来生成所有有界的合成

$$t=r_s+\cdots+r_1+r_0,\text{ 其中对于}s\geqslant j\geqslant 0,\text{ 有}0\leqslant r_j\leqslant m_j$$

30 61. [*32*] 试证明，通过在每步中仅改动两个部分，可以生成所有有界的合成。

▶ 62. [*M27*] 一个偶然性表是非负整数(a_{ij})的一个$m\times n$矩阵，并且有给出的行和$r_i=\sum_{j=1}^{n}a_{ij}$与列和$c_j=\sum_{i=1}^{m}a_{ij}$，其中$r_1+\cdots+r_m=c_1+\cdots+c_n$。

a) 试证明$2\times n$偶然性表等价于有界合成。

b) 当以从左到右和从顶到底的行方式来读矩阵的条目时，即以$(a_{11},a_{12},\cdots,a_{1n},a_{21},\cdots,$

a_{mn})的顺序读时，对于($r_1, \cdots, r_m$; $c_1, \cdots, c_n$)，什么是在词典顺序下最大的偶然性表？

c) 当以从顶到底和从左到右的列方式来读矩阵的条目时，即以(a_{11}, a_{21}, $\cdots$, a_{m1}, a_{12}, $\cdots$, a_{mn})的顺序读时，对于($r_1, \cdots, r_m$; $c_1, \cdots, c_n$)，什么是在词典顺序下最大的偶然性表？

d) 在以行方式和以列方式读时，对于 ($r_1, \cdots, r_m$; $c_1, \cdots, c_n$)，什么是在词典顺序下最小的偶然性表？

e) 试说明如何在词典顺序下生成对于($r_1, \cdots, r_m$; $c_1, \cdots, c_n$)的所有偶然性表？

63. [*M41*] 试证明对于($r_1, \cdots, r_m$; $c_1, \cdots, c_n$)，所有偶然性表可以通过在每步中恰好改变矩阵的四个条目来生成。

▶ 64. [*M30*] 只使用变换$*0 \leftrightarrow 0*$，$*1 \leftrightarrow 1*$，$0 \leftrightarrow 1$，对于所有具有s个数字和t个星号的 $2^s\binom{s+t}{t}$ 子立体，试构造一个广义词典顺序的格雷循环。例如，当$s=t=2$时这样的一个循环是

(00**, 01**, 0*1*, 0**1, 0**0, 0*0*, *00*, *01*, *0*1, *0*0, **00, **01, **11, **10, *1*0, *1*1, *11*, *10*, 1*0*, 1**0, 1**1, 1*1*, 11**, 10**)

65. [*M40*] 在习题64中只使用被允许的变换的子立体上，试枚举广义词典顺序格雷通路的总数。这些通路中有多少是循环的？

▶ 66. [*22*] 给定$n > t > 0$，试证明在每步中只改变一个二进位，有一条格雷通路通过习题12的所有典范基底($\alpha_1, \cdots, \alpha_t$)。例如，当$n=3$和$t=2$时，一条这样的通路是

001	101	101	001	001	011	010
010’	010’	110’	110’	100’	100’	100

67. [*46*] 试考虑习题13的艾辛配置，对于它有$a_0=0$。给定n、t和r，对于其中所有变换都有形式$0^k1 \leftrightarrow 10^k$或$01^k \leftrightarrow 1^k0$的这些配置，是否有一个格雷循环？例如，在$n=9$，$t=5$，$r=6$的情况下，有惟一的循环

(010101110, 010110110, 011010110, 011011010, 011101010, 010111010)

68. [*M01*] 如果α是一个t组合，什么是(a)$\partial^t\alpha$？(b) $\partial^{t+1}\alpha$？

▶ 69. [*M22*] t组合的最小集合A，对于它有$|\partial A|<|A|$，A有多大？

70. [*M25*] 对于$N > 0$，$\kappa_t N - N$的极大值是多少？

71. [*M20*] 有100万条边的图能有多少t团组？

▶ 72. [*M22*] 试证明，如果N有度数t的组合表示(57)，则每当$N < \binom{s+t}{t}$时，就存在一个容易的方法来求出补数 $M = \binom{s+t}{t} - N$ 的度数s的组合表示。

73. [*M23*] (安·约·威·希尔顿(A. J. W. Hilton)，1976。)令A是s组合的一个集合，而B是t组合的一个集合，两者都被包含在$U=\{0, \cdots, n-1\}$中，其中$n > s+t$。试证明，在对于所有$\alpha \in A$和$\beta \in B$时$\alpha \cap \beta \neq \phi$ 的意义下，如果A和B是穿插相交的，则在定理K中定义的集合Q_{Mns}和Q_{Nnt}也是，其中$M=|A|$且$N=|B|$。 31

74. [*M21*] 什么是在定理K中的$|\varrho P_{Nt}|$和$|\varrho Q_{Nnt}|$？

75. [*M20*] (60)的右边不总是$\kappa_t N$的度数$(t-1)$的组合表示，因为$v-1$可能为零。然而，试证明，如果我们允许在(57)中有$v=0$，则一个正整数N至多有两个表示，而且按照(60)这两个表示产生相同的$\kappa_t N$的值。因此对于$1 < k < t$

$$\kappa_k\kappa_{k+1}\cdots\kappa_t N=\binom{n_t}{k-1}+\binom{n_{t-1}}{k-2}+\cdots+\binom{n_v}{k-1+v-t}$$

76. [*M20*] 对于$\kappa_t(N+1)-\kappa_t N$，试求一个简单的公式。

▶ 77. [*M26*] 如果不假定定理K，通过二项式系数相乘，试证明κ函数的以下性质：

a) $\kappa_t(M+N)\leqslant\kappa_t M+\kappa_t N$。

b) $\kappa_t(M+N)\leqslant\max(\kappa_t M, N)+\kappa_{t-1}N$。

提示：$\binom{m_t}{t}+\cdots+\binom{m_1}{1}+\binom{n_t}{t}+\cdots+\binom{n_1}{1}$等于$\binom{m_t\vee n_t}{t}+\cdots+\binom{m_1\vee n_1}{1}+\binom{m_t\wedge n_t}{t}+\cdots+\binom{m_1\wedge n_1}{1}$，其中$\vee$和$\wedge$表示极大和极小。

78. [*M22*] 试证明在上道习题中，定理K很容易从不等式(b)得出。反之，两个不等式都是定理K的简单结果。提示：t组合的任何集合A可被写成$A=A_1+A_0 0$，其中$A_1=\{\alpha\in A|0\notin\alpha\}$。

79. [*M23*] 试证明如果$t\geqslant 2$，我们有$M\geqslant\mu_t N$当且仅当$M+\lambda_{t-1}M\geqslant N$。

80. [*HM26*] (拉·罗瓦斯泽(L. Lovász)，1979。)当x从$t-1$增长到∞时，函数$\binom{x}{t}$单调地从0增长到∞；因此我们可以定义

$$\underline{\kappa}_t N=\binom{x}{t-1}\text{，如果}N=\binom{x}{t}\text{和}x\geqslant t-1$$

试证明对于所有整数$t\geqslant 1$和$N\geqslant 0$，$\kappa_t N\geqslant\underline{\kappa}_t N$。提示：当$x$是整数时，等式成立。

▶ 81. [*M27*] 试证明定理M中极小阴影的大小由(64)给出。

82. [*HM31*] 对于$0<x<1$，图27的高木贞治函数由公式

$$\tau(x)=\sum_{k=1}^{\infty}\int_0^x r_k(t)\,dt$$

定义，其中$r_k(t)=(-1)^{\lfloor 2^k t\rfloor}$是等式7.2.1.1-(16)的拉德马彻函数。

a) 试证明$\tau(x)$在区间[0, 1]中是连续的，但它的导数在任何点中不存在。

b) 试证明$\tau(x)$是对于$0<x<1$，满足$\tau\left(\frac{1}{2}x\right)=\tau\left(1-\frac{1}{2}x\right)=\frac{1}{2}x+\frac{1}{2}\tau(x)$的惟一连续函数。

c) 当ϵ很小时，$\tau(\epsilon)$的渐近值为多少？

d) 试证明当x是有理数时，$\tau(x)$是有理数。

e) 试求方程$\tau(x)=1/2$的所有根。

f) 试求方程$\tau(x)=\max_{0\leqslant x\leqslant 1}\tau(x)$的所有根。

83. [*HM46*] 试确定使得方程$\tau(x)=r$有不可数多个解的所有有理数r的集合R。如果$\tau(x)$是
32 有理数而且x是无理数，$\tau(x)\in R$是否为真？(警告：这个问题很容易使人着迷。)

84. [*HM27*] 如果$T=\binom{2t-1}{t}$，对于$0\leqslant N\leqslant T$，试证明近函数

$$\kappa_t N-N=\frac{T}{t}\left(\tau\left(\frac{N}{T}\right)+O\left(\frac{(\log t)^3}{t}\right)\right)$$

85. [*HM21*] 把函数$\lambda_t N$和$\mu_t N$同高木贞治函数$\tau(x)$关联起来。

86. [*M20*] 证明散布/核心对偶性定律，即$X^{-+}=X^{\circ\sim}$。

87. [*M21*] 真或假：(a) $X\subseteq Y^{\circ}$ 当且仅当 $Y^{-}\subseteq X^{-\circ}$；(b) $X^{\circ+\circ}=X^{\circ}$；(c) $\alpha M\leqslant N$当且仅当$M\leqslant\beta N$。

88. [*M20*] 通过完成引理S的证明，来说明为什么交叉顺序是有用的。

89. [*16*] 对于$2\times2\times3$圆环体(69)，计算α和β函数。

90. [*M22*] 试证明基本压缩引理(85)。

91. [*M24*] 对于二维圆环体$T(l,m)$, $l\leqslant m$，证明定理W。

92. [*M28*] 令$x=x_1\cdots x_{n-1}$是圆环体$T(m_1,\cdots,m_{n-1})$的第N个元素，而且令S是在交叉顺序下$\preceq x_1\cdots x_{n-1}(m-1)$的$T(m_1,\cdots,m_{n-1},m)$的所有元素的集合。如果对于$0\leqslant a<m$，$S$的$N_a$个元素有最后的分量$a$，试证明对于$1\leqslant a<m$，$N_{m-1}=N$且$N_{a-1}=\alpha N_a$，其中$\alpha$是在$T(m_1,\cdots,m_{n-1})$中标准集合的散布函数。

93. [*M25*] (a)求这样一个N的值，对于它而言，当参数$m_1, m_2,\cdots,m_n$还未被排序成非递减顺序时，定理W的结论不成立。(b)在这个定理的证明中，什么地方使用$m_1\leqslant m_2\leqslant\cdots\leqslant m_n$的假设？

94. [*M20*] 试证明推论C的∂这半来自于ϱ这半。提示：相对于U，多重组合(92)的补是3211, 3210, 3200, 3110, 3100, 3000, 2110, 2100, 2000, 1100, 1000。

95. [*17*] 说明为什么定理K和定理M由推论C得出。

▶ 96. [*M22*] 如果S是正整数的无穷序列$(s_0,s_1,s_2,\cdots)$，令

$$\binom{S(n)}{k}=[z^k]\prod_{j=0}^{n-1}(1+z+\cdots+z^{s_j})$$

因此如果$s_0=s_1=s_2=\cdots=1$，则$\binom{S(n)}{k}$是通常的二项式系数$\binom{n}{k}$。

推广组合数系统，证明每一个非负整数N有一个惟一的表示

$$N=\binom{S(n_t)}{t}+\binom{S(n_{t-1})}{t-1}+\cdots+\binom{S(n_1)}{1}$$

其中$n_t>n_{t-1}>\cdots>n_1>0$且$\{n_t,n_{t-1},\cdots,n_1\}\subseteq\{s_0\cdot0,s_1\cdot1,s_2\cdot2,\cdots\}$。使用这个表示来给出推论C中数$|\partial P_{Nt}|$的一个简单公式。

▶ 97. [*M26*] 正文中解释说一个凸多边体的顶点可以稍微地扰动，使得它的所有面为单纯形。一般地，包含所有它的元素的阴影的组合的任何集合叫做一个*简单化的复合*；因此C是一个简单化的复合，当且仅当$\alpha\subseteq\beta$而且$\beta\in C$意味着$\alpha\in C$，当且仅当相对于集合包含C是一个理想的序。

当C恰包含大小为t的N_t个组合时，n个顶点上一个简单化的复合C的大小向量是$(N_0,N_1,\cdots,N_n)$。

a) 当五个规则固体(四面体，立方体，八面体，十二面体，二十面体)的顶点稍微被扭动时，它们的大小向量是什么？ 33

b) 对于大小向量(1, 4, 5, 2, 0)，试构造一个简单化的复合。

c) 求使一个给定的大小向量$(N_0,N_1,\cdots,N_n)$为能行的必要和充分条件。

d) 试证明$(N_0, \cdots, N_n)$是能行的，当且仅当它的“对偶”向量$(\overline{N}_0,\cdots,\overline{N}_n)$是能行的，其中我们定义$\overline{N}_t=\binom{n}{t}-N_{n-t}$。

e) 列出所有能行的大小向量$(N_0, N_1, N_2, N_3, N_4)$及它们的对偶，它们中哪些是自对偶的?

98. [*30*] 继续习题97，当$n<100$时，试求计算能行大小向量$(N_0, N_1, \cdots, N_n)$的个数的一个有效方法。

99. [*M*25] 在下列意义下，即$\alpha\subseteq\beta$ 和$\alpha, \beta\in C$，意味着$\alpha=\beta$。一个杂体(clutter)是不相兼容的组合的集合C。一个杂体的大小向量的定义见习题97。

a) 求使$(M_0, M_1, \cdots, M_n)$成为一个杂体的大小向量的必要和充分条件。

b) 在$n=4$的情况下，列出所有这样的大小向量。

▶ 100. [*M30*] (克莱门兹和林德斯特洛姆。) 设A是一个“简单化的多重复合”，即A是在具有性质$\partial A\subseteq A$的推论C中，多重集合U的子多重集合的集合。当$|A|=N$时，总共的权$\nu A=\sum\{|\alpha|\,|\,\alpha\in A\}$可以有多大?

101. [*M*25] 如果$f(x_1, \cdots, x_n)$是一个布尔公式，当每个变量x_j独立地以p的概率为1时，令$F(p)$是$f(x_1, \cdots, x_n)=1$的概率。

a) 对于布尔公式$g(w, x, y, z)=wxz\vee wyz\vee xy\bar{z}$ 和$h(w, x, y, z)=wyz\vee xyz$，计算$G(p)$和$H(p)$。

b) 证明有一个使得$F(p)=G(p)$的单调的布尔函数$f(w, x, y, z)$，但对于$F(p)=H(p)$却没有这样的函数。试说明一般如何测试这个条件。

102. [H*M*35] (弗·索·麦考莱，1927。)变量$\{x_1, \cdots, x_s\}$中的一个*多项式理想I*是在加法同一个常数的乘法，以及同任何变量的乘法操作之下封闭的一个多项式集合。如果它是由一组同族多项式的所有线性组合组成，它称为是同族的，即这些多项式的项全是像$xy+z^2$这样有相同次数。集合N_t是在I中次数为t的线性无关元素的极大个数。例如，如果$s=2$，所有$\alpha(x_0, x_1, x_2)(x_0x_1^2-2x_1x_2^2)+\beta(x_0, x_1, x_2)\,x_0\,x_1\,x_2^2$的集合（其中$\alpha$和$\beta$跑遍$\{x_0, x_1, x_2\}$中所有可能的多项式)，是对于$N_0=N_1=N_2=0$，$N_3=1$，$N_4=4$，$N_5=9$，$N_6=15$，…的同族多项式理想。

a) 证明，对于任何这样的理想I，存在另一个理想I'，其中次数t的所有同族多项式是N_t个无关的*单项式*的线性组合。(一个单项式是变量的乘积，像$x_1^3x_2x_5^4$这样。)

b) 使用定理M和(64)来证明对于所有$t\geq 0$，$N_{t+1}\geq N_t+\kappa_sN_t$。

c) 试证明$N_{t+1}>N_t+\kappa_sN_t$仅对有限多个t出现。(这个命题等价于由戴维·希尔伯特(David Hilbert)在[*Göttinger Nachrichten*(1888), 450-457; *Math. Annalen* **36** (1890)，473-534]所证明的“希尔伯特基础定理”。)

▶ 103. [*M38*] 一个子立体$a_1\cdots a_n$的阴影，其中每个a_j或为0或为1或为*，可通过以0或1代替某个*而得到。例如

$$\partial 0{*}11{*}0=\{0011{*}0, 0111{*}0, 0{*}1100, 0{*}1110\}$$

求一个集合P_{Nst}，使得如果A是有s个数字和t个星号的N个子立体$a_1\cdots a_n$的任何集合，则$|\partial A|\geq$
34 $|P_{Nst}|$。

104. [*M41*] 一个二进制串$a_1\cdots a_n$的阴影通过删去它的一个二进位而得到，例如，

$$\partial 110010010=\{10010010, 11010010, 11000010, 11001000, 11001001\}$$

试求一个集合P_{Nn}，使得如果A是N个二进制串$a_1 \cdots a_n$的任何集合，则$|\partial A| \geq |P_{Nn}|$。

105. [*M20*] 对于$\{0, 1, \cdots, n-1\}$，t组合的一个万有循环是$\binom{n}{t}$个数的这样一个循环，其t个连续元素的块跑遍每个t组合$\{c_1, \cdots, c_t\}$。例如，

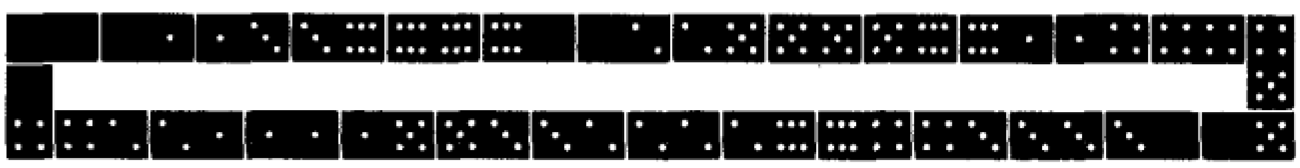
$$(02145061320516243152630425364103546)$$

是当t=3和n=7时的一个万有循环。

试证明，除非$\binom{n}{t}$是n的一个倍数，否则不可能有这样一个循环。

106. [*M21*] (路·庞索特(L. Poinsot)，1809。)对于$\{0, 1, \cdots, 2m\}$，求2组合的一个“优美”的万有循环。提示：考虑连续元素的差mod $(2m+1)$。

107. [22] (奥·特奎姆(O. Terquem)，1849。)庞索特定理意味着，传统的“双重6”集合的所有28个多米诺骨牌可以以一个循环来加以安排，使得相邻的多米诺骨牌上的点彼此匹配。



有多少这样的循环是可能的?

108. [*M31*] 当n mod 3 ≠ 0时，对于集合$\{0, \cdots, n-1\}$，求3组合的万有循环。

109. [*M31*] 当n mod 3 ≠ 0时(即对于允许重复的组合$(d_1 d_2 d_3)$)，求$\{0, 1, \cdots, n-1\}$的3多重组合的万有循环。例如，当n=5时

$$(000122411123302223441333400244401113)$$

是这样一个循环。

▶ 110. [*26*] 积分(原名为格里伯吉(Cribbage))游戏是用52张牌来玩的一种游戏，其中每张牌有黑桃、红桃、梅花、方片四种花色和(A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q和K)这些面值。这种游戏的一个特征是计算5张牌的组合$C=\{c_1, c_2, c_3, c_4, c_5\}$的积分，其中一张牌$c_k$叫做头牌。积分是如下计算的点数之和，对于$C$的每个子集$S$和$k$的每种选择，令$|S|=s$。

i) 15：如果$\sum\{v(c) | c \in S\}=15$，其中$(v(\text{A}), v(2), v(3), \cdots, v(9), v(10), v(\text{J}), v(\text{Q}), v(\text{K}))=(1, 2, 3, \cdots, 9, 10, 10, 10, 10)$积分两点。

ii) 对子：如果s=2，而且两张牌有相同的面值，则积分两点。

iii) 跑点：如果$s \geq 3$，而且面值是连续的，而且如果C不含长度为s+1的跑点，则积s点。

iv) 刷新：如果s=4，而且S的所有牌都有相同花色，而且如果$c_k \notin S$，则积4+[c_k有和其他相同的花色]。

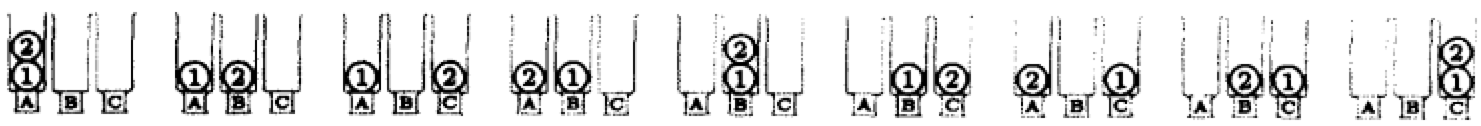
v) 脑袋瓜：如果s=1，而且$c_k \notin S$，如果这张牌是和c_k相同花色的J，则积分为1。

例如，如果你手上掌握{J♣，5♣，5♢，6♡}，而且如果4♣是头牌，对于15，积分为4×2分；对于一个对子，积分为2；对于跑点，积分为2×3；加上对于脑袋瓜的积分1，因此总分为17。

对于x=0, 1, 2, ⋯，问恰好有多少组合和头牌选择导致x点的积分? 35

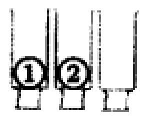
7.2.1.4 生成所有分划

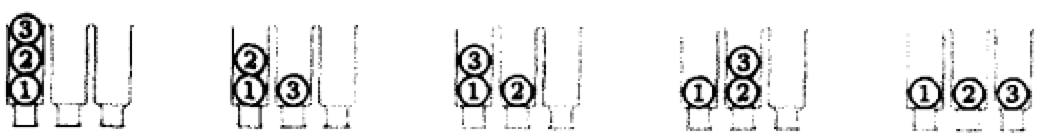
理查德·斯坦利的名著*Enumerative Combinatorics* (1986)以讨论“12种方法”开始，这是在实践中经常出现的基本组合问题的2 × 2 × 3数组（见表1)。所有12个斯坦利的基本问题都可以借助于一个给定个数的球可以被放置到给定个数的罐中的方法来加以描述。例如，如果球和罐都标了号，则把2个球放进3个罐中有9种方法:



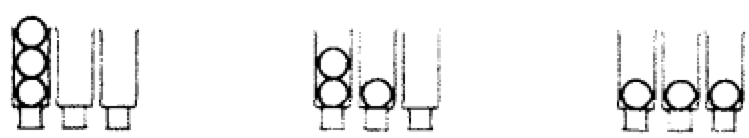
(在罐内，诸球的次序予以忽略)。但如果不对球加标号，则这些安排中的某些个是无区别的，因此只可能有6种方式:


(1)

如果罐未加标号，则像  这样的安排和  实际上相同，因此原来的9种安排中只有2种是可加区别的。如果我们已有3个加了标号的球，则把它们放进3个未加标号的罐中，仅有的不同方法是:


(2)

最后，如果既不对球也不对罐加标号，则这5种可能性就减少到只有3种:


(3)

12种方法考虑当球和罐加标号或不加标号时，以及当罐子可能被有选择地要求至少含一个球或至多含一个球时，所有可能的安排。

表1 12种方法

每罐的球	无限制	≤1	≥1
n个带标号的球 m个带标号的罐	m个事物的n元组	m个事物的n排列	把$\{1, \cdots, n\}$分成m个有序部分的分划
n个未标号的球 m个带标号的罐	m个事物的n个多重组合	m个事物的n组合	把n分成m个有序部分的合成
n个带标号的球 m个未标号的罐	把$\{1, \cdots, n\}$分成≤m的部分的分划	n只鸽子进入m个洞	把$\{1, \cdots, n\}$分成m部分的分划
n个未标号的球 m个未标号的罐	把n分成≤m的部分的分划	n只鸽子进入m个洞	把n分成m部分的分划

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在本章以前的各节中，我们已经学习过n元组、排列、组合以及合成；在表1的12个条目中，有两个是平凡的(即同“鸽子”有关的那些)。因此我们可以通过学

习表中剩下的5个条目来完成对于经典组合数学的研究，这5个都涉及分划。

让我们以承认“分划”一词在数学中有许多意义开始。
任何时候，当把某个对象分成一些小部分时，分划
一词就很可能弹出来。
——乔治·安德鲁斯(George Andrews)，*The theory of Partitions* (1976)

两个十分不同的概念共享同一个名字：一个**集合的分划**是把这个集合分成不相交的子集；因此(2)图示了{1, 2, 3}的5 种分划，即

$$\{1,2,3\},\ \{1,2\}\{3\},\ \{1,3\}\{2\},\ \{1\}\{2,3\},\ \{1\}\{2\}\{3\} \qquad (4)$$

而一个**整数的分划**是把此整数写成为正整数之和的方法，而不理会其顺序；因此(3)图示了3的3种分划，即

$$3,\ 2+1,\ 1+1+1 \qquad (5)$$

我们将把整数分划简单地称作“分划”，而无须任何定性的形容词，而在今后将把另一类叫做“集合分划”以示区别。两种类型的分划都很重要，因此我们将依次来研究它们。

生成一个整数的所有分划。n的一个分划可以形式地定义为非负整数$a_1 \geqslant a_2 \geqslant \cdots$的一个序列，使得$n=a_1+a_2+\cdots$；例如，7的一个分划有$a_1=a_2=3$，$a_3=1$，$a_4=a_5=\cdots=0$。非零项的个数称为**部分的数目**；而为零的项通常被删去。于是当上下文清楚时，我们写7=3+3+1或简单地写331，以节省空间。

生成所有分划的最简单的方法，而且也是最快的方法之一，是以颠倒的词典顺序来访问它们，以“n”开始并以“11⋯1”结尾。例如，当以这个顺序列出时，8的分划是

$$8,\ 71,\ 62,\ 611,\ 53,\ 521,\ 5111,\ 44,\ 431,\ 422,\ 4211,\ 41111,\ 332,\ 3311,\ 3221,\ 32111,\ 311111,\ 2222,\ 22211,\ 221111,\ 2111111,\ 11111111 \qquad (6)$$

如果一个分划不全为1，则对于某个$x \geqslant 1$，它以$x+1$后边跟着0个或多个1结尾；因此在词典顺序下，下一个最小的分划通过对某个适当的剩余$r \leqslant x$，以$x \cdots x\ r$代替后缀$(x+1)1\cdots1$来得到。如同约·基·斯·麦凯(J. K. S. McKay)[*CACM* 13 (1970), 52]所建议的，如果我们记住使$a_q \neq 1$的最大下标q，则这个过程十分有效。 37

算法P(*在颠倒的词典顺序下的分划*) 假设$n \geqslant 1$，这个算法生成所有分划$a_1 \geqslant a_2 \geqslant \cdots \geqslant a_m \geqslant 1$且$a_1+a_2+\cdots+a_m=n$和$1 \leqslant m \leqslant n$。

P1. [初始化。] 对于$n \geqslant m>1$，置$a_m \leftarrow 1$。然后置$a_0 \leftarrow 0$和$m \leftarrow 1$。

P2. [存最后部分。] 置$a_m \leftarrow n$和$q \leftarrow m-[n=1]$。

P3. [访问。] 访问分划$a_1 a_2 \cdots a_m$。如果$a_q \neq 2$则转到P5。

P4. [把2变成1+1。] 置$a_q \leftarrow 1$，$q \leftarrow q-1$，$m \leftarrow m+1$，并返回P3（在这一点对于$q<k \leqslant n$，我们有$a_k=1$。）

P5. [减a_q。] 如果q=0则结束此算法。否则置$x \leftarrow a_q - 1$，$a_q \leftarrow x$，$n \leftarrow m - q + 1$，以及$m \leftarrow q + 1$。

P6. [如有必要则复制x。] 如果$n \leqslant x$，则返回步骤P2。否则置$a_m \leftarrow x$，$m \leftarrow m + 1$，$n \leftarrow n - x$，并重复本步骤。 ▌

注意如果存在一个2，则从一个分划转到另一个分划的操作是容易的；然后步骤P4就只不过把最右的2改变成一个1和在右边附加另一个1而已。幸而，这令人高兴的情况是最普通的。例如，当n=100时，在所有分划中的79%都包含一个2。

当我们要生成把n分成固定个数的部分的所有分划时，可利用另一个简单的算法。以下的方法，它在卡·弗·辛登伯格(C. F. Hindenburg)18世纪的论文中起重要作用[*Infinitinomii Dignitatum Exponentis Indeterminati* (Göttingen, 1779), 73-91]，以协词典顺序，即在反射序列$a_m \cdots a_2\ a_1$的词典顺序下访问分划。

算法H(分划成m部分) 假定$n \geqslant m \geqslant 2$，这个算法生成所有整数$m$元组$a_1 \cdots a_m$，使得$a_1 \geqslant \cdots \geqslant a_m \geqslant 1$且$a_1 + \cdots + a_m = n$。

H1. [初始化。] 置$a_1 \leftarrow n - m + 1$且对于$1 < j \leqslant m$，$a_j \leftarrow 1$。还置$a_{m+1} \leftarrow -1$。

H2. [访问。] 访问分划 $a_1 \cdots a_m$。然后如果$a_2 \geqslant a_1 - 1$，则转到H4。

H3. [扭动a_1和a_2。] 置$a_1 \leftarrow a_1 - 1$和$a_2 \leftarrow a_2 + 1$，并返回H2。

H4. [求j。] 置$j \leftarrow 3$和$s \leftarrow a_1 + a_2 - 1$，然后如果$a_j \geqslant a_1 - 1$，则置$s \leftarrow s + a_j$，$j \leftarrow j + 1$，并且重复直到$a_j < a_1 - 1$为止。(现在$s = a_1 + \cdots + a_{j-1} - 1$。)

H5. [增加a_j。] 如果$j > m$，则结束。否则置$x \leftarrow a_j + 1$，$a_j \leftarrow x$，$j \leftarrow j - 1$。

H6. [扭动$a_1 \cdots a_j$。] 当$j > 1$时，置$a_j \leftarrow x$，$s \leftarrow s - x$以及$j \leftarrow j - 1$，最后置$a_1 \leftarrow s$，并且返回H2。 ▌

例如，当n=11和m=4时，所访问的逐次的分划为

$$8111, 7211, 6311, 5411, 6221, 5321, 4421, 4331, 5222, 4322, 3332 \qquad (7)$$

基本思想是，协词典顺序通过找使得a_j可被增加的最小的j开始，而不改动$a_{j+1} \cdots a_m$，由此从一个分划$a_1 \cdots a_m$进行到下一个分划。新的分划$a_1' \cdots a_m'$将有$a_1' \geqslant \cdots \geqslant a_j' = a_j + 1$

38 以及$a_1' + \cdots + a_j' = a_1 + \cdots + a_j$，而且这些条件可实现当且仅当$a_j < a_1 - 1$。其次，在协词典顺序下最小的这样的分划$a_1' \cdots a_m'$有$a_2' = \cdots = a_j' = a_j + 1$。

步骤H3处理简单情况j=2，它是迄今为止最为普通的。而且其实，j的值总是十分小的；后边我们将证明，算法H的总的运行时间至多是一个小常数乘以所访问的分划数，加上$O(m)$。

分划的其他表示。我们已经把分划定义为非负整数的一个序列$a_1 a_2 \cdots$且有$a_1 \geqslant a_2 \geqslant \cdots$和$a_1 + a_2 + \cdots = n$，但我们也可以把它当作非负整数的一个元组$c_1 c_2 \cdots c_n$，使得

$$c_1 + 2c_2 + \cdots + nc_n = n \qquad (8)$$

这里c_j是整数j在序列$a_1 a_2 \cdots$中出现的次数；例如，分划331对应于c_1=1，c_2=0，c_3=2，$c_4 = c_5 = c_6 = c_7 = 0$。部分的数目则为$c_1 + c_2 + \cdots + c_n$。类似于算法P的一个过程可以容易地

设计出来，以生成在部分计数形式下的分划。

我们已经在诸如等式1.2.9- (38)这样的公式中看到含蓄的部分计数表示，它把对称函数

$$h_n = \sum_{N > d_n > \cdots > d_2 > d_1 > 1} x_{d_1} x_{d_2} \cdots x_{d_n} \tag{9}$$

表示为

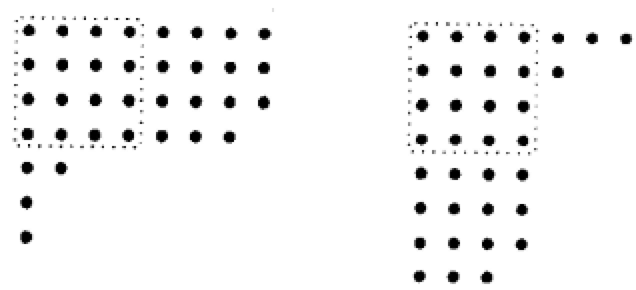
$$\sum_{\substack{c_1, c_2, \cdots, c_n > 0 \\ c_1+2c_2+\cdots+n c_n = n}} \frac{S_1^{c_1}}{1^{c_1} c_1!} \frac{S_2^{c_2}}{2^{c_2} c_2!} \cdots \frac{S_n^{c_n}}{n^{c_n} c_n!} \tag{10}$$

其中S_j是对称函数$x_1^j + x_2^j + \cdots + x_N^j$。(9) 中的和实际上是取遍$N$的所有$n$多重组合，而(10)中的和是对于$n$的所有分划来取的。因此，例如，$h_3 = \frac{1}{6}S_1^3 + \frac{1}{2}S_1S_2 + \frac{1}{3}S_3$，而且当$N$=2时，我们有

$$x^3 + x^2y + xy^2 + y^3 = \frac{1}{6}(x+y)^3 + \frac{1}{2}(x+y)(x^2+y^2) + \frac{1}{3}(x^3+y^3)$$

对分划的其他和出现于习题1.2.5-21，1.2.9-10，1.2.9-11，1.2.10-12等当中。由于这个原因，因此分划在对称函数——即一般地在数学中常见的一类函数——的研究中十分重要。[理查德 · 斯坦利所著的*Enumerative Combinatorics* 2 (1999)的第7章对对称函数理论的高级方面作了精彩的介绍。]

通过考虑n个点的一个数组，并且在顶部行有a_1个点，在下一行有a_2个点，等等，可以用吸引人的方式来想像分划。诸点的这样一种安排称为分划的**费尔利斯框图**，以纪念诺 · 麦 · 费尔利斯(N. M. Ferrers) [参见*Philosophical Mag*. 5 (1853), 199-202]。而它所包含的点的最大平方子数组叫做**德尔菲方块**，以纪念威 · 皮 · 德尔菲(W. P. Durfee)[参见*Johns Hopkins Univ. Circular* 2 (1882年12月)， 23]。例如，8887211的费尔利斯框图连同它的4 × 4德尔菲方块如图28a所示。 39



a) 8887211　　b) 75444443

图28 两个共轭分划的费尔利斯框图及德尔菲方块

当k是使得$a_k > k$的最大下标时，德尔菲方块包含k^2个点，我们可以把k叫做分划的**迹** (trace)。

如果α是任何分划$a_1 a_2 \cdots$，它的**共轭**$\alpha^T = b_1 b_2 \cdots$是通过把对应的费尔利斯框图的行和列转置而得到。例如，图28b示出，$(8887211)^T$=75444443。当$\beta=\alpha^T$时，我们显然有$\alpha=\beta^T$。分划β有a_1部分，α有b_1部分。确实，在α的部分计数表示$c_1 \cdots c_n$和共轭分划$b_1 b_2 \cdots$之间有一个简单表示，即对于所有$j > 1$，

$$b_j - b_{j+1} = c_j \tag{11}$$

这个关系使得计算一个给定分划的共轭，或者通过观察来写下它变得很容易(参见习题6)。

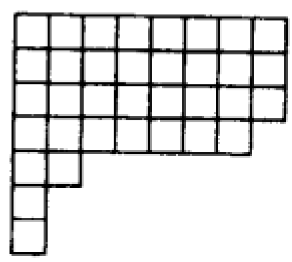
共轭的思想通常说明了看似十分神秘的分划的性质。例如，既然我们知道α^T的定义，我们就很容易地看到，在算法H的步骤H5的$j-1$值，刚好是共轭分划$(a_1\cdots a_m)^T$的第二个最小的部分。因此，在步骤H4和H6中需要加以完成的平均工作量，实质上和最大部分为m的一个随机分划的第二个最小部分的平均大小成正比。因此下边我们将看到，第二个最小部分几乎总是十分小的。

而且，算法H以它们的共轭的词典顺序来产生这些分划。例如，(7)的共轭分别为

$$\begin{array}{l} 41111111\ \ 4211111,\ \ 422111,\ \ 42221,\ \ 431111, \\ 43211,\ \ 4322,\ \ 4331,\ \ 44111,\ \ 4421,\ \ 443 \end{array} \tag{12}$$

这些是n=11且最大部分为4的分划。生成n的所有分划的一种方法是以平凡分划“n”开始，然后依次对于m=2, 3, ⋯, n来运行算法H；这个过程以α^T的词典顺序产生所有α(参见习题7)。因此算法H可以被认为是算法P的对偶。

至少有一种更有用的表示分划的方法，它叫做*边缘表示*（rim representation）。假设以盒子代替费尔利斯框图的点，由此得到如同我们在5.1.4节所做的一个表景形状；例如。图28a中的分划8887211变成为



(13)

40

这个形状右边的边界可以看做一个$n\times n$的方块从左下角到右上角的一条通路。而且我们从表7.2.1.3-1知道，这样一条通路对应于一个(n, n)组合。

例如，(13)对应于70个二进位的串：

$$0\cdots 01001011111010001\cdots 1=0^{28}1^{1}0^{2}1^{1}0^{1}1^{5}0^{1}1^{1}0^{3}1^{27} \tag{14}$$

其中我们在开始处放置足够的0，在末尾放置足够多的1来使每一个串恰好有n个二进位。0表示这条通路向上的步骤，而1表示向右的步骤。容易看出，以这种方式定义的二进位串恰好有n个反演。反之，恰好有n个反演的多重集合$\{n\cdot 0, n\cdot 1\}$的每个排列对应于n的一个分划。当分划有t个不同部分时，它的二进位串就可以以如下形式

$$0^{n-q_1-q_2-\cdots-q_t}1^{p_1}0^{q_1}1^{p_2}0^{q_2}\cdots 1^{p_t}0^{q_t}1^{n-p_1-p_2-\cdots-p_t} \tag{15}$$

写出，其中p_j和q_j是正整数。于是分划的标准表示是

$$a_1a_2\cdots=(p_1+\cdots+p_t)^{q_t}(p_1+\cdots+p_{t-1})^{q_{t-1}}\cdots(p_1)^{q_1} \tag{16}$$

即在我们的例子中$(1+1+5+1)^3(1+1+5)^1(1+1)^1(1)^2$=8887211。

分划的个数。1740年，列昂哈德·欧拉(Leonhard Euler)受到菲律浦·诺德

(Philipp Naudé)向他提出的一个问题的触动，写了两篇基础性的论文，其中他通过研究它们的生成函数来计算各种类型的分划的个数[*Commentarii Academiæ Scientiarum Petropolitanæ* **13** (1741)，64-93; *Novi Comment. Acad. Sci. Pet.* **3** (1750)，125-169]。他发现，在无穷乘积

$$(1+z+z^2+\cdots+z^j+\cdots)(1+z^2+z^4+\cdots+z^{2k}+\cdots)(1+z^3+z^6+\cdots+z^{3l}+\cdots)\cdots$$

中，z^n 的系数是方程$j+2k+3l+\cdots=n$的非负整数解的个数；而且$1+z^m+z^{2m}+\cdots$是$1/(1-z^m)$。因此如果我们写

$$P(z)=\prod_{m=1}^{\infty}\frac{1}{1-z^m}=\sum_{n=0}^{\infty}p(n)z^n \tag{17}$$

则n的分划数是$p(n)$。函数$P(z)$被证实具有微妙的数学性质。

例如，欧拉发现，当把$P(z)$的分母乘出时，就出现大量的删除:

$$\begin{aligned}(1-z)(1-z^2)(1-z^3)\cdots&=1-z-z^2+z^5+z^7-z^{12}-z^{15}+z^{22}+z^{26}-\cdots\\&=\sum_{-\infty<n<\infty}(-1)^n z^{(3n^2+n)/2}\end{aligned} \tag{18}$$

基于费尔利斯框图，这个著名的恒等式的一个组合证明出现在习题5.1.1-14中；我们通过在更加著名的雅可比恒等式 41

$$\prod_{k=1}^{\infty}(1-u^k v^{k-1})(1-u^{k-1}v^k)(1-u^k v^k)=\sum_{n=-\infty}^{\infty}(-1)^n u^{\binom{n}{2}}v^{\binom{-n}{2}} \tag{19}$$

中置$u=z$和$v=z^2$，也可以证明它，因为右边变成为 $\prod_{k=1}^{\infty}(1-z^{3k-2})(1-z^{3k-1})(1-z^{3k})$ ；参见习题5.1.1-20。欧拉恒等式(18)意味着分划个数满足递归式:

$$p(n)=p(n-1)+p(n-2)-p(n-5)-p(n-7)+p(n-12)+p(n-15)-\cdots \tag{20}$$

由此我们可以比在(17)中执行幂级数计算更快速地计算它们的值

$n=$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$p(n)=$	1	1	2	3	5	7	11	15	22	30	42	56	77	101	135	176

我们从1.2.8节知道，斐波那契递归式$f(n)=f(n-1)+f(n-2)$的解以指数方式增长，而且当$f(0)$和$f(1)$为正时，有$f(n)=\Theta(\phi^n)$。然而，在(20)中的附加的项"$-p(n-5)-p(n-7)$"对分划数有一个压制的效果；事实上，如果在那里停止递归，则得到的序列将在正值和负值之间振荡。进一步的项"$+p(n-12)+p(n-15)$"重新恢复指数增长。

对于某个常数A，$p(n)$的实际增长率结果是有阶 $A^{\sqrt{n}}/n$ 。例如，习题33直接证明$p(n)$至少和 $e^{2\sqrt{n}}/n$ 的增长一样快。而且一个相当容易的获取合适的上界的方法是在（17）中取对数，

$$\ln P(z)=\sum_{m=1}^{\infty}\ln\frac{1}{1-z^m}=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{z^{mn}}{n} \tag{21}$$

而后通过置$z=e^{-t}$来观察靠近$z=1$处的特性。

$$\ln P(e^{-t})=\sum_{m,n\geq 1}\frac{e^{-mnt}}{n}=\sum_{n\geq 1}\frac{1}{n}\frac{1}{e^{tn}-1}<\sum_{n\geq 1}\frac{1}{n^2t}=\frac{\zeta(2)}{t} \tag{22}$$

结果，由于$p(n)\leq p(n+1)<p(n+2)<\cdots$和$e^t>1$，对于所有$t>0$，我们有

$$\frac{p(n)}{1-e^{-t}}<\sum_{k=0}^{\infty}p(k)e^{(n-k)t}=e^{nt}P(e^{-t})<e^{nt+\zeta(2)/t} \tag{23}$$

置$t=\sqrt{\zeta(2)/n}$给出

$$p(n)<Ce^{2C\sqrt{n}}/\sqrt{n},\text{其中 }C=\sqrt{\zeta(2)}=\pi/\sqrt{6} \tag{24}$$

通过使用欧拉的求和公式(1.2.11.2小节)或默林变换(5.2.2节)，我们可以得到关于$P(e^{-t})$的大小更精确的信息；参见习题25。但迄今我们所见到的诸方法都不够强有力来导出$P(e^{-t})$的精确特性，因此，现在是我们在自己的技术武库中增加一种新
42 武器的时候了。

欧拉的生成函数$P(z)$很适合于泊松的求和公式[*J. École Royale Polytechnique* 12 (1823)，404-509，§63]，根据这个公式，每当f是一个"特性良好"的函数时，

$$\sum_{n=-\infty}^{\infty}f(n+\theta)=\lim_{M\to\infty}\sum_{m=-M}^{M}e^{2\pi mi\theta}\int_{-\infty}^{\infty}e^{-2\pi miy}f(y)dy \tag{25}$$

这个公式是以下列事实为基础的，即左边是θ的一个周期函数，而右边是该函数的一个傅里叶级数展开。如果，比如说$\int_{-\infty}^{\infty}|f(y)|dy<\infty$，而且或者

i) $f(n+\theta)$是对于某个$\varepsilon>0$和$0\leq\Re\theta\leq 1$，在区域$|\Im\theta|\leq\varepsilon$中复变量$\theta$的一个解析函数，而且左半边在该矩形中一致收敛；或者

ii) 对所有实数θ，$f(\theta)=\frac{1}{2}\lim_{\varepsilon\to 0}(f(\theta-\varepsilon)+f(\theta+\varepsilon))=g(\theta)-h(\theta)$，其中$g$和$h$是单调递增的，而且$g(\pm\infty)$和$h(\pm\infty)$是有限的。

则函数f是充分优美的。[参见彼得·亨利奇(Peter Henrici)，*Applied and Computational Complex Analysis* 2 (New York:Wiley, 1977)，定理10.6.2]。对各种求和问题，泊松公式并非是一种万能药，但如同我们将见到的那样，当它确实可应用时，结果却可以很漂亮。

为了"完成平方"，我们以$z^{1/24}$来乘欧拉公式(18):

$$\frac{z^{1/24}}{P(z)}=\sum_{n=-\infty}^{\infty}(-1)^n z^{\frac{3}{2}(n+\frac{1}{6})^2} \tag{26}$$

于是对于所有$t>0$，我们有$e^{-t/24}/P(e^{-t})=\sum_{n=-\infty}^{\infty}f(n)$，其中

$$f(y)=e^{-\frac{3}{2}t(y+\frac{1}{6})^2}\cos\pi y \tag{27}$$

而且在上述准则(i)和(ii)之下，这个函数f 对于泊松求和公式来说是合格的。因此我们可以尝试来对$e^{-2\pi miy}f(y)$进行积分，而且对于m=0，结果是

$$\int_{-\infty}^{\infty}f(y)dy=\sqrt{\frac{\pi}{2t}}e^{-\pi^2/6t} \tag{28}$$

我们必须往这公式加上

$$\sum_{m=1}^{\infty}\int_{-\infty}^{\infty}(e^{2\pi miy}+e^{-2\pi miy})f(y)dy=2\sum_{m=1}^{\infty}\int_{-\infty}^{\infty}f(y)\cos 2\pi my\,dy \tag{29}$$

再一次，这个积分证实是可进行的。而且这些结果(参见习题27)十分漂亮地合并在一起，并给出

$$\frac{e^{-t/24}}{P(e^{-t})}=\sqrt{\frac{2\pi}{t}}\sum_{n=-\infty}^{\infty}(-1)^n e^{-6\pi^2(n+\frac{1}{6})^2/t}=\sqrt{\frac{2\pi}{t}}\frac{e^{-\pi^2/6t}}{P(e^{-4\pi^2/t})} \tag{30}$$

令人惊讶!我们已经证明了关于$P(z)$的另一个重要事实: 43

定理D 当$\Re t>0$时，对于分划的生成函数(17)满足函数关系

$$\ln P(e^{-t})=\frac{\zeta(2)}{t}+\frac{1}{2}\ln\frac{t}{2\pi}-\frac{t}{24}+\ln P(e^{-4\pi^2/t}) \tag{31}$$

这个定理是由理查德·迪德金(Richard Dedekind)[*Crelle* 83 (1877)，265-292，§6]发现的。当$z=e^{2\pi i\tau}$时，他写$\eta(\tau)$表示函数$z^{1/24}/P(z)$。他的证明是以更复杂得多的椭圆函数理论为基础的。注意，当t是一小的正数时，$\ln P(e^{-4\pi^2/t})$极端地小；例如，当t=0.1时，我们有$\exp(-4\pi^2/t)\approx 3.5\times 10^{-172}$。因此定理D实际上告诉我们当$z$接近于1时，关于$P(z)$的值我们需要知道的每件事情。

戈·哈·哈迪(G. H. Hardy)和斯·拉曼奴燕(S. Ramanujan)使用这个事实，来推导对于很大的n的$p(n)$的渐近特性，而且他们的工作在许多年之后由汉斯·拉德曼彻进行了推广，他发现了不仅渐近而且收敛的一个级数[*Proc. London Math. Soc.* (2)17 (1918)，75-115; 43(1937)，241-254]。哈迪、拉曼奴燕及拉德曼彻提出的关于$p(n)$的公式肯定是已经发现的最惊人的恒等式之一；它指出

$$p(n)=\frac{\pi}{2^{5/4}3^{3/4}(n-1/24)^{3/4}}\sum_{k=1}^{\infty}\frac{A_k(n)}{k}I_{3/2}\left(\sqrt{\frac{2}{3}}\frac{\pi}{k}\sqrt{n-1/24}\right) \tag{32}$$

这里$I_{3/2}$表示修改过的球面贝塞尔函数

$$I_{3/2}(z)=\left(\frac{z}{2}\right)^{3/2}\sum_{k=0}^{\infty}\frac{1}{\Gamma(k+5/2)}\frac{(z^2/4)^k}{k!}=\sqrt{\frac{2z}{\pi}}\left(\frac{\cosh z}{z}-\frac{\sinh z}{z^2}\right) \tag{33}$$

而且系数$A_k(n)$由公式

$$A_k(n)=\sum_{h=0}^{k-1}[h\perp k]\exp\left(2\pi i\left(\frac{\sigma(h,k,0)}{24}-\frac{nh}{k}\right)\right) \tag{34}$$

定义，其中$\sigma(h,k,0)$是由等式3.3.3- (16)中定义的迪德金和。我们有

$$A_1(n)=1,\qquad A_2(n)=(-1)^n,\qquad A_3(n)=2\cos\frac{(24n+1)\pi}{18} \tag{35}$$

而且一般地，$A_k(n)$位于$-k$和k之间。

公式(32)的一个证明将使我们离题太远了，但基本思想是使用在7.2.1.5节中讨论的“马鞍点方法”。对于$k=1$的项，是当z接近1时由$P(z)$的特性推导出来的，第二项是当z接近于-1时由特性导出的，其中类似于(31)的一个转换可加以应用。一般地说，(32)的第k项考虑了对于带有分母k的不可约分数h/k，当z趋近$e^{2\pi ih/k}$时$P(z)$的动作方式；1的每第k个根是在对于$P(z)$的无穷乘积中，分数$1/(1-z^k)$，$1/(1-z^{2k})$，$1/(1-z^{3k})$，
44 …中每一个的因子。

如果我们只想要一个粗略的近似，则(32)的前导项可大为简化

$$p(n)=\frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}(1+O(n^{-1/2})) \tag{36}$$

或者，如果我们选择保留多一些细节，则有

$$p(n)=\frac{e^{\pi\sqrt{2n'/3}}}{4n'\sqrt{3}}\left(1-\frac{1}{\pi}\sqrt{\frac{3}{2n'}}\right)\left(1+O(e^{-\pi\sqrt{n/6}})\right),\quad n'=n-\frac{1}{24} \tag{37}$$

例如，$p(100)$有精确值190 569 292；公式(36)告诉我们$p(100)\approx 1.993\times 10^8$，而(37)给出更好得多的估计190 568 944.783。

安德鲁·奥德里兹科(Andrew Odlyzko)已经发现，当n很大时，哈迪-拉曼奴燕-拉德曼彻公式实际上给出了计算$p(n)$的精确值的一个近乎最优的方法，因为算术运算可以在接近于$O(\log p(n))=O(n^{1/2})$步内来进行。公式(32)最初的一些项作出主要贡献，然后这个级数就设置阶为$k^{-3/2}$的一些项以及通常为k^{-2}的一些项。其次，大约一半的$A_k(n)$系数结果为零(参见习题28)。例如，当$n=10^6$时，对于$k=1$、2和3的项分别是$\approx 1.47\times 10^{1107}$，$1.23\times 10^{550}$和$-1.23\times 10^{364}$。前250项之和是$\approx$ 1471684986…73818.01，而真正的值是1471684986…73818；这250项中的123项为零。

部分的个数。对于恰好有m个部分的n的分划的个数，引进记号

$$\left|\begin{matrix}n\\m\end{matrix}\right| \tag{38}$$

是方便的。于是对于所有整数m和n，递归式

$$\left|\begin{matrix}n\\m\end{matrix}\right|=\left|\begin{matrix}n-1\\m-1\end{matrix}\right|+\left|\begin{matrix}n-m\\m\end{matrix}\right| \tag{39}$$

成立，因为$\left|\begin{matrix}n-1\\m-1\end{matrix}\right|$计算了其最小部分为1的分划而$\left|\begin{matrix}n-m\\m\end{matrix}\right|$计算了其他部分。(如果最小部分为2或更多，我们可从每个部分减1，并且得到$n-m$分成m部分的一个分划。)通过类似的推理我们可以得出结论，$\left|\begin{matrix}m+n\\m\end{matrix}\right|$是$n$至多分成$m$个部分的分划的个数，即分成$m$个非负的求和项。通过考虑费尔利斯框图，我们还知道，$\left|\begin{matrix}n\\m\end{matrix}\right|$是其最大部分为$m$的$n$的分划个数，因此$\left|\begin{matrix}n\\m\end{matrix}\right|$是应知道的一个好的数。边界条件为，对于$m<0$或$n<0$

$$\left|\begin{matrix}n\\0\end{matrix}\right| = \delta_{n0} \text{ 和 } \left|\begin{matrix}n\\m\end{matrix}\right| = 0 \tag{40}$$

这使得对于小的参数值来造$\left|\begin{matrix}n\\m\end{matrix}\right|$的表很容易，因此我们得到对于$\binom{n}{m}$、$\left[\begin{matrix}n\\m\end{matrix}\right]$、$\left\{\begin{matrix}n\\m\end{matrix}\right\}$和$\left\langle\begin{matrix}n\\m\end{matrix}\right\rangle$熟悉的三角，这些我们以前都已见过了；参见表2。生成函数为

$$\sum_n \left|\begin{matrix}n\\m\end{matrix}\right| z^n = \frac{z^m}{(1-z)(1-z^2)\cdots(1-z^m)} \tag{41}$$ 45

表2 分划数

n	$\left\|\begin{matrix}n\\0\end{matrix}\right\|$	$\left\|\begin{matrix}n\\1\end{matrix}\right\|$	$\left\|\begin{matrix}n\\2\end{matrix}\right\|$	$\left\|\begin{matrix}n\\3\end{matrix}\right\|$	$\left\|\begin{matrix}n\\4\end{matrix}\right\|$	$\left\|\begin{matrix}n\\5\end{matrix}\right\|$	$\left\|\begin{matrix}n\\6\end{matrix}\right\|$	$\left\|\begin{matrix}n\\7\end{matrix}\right\|$	$\left\|\begin{matrix}n\\8\end{matrix}\right\|$	$\left\|\begin{matrix}n\\9\end{matrix}\right\|$	$\left\|\begin{matrix}n\\10\end{matrix}\right\|$	$\left\|\begin{matrix}n\\11\end{matrix}\right\|$
0	1	0	0	0	0	0	0	0	0	0	0	0
1	0	1	0	0	0	0	0	0	0	0	0	0
2	0	1	1	0	0	0	0	0	0	0	0	0
3	0	1	1	1	0	0	0	0	0	0	0	0
4	0	1	2	1	1	0	0	0	0	0	0	0
5	0	1	2	2	1	1	0	0	0	0	0	0
6	0	1	3	3	2	1	1	0	0	0	0	0
7	0	1	3	4	3	2	1	1	0	0	0	0
8	0	1	4	5	5	3	2	1	1	0	0	0
9	0	1	4	7	6	5	3	2	1	1	0	0
10	0	1	5	8	9	7	5	3	2	1	1	0
11	0	1	5	10	11	10	7	5	3	2	1	1

n的几乎所有分划都有$\Theta(\sqrt{n}\log n)$个部分。由鲍·厄尔多斯和约·勒纳所发现的[*Duke Math. J.* 8 (1941), 335-345]这一事实，有一个非常有教益的证明：

定理E 令$C=\pi/\sqrt{6}$，并令$m=\frac{1}{2C}\sqrt{n}\ln n + x\sqrt{n} + O(1)$，则对于所有$\varepsilon>0$和所有固定的$x$，当$n\to\infty$时，

$$\frac{1}{p(n)}\left|\begin{matrix}m+n\\m\end{matrix}\right| = F(x)(1+O(n^{-1/2+\varepsilon})) \tag{42}$$

其中

$$F(x) = e^{-e^{-Cx}}/C \tag{43}$$

当$x\to-\infty$时，这个函数$F(x)$非常快速地趋于0，而当$x\to+\infty$时，它快速地增加到1；所以它是一个概率分布函数。图29b表明，对应的密度函数$f(x)=F'(x)$大部分

密集于区域$-2\leqslant x\leqslant 4$中。图29a示出，当n=100时$\left|{n \atop m}\right| = \left|{m+n \atop m}\right| - \left|{m-1+n \atop m-1}\right|$的值，以供比较，在此情况下，$\frac{1}{2C}\sqrt{n}\ln n \approx 18$。

证明 我们将用$\left|{m+n \atop m}\right|$是$n$（其最大部分$\leqslant m$)的分划个数这一事实。于是，由容斥原理，即等式1.3.3-(29)，我们有

$$\left|{m+n \atop m}\right| = p(n) - \sum_{j>m} p(n-j) + \sum_{j_2>j_1>m} p(n-j_1-j_2) - \sum_{j_3>j_2>j_1>m} p(n-j_1-j_2-j_3) + \cdots$$

因为$p(n-j_1-\cdots-j_r)$是至少使用每个部分$\{j_1,\cdots,j_r\}$一次的n的分划个数。让我们把这写作

46 $$\frac{1}{p(n)}\left|{m+n \atop m}\right| = 1-\Sigma_1+\Sigma_2-\Sigma_3+\cdots, \qquad \Sigma_r = \sum_{j_r>\cdots>j_1>m} \frac{p(n-j_1-\cdots-j_r)}{p(n)} \tag{44}$$

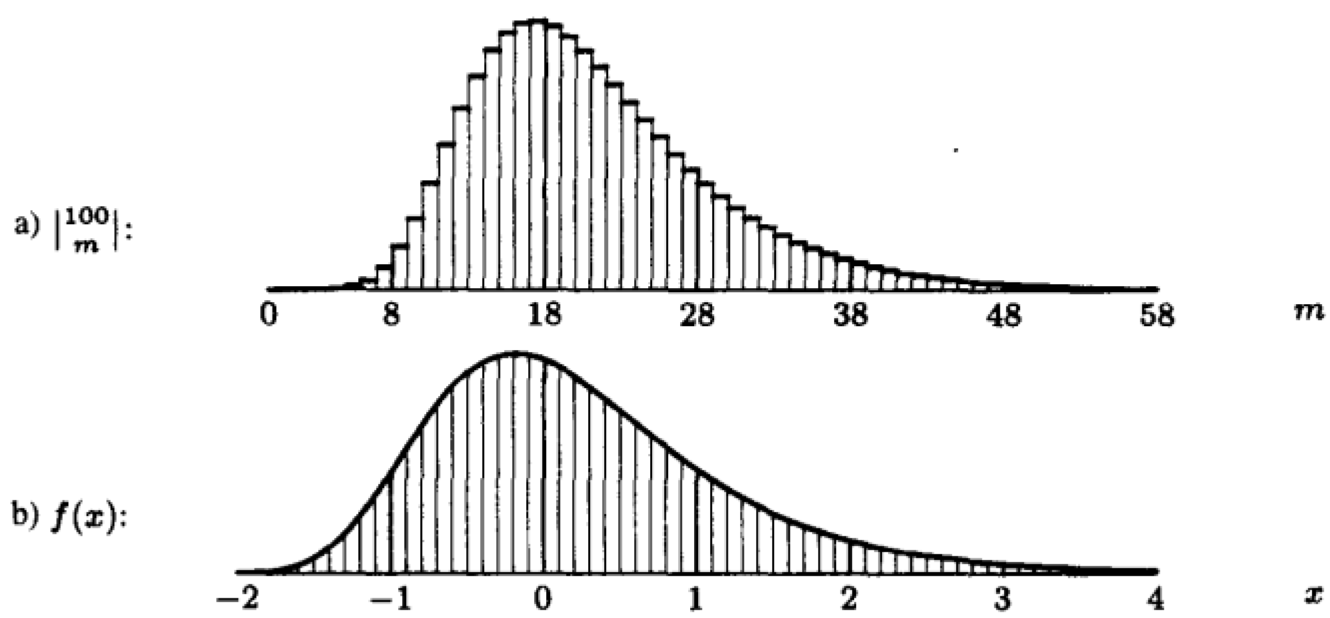


图29 当 a) n=100时；b) $n\to\infty$时，有m个部分的n的分划(参见定理E)

为了计算Σ_r，我们需要有比值$p(n-t)/p(n)$的一个好的估计。而且我们运气不错，因为等式(36)意味着，如果$0\leqslant t\leqslant n^{1/2+\epsilon}$

$$\begin{aligned}\frac{p(n-t)}{p(n)} &= \exp(2C\sqrt{n-t} - \ln(n-t) + O((n-t)^{-1/2}) - 2C\sqrt{n} + \ln n) \\ &= \exp(-Ctn^{-1/2} + O(n^{-1/2+2\epsilon}))\end{aligned} \tag{45}$$

而且，如果$t\geqslant n^{1/2+\epsilon}$，我们有$p(n-t)/p(n) \leqslant p(n-n^{1/2+\epsilon})/p(n) \approx \exp(-Cn^{\epsilon})$，这是一个渐近地小于$n$的任何次幂的值。因此对于所有$t\geqslant 0$的值，我们可以放心地使用近似值

$$\frac{p(n-t)}{p(n)} \approx \alpha^t, \qquad \alpha = \exp(-Cn^{-1/2}) \tag{46}$$

例如，我们有

$$\Sigma_1=\sum_{j>m}\frac{p(n-j)}{p(n)}=\frac{\alpha^{m+1}}{1-\alpha}(1+O(n^{-1/2+2\varepsilon}))+\sum_{n\geqslant j>n^{1/2+\varepsilon}}\frac{p(n-j)}{p(n)}$$
$$=\frac{e^{-Cx}}{C}(1+O(n^{-1/2+2\varepsilon}))+O(ne^{-Cn^{\varepsilon}})$$

因为$\alpha/(1-\alpha)=n^{1/2}/C+O(1)$，而且$\alpha^m=n^{-1/2}e^{-Cx}$。一个类似的论断(参见习题36)证明，如果$r=O(\log n)$，则

$$\Sigma_r=\frac{e^{-Crx}}{C^r r!}(1+O(n^{-1/2+2\varepsilon}))+O(e^{-n^{\varepsilon/2}}) \tag{47}$$

最后——而且这是一般容斥原理的一个美妙性质——(44)的部分和在下列意义之下，总把真正的值“括在括弧中”，即对于所有r

$$1-\Sigma_1+\Sigma_2-\cdots-\Sigma_{2r-1}\leqslant\frac{1}{p(n)}\left|\begin{matrix}m+n\\m\end{matrix}\right|\leqslant 1-\Sigma_1+\Sigma_2-\cdots-\Sigma_{2r-1}+\Sigma_{2r} \tag{48}$$

(参见习题37。)当$2r$接近于$\ln n$而且n很大时，项Σ_{2r}极其微小；因此除了以2ε来代替ε之外，我们得到(42)。 ▮ 47

定理E告诉我们，一个随机分划的最大部分几乎总是$\frac{1}{2C}\sqrt{n}\ln n+O(\sqrt{n})$，而且当$n$是合理地大时，其他部分也趋向于可预测。例如，假设我们取25的所有分划而且附加上它们的费尔利斯框图，并把点改变成盒（如同在边缘表示中那样）。哪些单元最经常被占用？图30示出一个结果：一个随机分划趋向于有一个典型的形状，当$n\to\infty$时，它逼近一个极限曲线。

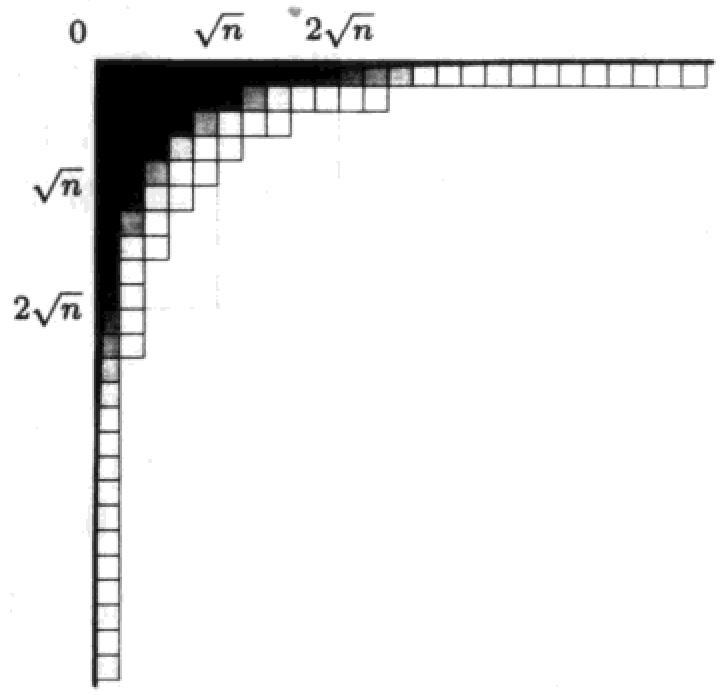


图30 一个随机分划的极限形状的坦佩里曲线(49)

哈·尼·瓦·坦佩里(H. N. V. Temperley)[*Proc. Cambridge Philos. Soc.* 48 (1952), 683-697]给出了启发式的原因，他相信一个大的随机分划$a_1\cdots a_m$的最大部分a_k将满足近似定律

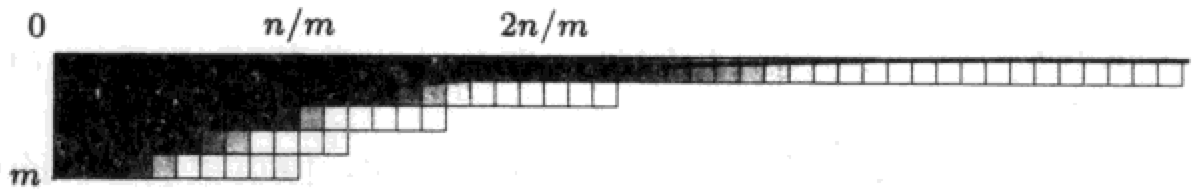
$$e^{-Ck/\sqrt{n}}+e^{-Ca_k/\sqrt{n}}\approx 1 \tag{49}$$

而且他的公式以一个强的形式随后被验证过了。例如，博里斯·皮梯尔(Boris Pittel)的一个定理[*Advances in Applied Math*. 18 (1997)，432-488]允许我们得出结论：一个随机分划的迹几乎总是 $\frac{\ln 2}{C}\sqrt{n}\approx 0.54\sqrt{n}$ ，这同(49)相一致，而且至多有 $O(\sqrt{n}\ln n)^{1/2}$ 的误差。因此所有费尔利斯的点中有29%位于德尔菲方块之内。

另一方面，如果我们仅仅考察有m个部分的n的分划，其中m是固定的，则极限形状稍有不同：如果m合理地大，则几乎所有这样的分划都有

$$a_k\approx\frac{n}{m}\ln\frac{m}{k} \tag{50}$$

图31示出n=50，m=5的情况。事实上，当m随n增大时，相同的极限成立，但以比起$\sqrt{n}$要慢些的速度收敛[参见弗尔希克(Vershik)和雅库波维奇(Yakubovich)，*Moscow Math. J*.1 (2001)，457-468]。



48 图31 当有m个部分时的极限形状(50)

分划的边缘表示给了我们有关加倍地有界的分划的进一步信息，这是在以下意义下，即我们不仅限制分划的个数，而且限制每个部分的大小。至多有m个部分，而每一个部分的大小至多为l的一个分划，可装入一个$m\times l$的盒子。所有这样的分划对应于恰好有n个反演的多重集合$\{m\cdot 0, l\cdot 1\}$的排列，而且我们在习题5.1.2-16中已经研究了多重集合排列的反演。特别地，该习题推导了可发生n个反演的方法数的一个不明显的公式：

定理C 若n的部分不多于m个且任何部分都不大于l，则n的分划的个数为

$$[z^n]\binom{l+m}{m}_z=[z^n]\frac{(1-z^{l+1})}{(1-z)}\frac{(1-z^{l+2})}{(1-z^2)}\cdots\frac{(1-z^{l+m})}{(1-z^m)} \tag{51}$$

这个结果是由奥·柯西给出的[*Comptes Rendus Acad. Sci*. 17 (Paris，1843)，523-531]。注意，当$l\to\infty$时，分子简单地变为1。一个更一般的结果的有趣组合证明见本小节末的习题39。 ▮

算法的分析。现在我们对于分划的定量方面知道的比所需要的还多，可以十分准确地推导算法P的特性。假设算法的步骤P1，…，P6，分别被执行$T_1(n)$，…，$T_6(n)$次。我们显然有$T_1(n)$=1和$T_3(n)=p(n)$；而且克希霍夫定律告诉我们，$T_2(n)=T_5(n)$和

$T_4(n)+T_5(n)=T_3(n)$。对于包含一个2的每个分划，我们到达步骤P4一次，而这显然是$p(n-2)$。

因此关于算法P的运行时间，惟一可能的神秘是我们必须执行步骤P6的次数，它循环返回到自身。然而，稍经思考可发现，算法仅在步骤P2和P6把一个$\geqslant 2$的值存入数组$a_1a_2\cdots$；而且每个这样的值或者在步骤P4或者在步骤P5中最终减1，因此

$$T_2''(n)+T_6(n)=p(n)-1 \tag{52}$$

其中$T_2''(n)$是步骤P2把a_m置为$\geqslant 2$的值的次数。令$T_2(n)=T_2'(n)+T_2''(n)$，使得$T_2'(n)$是步骤P2把$a_m\leftarrow 1$的次数，则$T_2'(n)+T_4(n)$是在1中结尾的分划的个数，因此

$$T_2'(n)+T_4(n)=p(n-1) \tag{53}$$

啊哈!我们已经找到足够的方程来确定所有需要的量:

$$\begin{aligned}&(T_1(n),\cdots,T_6(n))=\\&(1,p(n)-p(n-2),p(n),p(n-2),p(n)-p(n-2),p(n-1)-1)\end{aligned} \tag{54}$$

而且从$p(n)$的渐近式，我们还知道每个分划的平均计算量:

$$\left(\frac{T_1(n)}{p(n)},\cdots,\frac{T_6(n)}{p(n)}\right)=\left(0,\frac{2C}{\sqrt{n}},1,1-\frac{2C}{\sqrt{n}},\frac{2C}{\sqrt{n}},1-\frac{C}{\sqrt{n}}\right)+O\left(\frac{1}{n}\right) \tag{55}$$

其中$C=\pi/\sqrt{6}\approx 1.283$（参见习题45）。因此，对于每一分划存储器访问的总次数仅仅是$4-3C/\sqrt{n}+O(1/n)$。 49

> 无论是谁想要来生成所有的分划，那不仅仅要使自己投身于巨大的劳苦之中，而且还必须使自己忍受痛苦来保持全神贯注，以便不致被严重地欺骗。
>
> ——列昂哈德·欧拉，*De Partitione Numerorum* (1750)

算法H更难以分析，但我们至少可以证明它的运行时间的一个合适的上限。关键的量是j的值，即使得$a_j<a_1-1$的最小的下标。当$m=4$和$n=11$时，j的逐次的值是$(2,2,2,3,2,2,3,4,2,3,5)$，而且当$b_1\cdots b_l$是共轭分划$(a_1\cdots a_m)^T$时，我们发现$j=b_{l-1}+1$(参见(7)和(12))。步骤H3跳出情况$j=2$，因为这个情况不仅最普通，而且它也特别容易处理。

令$c_m(n)$是$j-1$累加的总值，即对于由算法H生成的所有$\left|{n \atop m}\right|$个分划求和所得。例如，$c_4(11)=1+1+1+2+1+1+2+3+1+2+4=19$。我们可以认为$c_m(n)/\left|{n \atop m}\right|$是每个分划运行时间的一个好的标志，因为执行最耗时的步骤H4和H6所需时间大约和$j-2$成比例。这个比$c_m(n)/\left|{n \atop m}\right|$不是有界的，因为$c_m(m)=m$而$\left|{m \atop m}\right|=1$。但尽管如此，下列定理证明算法H是有效的:

定理H 算法H的代价测量至多为$3\left|{n \atop m}\right|+m$。

证明 当$1\leqslant n<m$时，如果我们人为地定义$c_m(n)=1$，则可以很容易地验证$c_m(n)$满足和$\left|{n \atop m}\right|$一样的递归，即对于$m,n\geqslant 1$，

$$c_m(n)=c_{m-1}(n-1)+c_m(n-m) \tag{56}$$

参见(39)。但现在边界条件是不同的:

$$c_m(0)=[m>0], \quad c_0(n)=0 \tag{57}$$

表3示出当m和n很小时，$c_m(n)$的特性。

表3 算法H的代价

n	$c_0(n)$	$c_1(n)$	$c_2(n)$	$c_3(n)$	$c_4(n)$	$c_5(n)$	$c_6(n)$	$c_7(n)$	$c_8(n)$	$c_9(n)$	$c_{10}(n)$	$c_{11}(n)$
0	0	1	1	1	1	1	1	1	1	1	1	1
1	0	1	1	1	1	1	1	1	1	1	1	1
2	0	1	2	1	1	1	1	1	1	1	1	1
3	0	1	2	3	1	1	1	1	1	1	1	1
4	0	1	3	3	4	1	1	1	1	1	1	1
5	0	1	3	4	4	5	1	1	1	1	1	1
6	0	1	4	6	5	5	6	1	1	1	1	1
7	0	1	4	7	7	6	6	7	1	1	1	1
8	0	1	5	8	11	8	7	7	8	1	1	1
9	0	1	5	11	12	12	9	8	8	9	1	1
10	0	1	6	12	16	17	13	10	9	9	10	1
11	0	1	6	14	19	21	18	14	11	10	10	11

为证明这个定理，我们实际上将证明一个更强的结果，对于$n\geqslant m\geqslant 2$，

$$c_m(n)\leqslant 3\left|{n \atop m}\right|+2m-n-1 \tag{58}$$

习题50表明，当$m\leqslant n\leqslant 2m$时，这个不等式成立，所以如果我们能够证明当$n>2m$时它成立，则证明就完成了。在后一种情况下，通过归纳法我们有

$$\begin{aligned}
c_m(n) &= c_1(n-m)+c_2(n-m)+c_3(n-m)+\cdots+c_m(n-m)\\
&\leqslant 1+\left(3\left|{n-m \atop 2}\right|+3-n+m\right)+\left(3\left|{n-m \atop 3}\right|+5-n+m\right)+\cdots\\
&\qquad +\left(3\left|{n-m \atop m}\right|+2m-1-n+m\right)\\
&= 3\left|{n-m \atop 1}\right|+3\left|{n-m \atop 2}\right|+\cdots+3\left|{n-m \atop m}\right|-3+m^2-(m-1)(n-m)\\
&= 3\left|{n \atop m}\right|+2m^2-m-(m-1)n-3
\end{aligned}$$

50 而且由于$n\geqslant 2m+1$，因此$2m^2-m-(m-1)n-3\leqslant 2m-n-1$。 ▌

***分划的格雷码**。当如同在习题5中那样，以部分计数形式$c_1\cdots c_n$生成分划时，在每步中c_j的值中至多有四个发生变化。但我们可能更喜欢选择极小化各个部分的改变，并且以这样一种方式来生成诸分划，即$a_1a_2\cdots a_n$的后继总是通过对于某个j和k，简单地设置$a_j\leftarrow a_j+1$和$a_k\leftarrow a_k-1$来得到，如同在7.2.1.3节的“转动门”算法中那样。结果证实这总是可能的；事实上，当$n=6$时，有惟一方法来做到这一点:

$$111111,\ 21111,\ 3111,\ 2211,\ 222,\ 321,\ 33,\ 42,\ 411,\ 51,\ 6 \qquad (59)$$

而且一般地，把n分成至多m部分的$\left|{m+n \atop m}\right|$个分划总是通过一个适当的格雷通路生成的。

注意，$\alpha \to \beta$是从一个分划到另一个分划的可允许的转换，当且仅当通过仅仅移动α的费尔利斯框图中的一个点，我们可得到β的费尔利斯框图。因此，$\alpha^T \to \beta^T$也是一个可允许的转换。由此得出，对于分划成至多m个部分的每个格雷码，对应于分划成不超过m部分的一个格雷码。我们将在后者的约束下进行工作。

分划的格雷码的总数是巨大的：当n=7时，有52种；而当n=8时，有652种；当n=9时，有298 896种；当n=10时，有2 291 100 484种。但现在实际上仍不知道简单的结构。原因大概是，有很少的分划仅有两个邻居，即当$1<d<n$且d是n的因子时，有$d^{n/d}$个分划。这样的分划必须以$\{(d+1)d^{n/d-2}(d-1),\ d^{n/d-1}(d-1)\ 1\}$居前，而且为它们所跟随；而这样的要求似乎排除了任何简单的递归方法。

卡拉·戴·萨维吉(Carla D. Savage) [*J. Algorithms* 10 (1989)，577-595]找到了仅通过适度数量的复杂性克服这些困难的一个方法。令

$$\mu(m,n) = \overbrace{m\ m\ \cdots\ m}^{\lfloor n/m \rfloor}\ (n \bmod m) \qquad (60)$$ 51

是n的按词典顺序最大的分划（且诸部分$\leqslant m$）；我们的目标将是从分划1^n到$\mu(m,n)$来递归地构造定义的格雷通路$L(m,n)$和$M(m,n)$，其中$L(m,n)$跑遍其分划部分以m为界的所有分划，而$M(m,n)$跑遍那些分划和更多一些的分划。假定其他部分全都严格地小于m，则$M(m,n)$还包括其最大部分为$m+1$的分划。例如，$L(3,8)$是11111111, 2111111, 311111, 221111, 22211, 2222, 3221, 32111, 3311, 332，而$M(3,8)$是

$$\begin{gathered}11111111, 2111111, 221111, 22211, 2222, 3221,\\ 3311, 32111, 311111, 41111, 4211, 422, 332\end{gathered} \qquad (61)$$

以4开头的另外的分划将在递归的其他部分给我们“摆动的门”。对于所有$n \geqslant 0$，我们将定义$L(m,n)$，但仅对于$n>2m$，我们定义$M(m,n)$。

对于m=5示出以简化记号的以下构造，几乎都有效：

$$L(5)=\begin{cases}L(3)\\4L(\infty)^R\\5L(\infty)\end{cases}\ \text{如果}n\leqslant 7;\quad \begin{cases}L(3)\\4L(2)^R\\5L(2)\\431\\44\\53\end{cases}\ \text{如果}n=8;\quad \begin{cases}M(4)\\54L(4)^R\\55L(5)\end{cases}\text{如果}n\geqslant 9; \qquad (62)$$

$$M(5)=\begin{cases}L(4)\\5L(4)^R\\6L(3)\\64L(\infty)^R\\55L(\infty)\end{cases}\ \text{如果}11\leqslant n\leqslant 13;\quad \begin{cases}L(4)\\5M(4)^R\\6L(4)\\554L(4)^R\\555L(5)\end{cases}\text{如果}n\geqslant 14 \qquad (63)$$

这里$L(m, n)$和$M(m, n)$中的参数n已被省略，因为它可从上下文推出。每个L或M被设想来生成以前部分已被减去后，剩下的无论多少数量的分划。因此，例如，(63)确定

$$M(5,14)=L(4,14), 5M(4,9)^R, 6L(4,8), 554L(4,0)^R, 555L(5,-1)$$

序列$L(5,-1)$实际上为零，$L(4,0)$是空串，所以$M(5,14)$的最后分划是$554=\mu(5,14)$，如同它应该的那样。记号$L(\infty)$代表$L(\infty, n)=L(n, n)$，即n的所有分划的格雷通路，它以1^n开始并以n^1结尾。

一般地说，如果我们分别以$m-3$，$m-2$，$m-1$，m和$m+1$代替(62)和(63)中的数字2, 3, 4, 5和6，对于所有$m\geqslant 3$，$L(m)$和$M(m)$实质上是通过相同规则定义的。范围$n\leqslant 7$，$n=8$，$n\geqslant 9$变成$n\leqslant 2m-3$，$n=2m-2$，$n\geqslant 2m-1$。范围$11\leqslant n\leqslant 13$和$n\geqslant 14$变成$2m+1\leqslant n\leqslant 3m-2$和$n\geqslant 3m-1$。序列$L(0)$，$L(1)$，$L(2)$有明显的定义，因为当$m\leqslant 2$时通路是惟一的。对于$n\geqslant 5$，序列$M(2)$是$1^n$, 21^{n-2}, 31^{n-3}, 221^{n-4},
52 2221^{n-6}, …, $\mu(2,n)$。

定理S　对于$m, n\geqslant 0$的格雷通路$L'(m, n)$和对于$n\geqslant 2m+1\geqslant 5$的$M'(m, n)$，具有上述性质的所有分划存在，但$L'(4, 6)$的情况除外。而且，$L'$和$M'$遵从相互递归的(62)和(63)，但一些情况例外。

证明　我们注意到(62)和(63)几乎都有效。读者可以验证，当(62)给出

$$\begin{aligned}L(4,6)&=L(2,6),\ 3L(1,3)^R,\ 4L(1,2),\ 321,\ 33,\ 42\\&=111111, 21111, 2211, 222, 3111, 411, 321, 33, 42\end{aligned}\qquad(64)$$

时，仅在$L(4, 6)$的情况下出现小的出入。如果$m>4$，那就成了，因为从$L(m-2, 2m-2)$的末尾到$(m-1)L(m-3, m-1)^R$的开始是从$(m-2)(m-2)2$到$(m-1)(m-3)2$。不存在令人满意的通路$L(4, 6)$，因为通过这9个分划的所有格雷码必定或者以411, 33, 3111, 222结束，或者以2211结束。

为了中立化这个异常性，我们需要在8处地方，也就是涉及“出毛病的子程序”$L(4, 6)$的地方，对$L(m, n)$和$M(m, n)$的定义进行修补。一个简单的方法是进行如下的定义:

$$\begin{aligned}L'(4,6)&=111111, 21111, 3111, 411, 321, 33, 42\\L'(3,5)&=11111, 2111, 221, 311, 32\end{aligned}\qquad(65)$$

于是，我们从$L(4, 6)$中省略222和2211；我们也对$L(3, 5)$重新编程，使得2111同221相邻。习题60表明，把在$L(4, 6)$中不出现的两个分划“嫁接”在一起总是容易的。▌

习　题

▶ 1. [*M21*] 试给出在“12种方法”的每个问题中可能性总数的公式。例如，m个事物的n元组的个数为m^n。(当适宜时，使用记号(38)，而且要注意确保即使当$m=0$或$n=0$时，你的公式仍然正确。)

▶ 2. [*20*] 试证明对于步骤H1的一个小改动可产生如下算法：将生成n的至多m个部分的所有分划。

3. [*M17*] n的一个m部分分划$a_1+\cdots+a_m$ $(a_1\ge\cdots\ge a_m)$是最优地平衡的——如果对于$1\le i$, $j\le m$有$|a_i-a_j|\le 1$。试证明每当$n\ge m\ge 1$时，恰有一个这样的分划，并且给出第j部分a_j作为j, m和n的函数的一个简单公式。

4. [*M22*] (吉迪安 · 厄尔里兹，1974。)什么是n的在词典顺序下最小的分划(其中所有部分都$\ge r$)? 例如，当n=19和r=5时，答案为766。

▶ 5. [*23*] 试设计在(8)的部分计数形式$c_1\cdots c_n$之下生成n的所有分划的算法。以协词典顺序生成它们，即在$c_n\cdots c_1$的词典顺序下，它等价于对应分划$a_1a_2\cdots$的词典顺序。为了有效性起见，还维持一个链接表$l_0l_1\cdots l_n$，使得如果使c_k>0的不同k值有$k_1<\cdots<k_t$，则我们有

$$l_0=k_1,\ l_{k_1}=k_2,\ \cdots,\ l_{k_{t-1}}=k_t,\ l_{k_t}=0$$

(因此分划331将通过$c_1\cdots c_7$=1020000，l_0=1，l_1=3和l_3=0表示；其他链接l_2, l_4, l_5, l_7可被置成任何方便的值。) 53

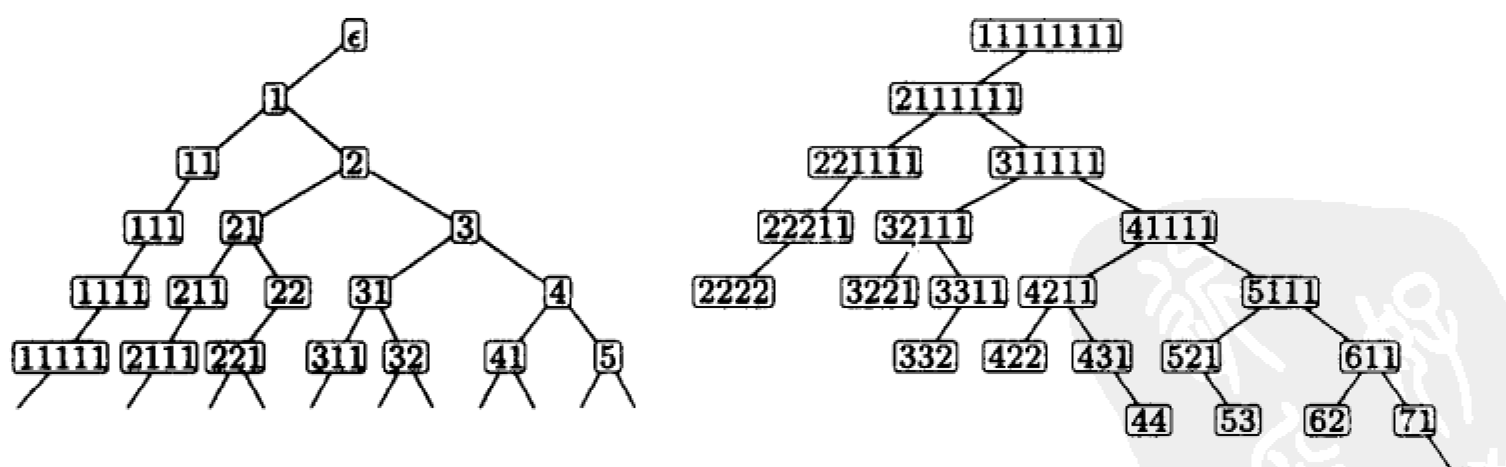
6. [*20*] 给定$a_1a_2\cdots$，试设计一个算法来计算$b_1b_2\cdots=(a_1a_2\cdots)^T$。

7. [*M20*] 设$a_1\cdots a_n$和$a_1'\cdots a_n'$是具有$a_1\ge\cdots\ge a_n\ge 0$和$a_1'\ge\cdots\ge a_n'\ge 0$的$n$的分划，并令它们的共轭分别是$b_1\cdots b_n=(a_1\cdots a_n)^T$和$b_1'\cdots b_n'=(a_1'\cdots a_n')^T$。试证明$b_1\cdots b_n<b_1'\cdots b_n'$，当且仅当$a_n\cdots a_1<a_n'\cdots a_1'$。

8. [*15*] 当$(p_1\cdots p_t,\ q_1\cdots q_t)$是如同在(15)和(16)中的一个分划$a_1a_2\cdots$的边缘表示时，什么是共轭分划$(a_1a_2\cdots)^T=b_1b_2\cdots$?

9. [22] 如果$a_1a_2\cdots a_m$和$b_1b_2\cdots b_m=(a_1a_2\cdots a_m)^T$是共轭分划，试证明多重集合$\{a_1+1,\ a_2+2,\ \cdots,\ a_m+m\}$和$\{b_1+1,\ b_2+2,\ \cdots,\ b_m+m\}$相等。

10. [*21*] 如下两种简单类型的二叉树有时对于分划的推理有帮助：(a)包括所有整数的所有分划的一棵树，(b)包括一个给定整数n的所有分划的一棵树。这里示出n=8时这两种类型的树。



a) b)

试推导出奠定这些构造的基础的一般规则。树的什么遍历顺序对应于分划的词典顺序?

11. [*M22*] 使用面值1, 2, 5, 10, 20, 50和/或100分的硬币，有多少种方法来支付1欧元? 如果每种硬币至多只允许使用两个，那又怎样?

▶ 12. [*M21*] (列 · 欧拉，1750。)使用生成函数来证明：把n分划成不同部分的方法数是把

n分划成奇数部分的方法数。例如，5=4+1=3+2；5=3+1+1=1+1+1+1+1。

[注：下两道习题使用组合技术来证明这个著名定理的扩展。]

▶ 13. [*M22*] (法·富兰克林，1882。)试找出在n的恰有k个部分且重复一次以上的分划和n的恰好有偶数个部分的分划之间的一一对应。(k=0的情况对应于欧拉的结果。)

▶ 14. [*M28*] (詹·约·希尔威斯特，1882。)试求在把n分成恰有k个"间隔"的不同部分$a_1>a_2>\cdots>a_m$(其中$a_j>a_{j+1}+1$)和把n分成恰有$k+1$个不同值的奇数个分划之间的一一对应关系。(例如，当k=0时这个构造证明，把n写成连续的整数之和的方法数是n的奇因子的个数。)

15. [*M20*] (詹·约·希尔威斯特。)试找出自共轭的分划(即使得$\alpha=\alpha^T$的分划)个数的一个生成函数。

54 16. [*M21*] 试找出迹k的分划的生成函数，并且对k求和，以得到一个非平凡的恒等式。

17. [*M26*] n的一个联合分划是满足

$$a_1\ge\cdots\ge a_r,\ b_1>\cdots>b_s\text{和}a_1+\cdots+a_r+b_1+\cdots+b_s=n$$

的正整数的一对序列$(a_1,\cdots,a_r;\ b_1,\cdots,b_s)$。

因此如果s=0，这是一个通常的分划，而如果r=0，这是一个被分划成不同部分的分划。

a) 试找出生成函数$\sum u^{r+s}v^s z^n$的一个简单公式，求和是对于有r个通常部分a_i和s个不同部分b_j的n的所有联合分划进行的。

b) 类似地，给定t的值，试找出当求和是对于恰好有$r+s=t$个总的部分的所有联合分划进行时，$\sum v^s z^n$的一个简单公式。

c) 你能导出什么恒等式？

▶ 18. [*M23*] (多伦·泽尔伯格(Doron Zeilberger)。)试证明，在使得

$$a_1\ge a_2\ge\cdots\ge a_r,\quad b_1>b_2>\cdots>b_s$$

的整数序列对$(a_1,a_2,\cdots,a_r;\ b_1,b_2,\cdots,b_s)$和使得

$$c_1\ge c_2\ge\cdots\ge c_{r+s},\quad\text{对于}1\le j\le r+s,\ d_j\in\{0,1\}$$

的整数序列对$(c_1,c_2,\cdots,c_{r+s};\ d_1,d_2,\cdots,d_{r+s})$之间有一个一一对应关系，它们通过多重集合等式

$$\{a_1,a_2,\cdots,a_r\}=\{c_j\mid d_j=0\}\text{和}\{b_1,b_2,\cdots,b_s\}=\{c_j+r+s-j\mid d_j=1\}$$

而相关。结果，我们得到有趣的恒等式

$$\sum_{\substack{a_1\ge\cdots\ge a_r>0\\ b_1>\cdots>b_s>0}}u^{r+s}v^s z^{a_1+\cdots+a_r+b_1+\cdots+b_s}=\sum_{\substack{c_1\ge\cdots\ge c_t>0\\ d_1,\cdots,d_t\in\{0,1\}}}u^t v^{d_1+\cdots+d_t}z^{c_1+\cdots+c_t+(t-1)d_1+\cdots+d_{t-1}}.$$

19. [*M21*] (爱·海尼(E. Heine)，1847。)证明四参数恒等式

$$\prod_{m=1}^{\infty}\frac{(1-wxz^m)(1-wyz^m)}{(1-wz^m)(1-wxyz^m)}=\sum_{k=0}^{\infty}\frac{w^k(x-1)(x-z)\cdots(x-z^{k-1})(y-1)(y-z)\cdots(y-z^{k-1})z^k}{(1-z)(1-z^2)\cdots(1-z^k)(1-wz)(1-wz^2)\cdots(1-wz^k)}$$

提示：在下列公式

$$\sum_{k,l\ge0}u^k v^l z^{kl}\frac{(z-az)(z-az^2)\cdots(z-az^k)}{(1-z)(1-z^2)\cdots(1-z^k)}\frac{(z-bz)(z-bz^2)\cdots(z-bz^l)}{(1-z)(1-z^2)\cdots(1-z^l)}$$

中进行对k或l的求和并考虑当$b=auz$时出现的简化。

▶ 20. [*M21*] 使用欧拉的递归式(20)，对于$1\leqslant n\leqslant N$，计算分划数$p(n)$的一张表，近似地要花费多长时间？

21. [*M21*] (伦·欧拉。)令$q(n)$是分划成不同部分的分划个数，如果你已经知道$p(1)$, …, $p(n)$值，则计算$q(n)$的一个好方法是什么？

22. [*HM21*] (伦·欧拉。)令$\sigma(n)$是正整数n的所有正因子之和。因此当n为奇数时，$\sigma(n)=n+1$，而当n是高度可合成的时，$\sigma(n)$可以比n大很多。尽管有这种稍微古怪的特性，试证明，$\sigma(n)$几乎满足和分划数相同的递归式(20)，对于$n\geqslant 1$

$$\sigma(n)=\sigma(n-1)+\sigma(n-2)-\sigma(n-5)-\sigma(n-7)+\sigma(n-12)+\sigma(n-15)-\cdots$$

当右边一项是“$\sigma(0)$”，其值用n代替时除外。例如，$\sigma(11)=1+11=\sigma(10)+\sigma(9)-\sigma(6)-\sigma(4)=18+13-12-7$；$\sigma(12)=1+2+3+4+6+12=\sigma(11)+\sigma(10)-\sigma(7)-\sigma(5)+12=12+18-8-6+12$。 55

23. [*HM25*] 使用雅可比的三元组乘积恒等式(19)来证明他发现的另一个公式

$$\prod_{k=1}^{\infty}(1-z^k)^3=1-3z+5z^3-7z^6+9z^{10}-\cdots=\sum_{n=0}^{\infty}(-1)^n(2n+1)z^{\binom{n+1}{2}}$$

24. [*M26*] (斯·拉曼奴燕，1919。)令$A(z)=\prod_{k=1}^{\infty}(1-z^k)^4$　。

a) 证明：当$n \bmod 5=4$时$[z^n]A(z)$是5的倍数。

b) 证明：如果B是任何具有整系数的幂级数，则$[z^n]A(z)B(z)^5$具有相同性质。

c) 因此当$n \bmod 5=4$时，$p(n)$是5的倍数。

25. [*HM27*] 通过使用a)欧拉求和公式和b)默林变换来估计$\ln P(e^{-t})$，对(22)进行改进。提示：去对数函数$\mathrm{Li}_2(x)=x/1^2+x^2/2^2+x^3/3^2+\cdots$满足$\mathrm{Li}_2(x)+\mathrm{Li}_2(1-x)=\zeta(2)-(\ln x)\ln(1-x)$。

26. [*HM22*] 在习题5.2.2-44和习题5.2.2-51中，我们研究两种方法来证明，对于所有$M>0$

$$\sum_{k=1}^{\infty}e^{-k^2/n}=\frac{1}{2}(\sqrt{\pi n}-1)+O(n^{-M})$$

试证明泊松求和公式给出更强得多的一个结果。

27. [*HM23*] 试对(29)求值并完成导致定理D的计算。

28. [*HM42*] (德·亨·莱默。)试证明，在(34)中定义的哈迪-拉曼奴燕-拉德曼彻系数$A_k(n)$有下列重要的性质：

a) 如果k为奇数，则$A_{2k}(km+4n+(k^2-1)/8)=A_2(m)A_k(n)$。

b) 如果p为素数，$p^e>2$，且$k\perp 2p$，则

$$A_{p^e k}(k^2m+p^{2e}n-(k^2+p^{2e}-1)/24)=(-1)^{[p^e=4]}A_{p^e}(m)A_k(n)$$

在这个公式中，如果p或k可为2或3整除，则$k^2+p^{2e}-1$是24的倍数。否则以24来除应通过modulo $p^e k$ 完成。

c) 如果p为素数，则$|A_{p^e}(n)|<2^{[p>2]}p^{e/2}$。

d) 如果p为素数，则$A_{p^e}(n)\neq 0$当且仅当$1-24n$是modulo p二次剩余，而且或者$e=1$或者$24n \bmod p\neq 1$。

e) 当k可恰好为$\geqslant 5$的t个素数整除而且n是一个随机整数时，$A_k(n)=0$的概率近似地为$1-2^{-t}$。

▶ 29. [*M16*] 推广(41)，并计算和$\sum_{a_1>a_2>\cdots>a_m>1} z_1^{a_1} z_2^{a_2} \cdots z_m^{a_m}$。

30. [*M17*] 求下列和的一个封闭形式。

$$\text{(a)} \sum_{k\ge 0} \left| {n-km \atop m-1} \right| \qquad \text{(b)} \sum_{k\ge 0} \left| {n \atop m-k} \right|$$

(它们是有限的，因为当k很大时，被求和的项为零。)

31. [*M24*] (奥 · 德 · 摩根，1843。)试证明$\left|{n \atop 2}\right| = \lfloor n/2 \rfloor$和$\left|{n \atop 3}\right| = \lfloor (n^2+6)/12 \rceil$；对于$\left|{n \atop 4}\right|$，试求一个类似的公式。

32. [*M15*] 试证明，对于所有$m, n \ge 0$，$\left|{n \atop m}\right| \le p(n-m)$。等式何时成立？

33. [*HM20*] 使用下列事实，即n分成m部分恰有$\binom{n-1}{m-1}$个合成，即等式7.2.1.3-(9)，来证

56 明$\left|{n \atop m}\right|$的一个下界。然后置$m = \lfloor \sqrt{n} \rfloor$来得到$p(n)$的一个基本的下界。

▶ 34. [*HM21*] 证明$\left|{n-m(m-1)/2 \atop m}\right|$是$n$分成$m$个不同部分的分划数。结果，当$m < n^{1/3}$时

$$\left|{n \atop m}\right| = \frac{n^{m-1}}{m!(m-1)!}\left(1+O\left(\frac{m^3}{n}\right)\right)$$

35. [*HM21*] 在厄尔多斯-勒纳概率分布(43)中，x的什么值是(a)最为可能的？(b)中值？(c)均值？(d)标准离差是什么？

36. [*HM24*] 试证明在定理E中需要的关键估计(47)。

37. [*M22*] 通过分析一个恰有超过m的q个不同部分的分划在第r个部分和中被计数多少次，来证明容斥加括号引理(48)。

38. [*M20*] n恰好有m个部分，以及最大部分为l的分划的生成函数是什么？

▶ 39. [*M25*] (法 · 富兰克林。)推广定理C，试证明，对于$0 \le k \le m$，

$$[z^n]\frac{(1-z^{l+1})\cdots(1-z^{l+k})}{(1-z)(1-z^2)\cdots(1-z^m)}$$

是n分成m个或更少的部分$a_1 a_2 \cdots$的个数，且有$a_1 \le a_{k+1}+l$的性质。

40. [*M22*] (奥 · 柯西。)分划成m个全都不同的部分且都小于l的生成函数是怎样的？

41. [*HM42*] 推广哈迪-拉曼奴燕-拉德曼彻公式(32)，来得到n分成至多m个部分的收敛序列，而且没有任何部分超过l。

42. [*HM42*] 对于n分成至多$\theta\sqrt{n}$个部分，且没有任何部分超过$\varphi\sqrt{n}$，并假定$\theta\varphi>1$，试求出类似于(49)的极限形状。

43. [*M21*] 给定n和k，n有多少分划有$a_1>a_2>\cdots>a_k$？

▶ 44. [*M22*] n有多少个分划是有两个相等的最小部分的？

45. [*HM21*] 计算$p(n-1)/p(n)$的渐近值且有相对误差$O(n^{-2})$。

46. [*M20*] 在正文对于算法P的分析中，$T_2'(n)$和$T_2''(n)$中哪一个更大？

▶ 47. [*HM22*] (阿 · 尼珍休斯和希 · 索 · 威尔弗，1975。)下列基于分划数$p(0), p(1), \cdots, p(n)$表的简单算法，使用(8)的部分计数表示$c_1 \cdots c_n$生成n的一个随机分划。试证明它以相等概率产生每个分划。

N1.[初始化。] 置$m \leftarrow n$和$c_1 \cdots c_n \leftarrow 0 \cdots 0$。

N2.[完成了吗？] 如果m=0则结束。

N3.[生成。] 在$0 \leqslant M < mp(m)$的范围内，生成一个随机整数M。

N4.[选择诸部分。] 置$s \leftarrow 0$。然后对于j=1, 2, …, n和k=1, 2, …, $\lfloor m/j \rfloor$，反复地置$s \leftarrow s+k\,p(m-jk)$直到$s>M$为止。

N5.[修改] 置$c_k \leftarrow c_k+j$，$m \leftarrow m-jk$，并返回N2。 ▌

提示：步骤N4是基于恒等式

$$\sum_{j=1}^{m} \sum_{k=1}^{\lfloor m/j \rfloor} kp(m-jk) = mp(m)$$

的，它以$kp(m-jk)/(mp(m))$的概率选择每个值对(j,k)。

48. [*HM40*] 试分析在上题中算法的运行时间。 57

▶ 49. [*HM26*] (a)什么是n的所有分划的最小部分之和的生成函数$F(z)$？(这个级数以$z+3z^2+5z^3+9z^4+12z^5+\cdots$开始。)

(b)试求$[z^n]F(z)$的渐近值且其相对误差为$O(n^{-1})$。

50. [*HM33*] 令在(56)和(57)的递归式中，$c(m)=c_m(2m)$。

a) 试证明对于$0 \leqslant k \leqslant m$，$c_m(m+k)=m-k+c(k)$。

b) 结果，如果对于所有m，$c(m)<3p(m)$，则对于$m \leqslant n \leqslant 2m$，(58)成立。

c) 证明$c(m)-m$是m的所有分划的次最小部分之和。

d) 试求n的次最小部分为k的所有分划与$\leqslant n$的数(最小部分为k+1)的所有分划间的一一对应。

e) 试描述生成函数 $\sum_{m \geqslant 0} c(m)z^m$ 。

f) 试得出结论，即对于所有$m \geqslant 0$，$c(m)<3\,p(m)$。

51. [*M46*] 试对算法H做一个详尽的分析。

▶ 52. [*M21*] 当n=64时，由算法P生成的第100万个分划是什么？提示：$p(64)=1741630=1000000+\binom{77}{13}+\binom{60}{10}+\binom{47}{8}+\binom{35}{5}+\binom{27}{3}+\binom{22}{2}+\binom{18}{1}+\binom{15}{0}$。

▶ 53. [*M21*] 当m=32和n=100时，由算法H生成的第100万个分划是什么？提示：$999999=\binom{80}{12}+\binom{66}{11}+\binom{50}{7}+\binom{41}{6}+\binom{33}{5}+\binom{26}{4}+\binom{21}{4}$。

▶ 54. [*M30*] 如果对于所有$k \geqslant 0$，$a_1+\cdots+a_k \geqslant b_1+\cdots+b_k$，则说分划$\alpha=a_1a_2\cdots$支配分划$\beta=b_1b_2\cdots$，写成$\alpha \succeq \beta$或$\beta \preceq \alpha$。

a) 真或假：$\alpha \succeq \beta$意味着$\alpha \geqslant \beta$(在词典顺序下)。

b) 真或假：$\alpha \succeq \beta$意味着$\beta^T \succeq \alpha^T$。

c) 试证明n的任何两个分划有最大下界$\alpha \wedge \beta$使得$\alpha \succeq \gamma$和$\beta \succeq \gamma$，当且仅当$\alpha \wedge \beta \succeq \gamma$。试说明如何计算$\alpha \wedge \beta$。

d) 类似地，试说明如何来计算最小上界$\alpha \vee \beta$，使得$\gamma \succeq \alpha$和$\gamma \succeq \beta$当且仅当$\gamma \succeq \alpha \vee \beta$。

e) 如果α有l个部分，β有m个部分，问$\alpha \wedge \beta$和$\alpha \vee \beta$有多少个部分？

f) 真或假：如果α有不同部分，β也有不同部分，则$\alpha \wedge \beta$和$\alpha \vee \beta$也有不同部分。

▶ 55. [*M37*] 继续上道题，如果$\alpha \succeq \beta$，$\alpha \neq \beta$和$\alpha \succeq \gamma \succeq \beta$意味着$\gamma=\alpha$或$\gamma=\beta$，则说$\alpha$覆盖$\beta$。例如，图32示出数12的诸分划之间的覆盖关系。

a) 如果对于所有$k \geqslant 1$和某个$l \geqslant 1$，$\alpha=a_1 a_2 \cdots$和$\beta=b_1 b_2 \cdots$都是分划且有$b_k=a_k-[k=l]+[k=l+1]$，则我们写$\alpha \vdash \beta$。试证明α覆盖β当且仅当$\alpha \vdash \beta$或$\beta^T \vdash \alpha^T$。

b) 试证明，通过考察α和β的边缘表示，有一个容易的方法来指出是否有α覆盖β。

c) 令$n=\binom{n_2}{2}+\binom{n_1}{1}$，其中$n_2>n_1 \geqslant 0$。试证明$n$的分划中没有一个能覆盖多于$n_2-2$个分划。

d) 如果没有分划λ存在使$\mu \vdash \lambda$，说分划μ为极小的。试证μ是极小的当且仅当μ^T有不同的部分。

e) 假设$\alpha=\alpha_0 \vdash \alpha_1 \vdash \cdots \vdash \alpha_k$且$\alpha=\alpha'_0 \vdash \alpha'_1 \vdash \cdots \vdash \alpha'_{k'}$，其中$\alpha_k$和$\alpha'_{k'}$是极小分划。试证明$k=k'$且$\alpha_k=\alpha'_{k'}$。

f) 试说明如何把在词典顺序下最小的分划计算成为支配一个给定的分划α的不同部分。

g) 试描述把n分成不同部分的在词典顺序下最小的分划λ_n，所有通路$n^1=\alpha_0 \vdash \alpha_1 \vdash \cdots \vdash \lambda_n^T$

58 的长度是多少？

h) 什么是形式$n^1=\alpha_0, \alpha_1, \cdots, \alpha_l=1^n$的最长和最短通路的长度，其中对于$0 \leqslant j<l$，$\alpha_j$覆盖$\alpha_{j+1}$。

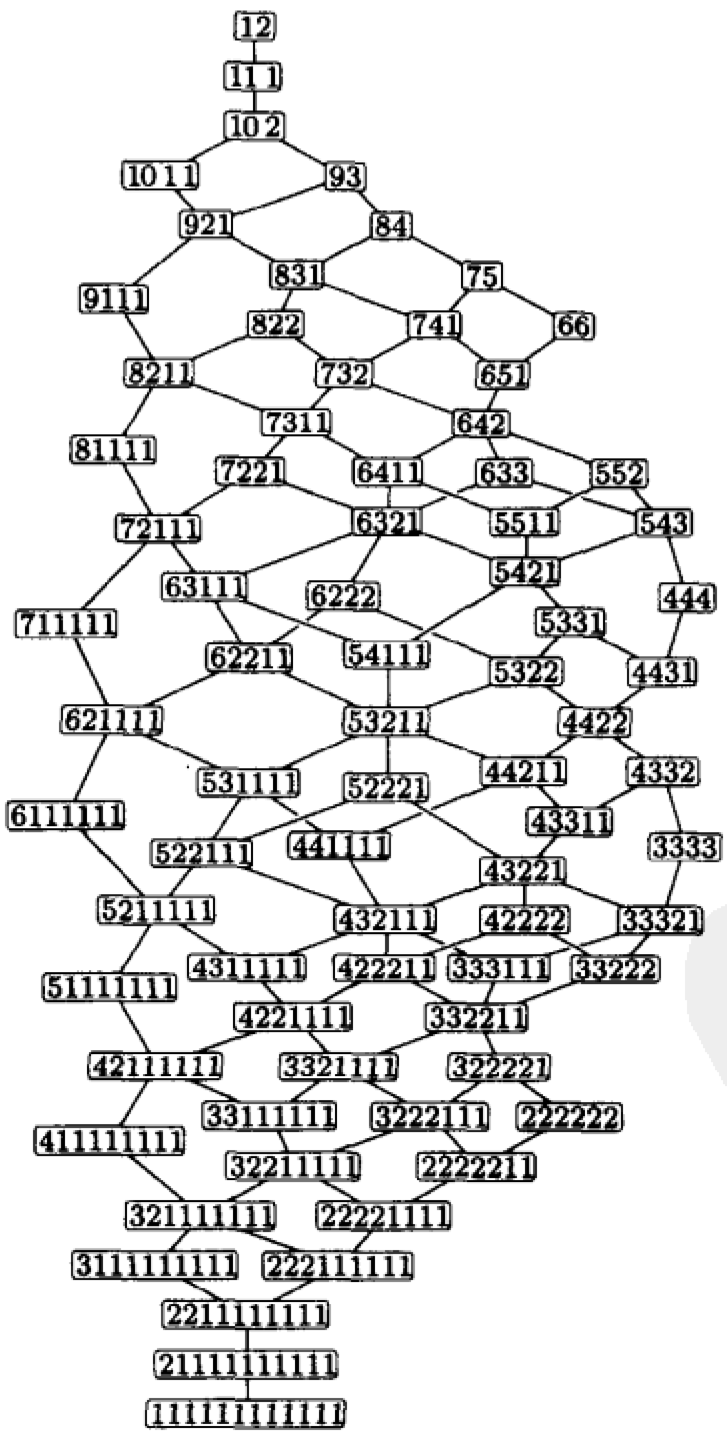


图32 对于12的分划的支配性格(参见习题54~58)

▶ 56. [*M27*] 给定分划λ和μ且$\lambda \preceq \mu$，试设计一个算法来生成所有分划α，使得$\lambda \preceq \alpha \preceq \mu$。

注：这样的算法有许多应用。例如，要生成有*m*个部分而且没有任何部分超过*l*的所有分划，可以令λ是最小的这样的分划，即如同在习题3中那样$\lceil n/m \rceil \ldots \lfloor n/m \rfloor$，并且令μ是最大的，即$((n-m+1)1^{m-1}) \wedge (l^{\lfloor n/l \rfloor}(n \bmod l))$。类似地，按照希·加·兰道(H.G. Landau) [*Bull. Math. Biophysics* 15 (1953)，143-148]的著名定理，使得

$$\left\lfloor \frac{m}{2} \right\rfloor^{\lfloor m/2 \rfloor} \left\lfloor \frac{m-1}{2} \right\rfloor^{\lceil m/2 \rceil} \preceq \alpha \preceq (m-1)(m-2)\ldots 21$$

的$\binom{m}{2}$的分划是一个循环淘汰赛的可能的“积分向量”，即使得第*j*个最强的选手打赢第a_j场比赛的分划$a_1 \cdots a_m$。

57. [*M22*] 假设0和1的一个矩阵(a_{ij})有行之和$r_i = \sum_j a_{ij}$和列之和$c_j = \sum_i a_{ij}$，则$\lambda = r_1 r_2 \cdots$和$\mu = c_1 c_2 \cdots$是$n = \sum_{i,j} a_{ij}$的分划。试证明这样的矩阵存在当且仅当$\lambda \preceq \mu^T$。 59

58. [*M23*] (对称均值。)令$\alpha = a_1 \cdots a_m$和$\beta = b_1 \cdots b_m$是*n*的分划，试证明对于变量$(x_1, \cdots, x_m)$所有非负的值，不等式

$$\frac{1}{m!}\sum x_{p_1}^{a_1} \cdots x_{p_m}^{a_m} \geqslant \frac{1}{m!}\sum x_{p_1}^{b_1} \cdots x_{p_m}^{b_m}$$

成立，其中求和是对于{1，…，*m*}的所有*m*!个排列进行，当且仅当$\alpha \succeq \beta$(例如，在$m=n$，$\alpha = n0 \ldots 0$和$\beta = 11 \ldots 1$，$x_j = y_j^{1/n}$的特殊情况下，这个不等式简化为$(y_1 + \cdots + y_n)/n \geqslant (y_1 \cdots y_n)^{1/n}$。

59. [*M22*] 在颠倒的序列6, 51, …, 111111和共轭序列$(111111)^T$, $(21111)^T$, …, $(6)^T$一样的意义下，格雷通路(59)是对称的。试求在这种方式下对称的所有格雷通路$\alpha_1, \cdots, \alpha_{p(n)}$。

60. [*23*] 通过修改在(62)和(63)中调用*L*(4, 6)的所有位置处*L*(*m*, *n*)和*M*(*m*, *n*)的定义，完成对定理S的证明。

61. [*26*] 试实现基于定理S的一个分划生成方案，并且总是确定在访问之间已经改变的两个部分。

62. [*46*] 证明或否定：对于所有充分大的整数*n*和$3 \leqslant m < n$，使得$n \bmod m \neq 0$，以及对于有$a_1 \leqslant m$的*n*的所有分划α，存在有部分$\leqslant m$的所有分划的格雷通路，并且以1^n开始和以α结束，除非$\alpha = 1^n$或$\alpha = 21^{n-2}$。

63. [*47*] 对于哪一个分划λ和μ，存在有通过所有分划α的一个格雷码，使得$\lambda \preceq \alpha \preceq \mu$?

▶ 64. [*32*] (二进制分划。)试设计一个无循环的算法，它访问把*n*分成为2的次幂的所有分划，其中每步以2^{k+1}代替2^k+2^k，或反过来。

65. [*23*] 众所周知，*m*个元素的每一个可交换群都可表示为一个带有7.2.1.3-(66)的加法操作的离散圆环体$T(m_1, \cdots, m_n)$，其中$m = m_1 \cdots m_n$且对于$1 \leqslant j < n$，m_j是m_{j+1}的一个倍数。例如，当$m = 360 = 2^3 \cdot 3^2 \cdot 5^1$时，对应于因式分解$(m_1, m_2, m_3) = (30, 6, 2)$, (60, 6, 1)，(90, 2, 2)，(120, 3, 1)，(180, 2, 1)和(360, 1, 1)，共有6个这样的群。

试说明如何通过一个算法在每步中恰改变因式m_j中的两个，系统地生成所有的因式。

▶ 66. [*M25*] (*P*分划。)不同于坚持$a_1 \geqslant a_2 \geqslant \cdots$，假设我们要来考虑满足一个给定的偏序的*n*的所有非负的合成。例如，珀·阿·麦克马洪(P. A. MacMahon)发现，“由顶向下”的不等式$a_4 \leqslant a_2 \geqslant a_3 \leqslant a_1$的所有解都可以分成5个不相重叠的类型：

$$a_1 \geqslant a_2 \geqslant a_3 \geqslant a_4; \qquad a_1 \geqslant a_2 \geqslant a_4 > a_3;$$

$$a_2 > a_1 \geqslant a_3 \geqslant a_4; \quad a_2 > a_1 \geqslant a_4 > a_3; \quad a_2 \geqslant a_4 > a_1 \geqslant a_3$$

这些类型的每一个都能容易地枚举，因为，例如$a_2 > a_1 \geqslant a_4 > a_3$等价于$a_2 - 2 \geqslant a_1 - 1 \geqslant a_4 - 1 \geqslant a_3$；对于$a_3 \geqslant 0$和$a_1+a_2+a_3+a_4=n$，解的个数是把$n-1-2-0-1$分成至多4个部分的分划数。

试说明如何来求解这类的一般问题：给定m个元素上的任何偏序关系$\prec$，考虑具有如
60 下性质，即当$j \prec k$时$a_j \geqslant a_k$的所有m元组$a_1 \cdots a_m$。假设已经对下标作了选择，使得$j \prec k$意味着$j < k$，试说明所有这些合意的m元组恰好落入N个类中，即拓扑排序算法7.2.1.2V的每一个输出在一个类中。对于非负的和加起来等于n的所有这样的$a_1 \cdots a_m$，生成函数是什么？你怎样才能把它们全都生成出来？

67. [*M25*] (珀·阿·麦克马洪，1886。)n 的一个完美分划是这样一个多重集合，它恰有$n+1$个子多重集合，而且这些多重集合是整数0, 1, …, n的分划。例如，多重集合{1, 1, 1, 1, 1}，{2, 2, 1}和{3, 1, 1}是5的完美分划。

试说明如何构造有最小元素的n的完美分划。

68. [*M23*] 当(a)给定m，(b)m任意时，n分成m个部分的什么分划有最大乘积$a_1 \cdots a_m$?

69. [*M30*] 试求使得方程$x_1+x_2+\cdots+x_n=x_1 x_2 \cdots x_n$在正整数$x_1 \geqslant x_2 \geqslant \cdots \geqslant x_n$中仅有惟一解的所有$n<10^9$。(例如，当$n$=2、3或4时，仅有惟一解；但却有5+2+1+1+1=5·2·1·1·1和3+3+1+1+1=3·3·1·1·1和2+2+2+1+1=2·2·2·1·1。)

70. [*M30*] (“保加利亚单人纸牌戏”。)取n张牌并任意把它们分成一堆或多堆，然后反复地从每堆删去一张牌并形成一个新堆。

试证明，如果$n=1+2+\cdots+m$，则这个过程总是达到一个自重复状态且有大小为{m, $m-1$, …, 1}的堆。例如，如果n=10，而且如果我们以堆大小为{3, 3, 2, 2}的堆开始，则得到分划序列:

$$3322 \to 42211 \to 5311 \to 442 \to 3331 \to 4222 \to 43111 \to 532 \to 4321 \to 4321 \to \cdots$$

对于n的其他值，什么是可能的循环状态?

71. [*M46*] 继续上一题，在n张牌的保加利亚单人纸牌戏达到一个循环状态之前，能出现的极大步骤数是什么?

72. [*M25*] 假设我们写下n的所有分划，例如当n=6时为

6, 51, 42, 411, 33, 321, 3111, 222, 2211, 21111, 111111

而且把k的每第j个出现改成j:

1, 11, 11, 112, 12, 111, 1123, 123, 1212, 11234, 123456

a) 试证明这个操作产生个别元素的一个排列。

b) 元素k总共出现多少次?

7.2.1.5 生成所有集合的分划

现在让我们改变话题，并专注于稍微不同类型的分划。一个集合的分划是把这个集合当作称为块的非空不相交子集的并的方法。例如，我们在7.2.1.4-(2)和

7.2.1.4-(4)中，即在上一小节的开始处，列出了{1, 2, 3}的5个实际上不同的分划。这5种分划也可以以如下形式更紧凑地写出:

$$123,\ 12|3,\ 13|2,\ 1|23,\ 1|2|3 \tag{1}$$

用垂直线来把一个块同其他块分开。在这个列表中，每个块的元素可以以任何顺序写出，而且诸块本身也是这样，因为“13|2”、“31|2”、“2|13”以及“2|31”全都表示相同的分划。但是我们可以通过约定，例如，每块的元素以递增顺序出现，并且以它们最小元素的递增顺序来安排诸块。通过这个约定，{1, 2, 3, 4}的分划是 61

$$\begin{aligned}&1234,\ 123|4,\ 124|3,\ 12|34,\ 12|3|4,\ 134|2,\ 13|24,\ 13|2|4,\\&14|23,\ 1|234,\ 1|23|4,\ 14|2|3,\ 1|24|3,\ 1|2|34,\ 1|2|3|4\end{aligned} \tag{2}$$

这是以所有可能的方式来把4安排到(1)的块中得到的。

集合分划在许多不同范畴中出现。例如，政治学家和经济学家经常把它们看作是“联盟”；计算机系统设计师可能把它们考虑为内存访问的“高速缓冲命中模式”；诗人则认为它们是“音节方案”(参见习题34~37)。在2.3.3节我们看到，对象之间的等价关系——即反身、对称和传递的任何二元关系——定义把这些对象分成所谓的“等价类”的一个分划。反之，每个集合分划定义一个等价关系。如果Π是$\{1, 2, \cdots, n\}$的一个分划，则每当j和k属于Π的同一块时，我们可以写:

$$j \equiv k \pmod{\Pi} \tag{3}$$

在一台计算机中，表示一个集合分划的最方便的方式之一是把它编码成一个限定增长的串，即这样一个非负整数的串$a_1 a_2 \cdots a_n$，其中我们有:

$$a_1=0\text{和对于}1 \le j<n,\ a_{j+1} \le 1+\max(a_1, \cdots, a_j) \tag{4}$$

这里的想法是置$a_j = a_k$当且仅当$j \equiv k$，而且每当j是它所在的块中的最小者时，就对a_j选择最小可利用的数。例如，对于(2)中的15个分划，限制增长的串分别是

$$\begin{aligned}&0000,\ 0001,\ 0010,\ 0011,\ 0012,\ 0100,\ 0101,\ 0102,\\&0110,\ 0111,\ 0112,\ 0120,\ 0121,\ 0122,\ 0123\end{aligned} \tag{5}$$

这一约定提议以下由乔治·哈特钦森(George Hutchinson)给出的简单生成方案[*CACM* 6 (1963), 613-614]:

算法H(*在词典顺序下的限制增长的串*) 给定$n \ge 2$，本算法通过访问满足限制增长条件(4)的所有串$a_1 a_2 \cdots a_n$，来生成$\{1, 2, \cdots, n\}$的所有分划。我们维持一个辅助数组$b_1 b_2 \cdots b_n$，其中$b_{j+1} = 1+\max(a_1, \cdots, a_j)$。为有效起见，$b_n$的值实际上被保持在分开的变量$m$中。

H1. [初始化。] 置$a_1 \cdots a_n \leftarrow 0 \cdots 0$，$b_1 \cdots b_{n-1} \leftarrow 1 \cdots 1$，且$m \leftarrow 1$。

H2. [访问。] 访问限制增长串$a_1 \cdots a_n$，它把一个分划表示成为$m+[a_n=m]$块，然后如果$a_n=m$，则转向H4。

H3. [增加a_n。] 置$a_n \leftarrow a_n+1$，并返回H2。

62 H4. [求j。] 置$j \leftarrow n-1$；然后当$a_j=b_j$时，置$j \leftarrow j-1$。

H5. [增加a_j。] 如果$j=1$则结束。否则置$a_j \leftarrow a_j+1$。

H6. [$a_{j+1} \cdots a_n$ 清空。] 置$m \leftarrow b_j+[a_j=b_j]$，并且$j \leftarrow j+1$。然后当$j<n$时，置$a_j \leftarrow 0$，$b_j \leftarrow m$及$j \leftarrow j+1$，最后置$a_n \leftarrow 0$并返回H2。 ▌

习题47证明，步骤H4~H6很少是需要的，因而在H4和H6中的循环几乎总是短的。这个算法的链接表版本见习题2。

集合分划的格雷码。快速地通过所有集合分划的一个方法是，在每步中仅仅改变限制增长串$a_1 \cdots a_n$的一个数字，因为对于a_j的一个改动只不过意味着元素j从一个块移动到另一个块。安排这样一个表的一个优雅方法是由吉迪安 · 厄尔里兹提出的[*JACM* **20** (1973)，507-508]：我们可以逐次地附加数字

$$0, m, m-1, \cdots, 1 \text{或} 1, \cdots, m-1, m, 0 \tag{6}$$

到$n-1$个元素的分划的表中的每个串$a_1 \cdots a_{n-1}$，其中$m=1+\max(a_1, \cdots, a_{n-1})$，在这两个情况之间交替地进行。因此对于$n=2$的表‘00, 01’变成对于$n=3$的‘000, 001, 011, 012, 010’；而且当我们把它扩展到$n=4$的情况时，这个表变成

0000，0001，0011，0012，0010，0110，0112，0111，
0121，0122，0123，0120，0100，0102，0101 (7)

习题14表明厄尔里兹的方案导致一个简单的算法，它实现这个格雷码顺序，而又不必比算法H做更多的工作。

然而，假设我们并不对所有分划感兴趣，我们仅想要恰有m个块的那些。我们能否跑遍限制增长串的这一小些的汇集，而每次仍然只改变一个数字？是的，弗朗克 · 拉斯基已经发现生成这样一个表的非常漂亮的方法[*Lecture Notes in Comp. Sci.* **762** (1993)，205-206]。他定义两个这样的序列A_{mn}和A'_{mn}，两者都以在词典顺序下最小的m块串$0^{n-m}01\cdots(m-1)$开始。如果$n>m+1$，两者的区别在于A_{mn}以$01\ldots(m-1)0^{n-m}$结束，而A'_{mn}以$0^{n-m-1}01\ldots(m-1)0$结束。当$1<m<n$时，拉斯基的递归规则如下：

$$A_{m(n+1)} = \begin{cases} A_{(m-1)n}(m-1), A^R_{mn}(m-1), \cdots, A^R_{mn}1, A_{mn}0 & \text{，如果}m\text{为偶数} \\ A'_{(m-1)n}(m-1), A_{mn}(m-1), \cdots, A^R_{mn}1, A_{mn}0 & \text{，如果}m\text{为奇数} \end{cases} \tag{8}$$

$$A'_{m(n+1)} = \begin{cases} A'_{(m-1)n}(m-1), A_{mn}(m-1), \cdots, A_{mn}1, A^R_{mn}0 & \text{，如果}m\text{为偶数} \\ A_{(m-1)n}(m-1), A^R_{mn}(m-1), \cdots, A_{mn}1, A^R_{mn}0 & \text{，如果}m\text{为奇数} \end{cases} \tag{9}$$

当然，基础情况只不过是一个元素的表，

63 $$A_{1n} = A'_{1n} = \{0^n\} \text{和} A_{nn} = \{01\ldots(n-1)\} \tag{10}$$

通过这些定义，$\{1,2,3,4,5\}$分成3块的$\left\{ {5 \atop 3} \right\}=25$个分划是

00012，00112，01112，01012，01002，01102，00102，

$$\begin{array}{l} 00122,\ 01122,\ 01022,\ 01222,\ 01212,\ 01202, \\ 01201,\ 01211,\ 01221,\ 01021,\ 01121,\ 00121, \\ 00120,\ 01120,\ 01020,\ 01220,\ 01210,\ 01200 \end{array} \qquad (11)$$

(关于有效的实现，请参见习题17。)

在厄尔里兹的方案(7)中，$a_1 \cdots a_n$ 最右边的数字最快速地改变，但在拉斯基的方案中，大多数的变化出现在靠近左边处。然而，在两种情况下，每一步仅影响一个数字a_j，而且这些改变都十分简单，或者a_j加减1而变，或者它在两个极端的值0和$1+\max(a_1, \cdots, a_{j-1})$之间跳动。在同样的限定下，序列 $A'_{1n}, A'_{2n}, \cdots, A'_{nn}$ 以块的号码的递增顺序跑遍所有分划。

集合分划的个数。我们已经看到，{1, 2, 3}有5个分划，而{1, 2, 3, 4}有15个。查·圣·珀西(C. S. Peirce)发现了计算这些计数的一个快速方法，他在(*American Journal of Mathematics* **3** (1880)，48页)上给出了以下数的三角：

$$\begin{array}{llllll} 1 \\ 2 & 1 \\ 5 & 3 & 2 \\ 15 & 10 & 7 & 5 \\ 52 & 37 & 27 & 20 & 15 \\ 203 & 151 & 114 & 87 & 67 & 52 \end{array} \qquad (12)$$

这里第n行条目$\varpi_{n1}, \varpi_{n2}, \cdots, \varpi_{nn}$满足简单递归式

$$\varpi_{nk} = \varpi_{(n-1)k} + \varpi_{n(k+1)}，\ 如果 1 \leqslant k<n；\quad \varpi_{nn} = \varpi_{(n-1)1}，\ 如果 n>1 \qquad (13)$$

而且$\varpi_{11}=1$。珀西的三角有许多值得注意的性质，习题26~31介绍了其中一些。例如，ϖ_{nk}是$\{1, 2, \cdots, n\}$的分划的个数，其中k是在它所在的块中最小的。

珀西三角对角线上的条目和头一列的条目告诉我们集合分划的全部个数，通常称为贝尔数。因为埃·坦·贝尔写了好多篇关于它们的有影响的论文[*AMM* **41** (1934)，411-419; *Annals of Math.* **35** (1934)，258-277; **39** (1938)，539-557]。我们将以ϖ_n来记贝尔数，追随路易斯·康姆特(Louis Comtet)的首创，以免同贝努利数B_n相混淆。最初的一些值是：

$n=$	0	1	2	3	4	5	6	7	8	9	10	11	12
$\varpi_n=$	1	1	2	5	15	52	203	877	4140	21147	115975	678570	4213597

注意这个序列快速地增长，但不像$n!$那样快；我们下边将证明$\varpi_n=\Theta(n/\log n)^n$。

对于$n>0$贝尔数$\varpi_n=\varpi_{n1}$必定满足递归公式

$$\varpi_{n+1} = \varpi_n + \binom{n}{1}\varpi_{n-1} + \binom{n}{2}\varpi_{n-2} + \cdots = \sum_k \binom{n}{k}\varpi_{n-k} \qquad (14)$$ 64

因为$\{1, \cdots, n+1\}$的每一个分划通过选择$\{1, \cdots, n\}$的k个元素来放进含有$n+1$的块中，以及通过对于某个k，以ϖ_{n-k}种方法把剩下的元素分划而得到。由松永良弼在18世纪

发现(参见7.2.1.7节)的这个递归式，导致了由威·阿·惠特沃什(W. A. Whitworth)发现的[*Choice and Chance*，第3版 (1878)，3 . XXIV]一个漂亮的生成函数：

$$\Pi(z)=\sum_{n=0}^{\infty}\varpi_n\frac{z^n}{n!}=e^{e^z-1} \tag{15}$$

因为如果我们以$z^n/n!$来乘（14）的两边并对n求和，我们得到

$$\Pi'(z)=\sum_{n=0}^{\infty}\varpi_{n+1}\frac{z^n}{n!}=\left(\sum_{k=0}^{\infty}\frac{z^k}{k!}\right)\left(\sum_{m=0}^{\infty}\varpi_m\frac{z^m}{m!}\right)=e^z\Pi(z)$$

而且（15）是这个微分方程和$\Pi(0)=1$的一个解。

由于数ϖ_n同这个公式有关的一些奇怪的性质，在惠特沃什指出它们同集合分划的组合联系之前很久，数ϖ_n就已被研究了许多年。例如，我们有

$$\varpi_n=\frac{n!}{e}[z^n]e^{e^z}=\frac{n!}{e}[z^n]\sum_{k=0}^{\infty}\frac{e^{kz}}{k!}=\frac{1}{e}\sum_{k=0}^{\infty}\frac{k^n}{k!} \tag{16}$$

[*Mat. Sbornik* **3** (1868)，62; 4 (1869)，39; G. Dobiński，*Archiv der Math. und Physik* **61** (1877)，333-336; **63** (1879)，108-110。]克里斯蒂安·克兰姆普在由卡·弗·辛登伯格主编的(*Polynomische Lehrsatz* (Leipzig: 1796)，112-113)中讨论了e^{e^z}的展开；他提出计算系数的两种方法，即或者是使用(14)，或者是使用$p(n)$项的求和。这里对于n的每个通常的分划取一个。(参见阿尔博加斯特(Arbogast)的公式，习题1.2.5-21。克兰姆普接近于发现这个公式，但他似乎更喜欢他的基于分划的方法，没有认识到当n变得越来越大时，它要求高于多项式的运行时间，而且对于z^{10}的系数，他算出116015而非115975。)

***渐近估计。**通过使用复数残数理论的最基本原理之一，我们可以知道ϖ_n的增长有多快。如果每当$|z|<r$时幂级数$\sum_{k=0}^{\infty}a_kz^k$收敛，而且如果积分是沿着围绕原点逆时钟方向进行的一个简单闭回路并且驻留在圆$|z|<r$之内进行，则

$$a_{n-1}=\frac{1}{2\pi i}\oint\frac{a_0+a_1z+a_2z^2+\cdots}{z^n}dz \tag{17}$$

令$f(z)=\sum_{k=0}^{\infty}a_kz^{k-n}$是被积函数。我们可自由地选择任何这样的通路，但当通路通过一个点z_0而微分$f'(z_0)$为零时，一些特殊技术经常适用。因为在这样一个点附近，我们有

$$f(z_0+\varepsilon e^{i\theta})=f(z_0)+\frac{f''(z_0)}{2}\varepsilon^2e^{2i\theta}+O(\varepsilon^3) \tag{18}$$

65 例如，如果$f(z_0)$和$f''(z_0)$为实数且为正，比如说$f(z_0)=u$且$f''(z_0)=2v$，这个公式表明，$f(z_0\pm\varepsilon)$的值近似于$u+v\varepsilon^2$，而$f(z_0\pm i\varepsilon)$近似于$u-v\varepsilon^2$。如果z从$z_0-i\varepsilon$移动到$z_0+i\varepsilon$，则$f(z)$的值上升到一个极大值u。然后再次下降；但是更大的值$u+v\varepsilon^2$出现在这个通

路的左边和右边。换言之，一个复杂平面上徒步旅行的登山者，当在点z的高度为$\Re f(z)$时，在z_0处遇到一个“关口”。在这一点处的地形好像是一个马鞍点。如果沿任何通路来进行，则$f(z)$的整个积分将是相同的，但是不通过这个关口的通路就不一定那样好，因为它将必须删除可能加以避免的$f(z)$的某些高值。因此，我们倾向于通过在增加虚数部分的方向上选择通过z_0的一条通路，来获得最好的结果。由彼·德拜[*Math. Annalen* 67 (1909)，535-558]给出的这一重要技术称为“马鞍点方法”。

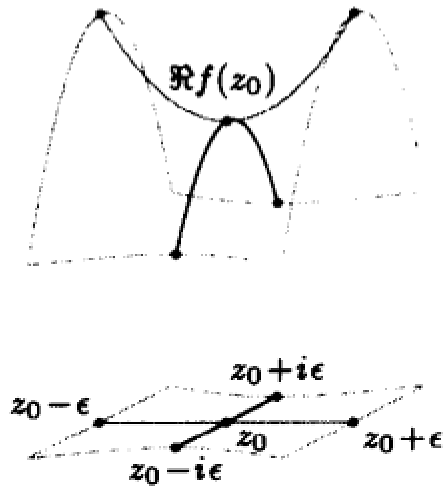


图33 靠近马鞍点的一个解析函数的特性

我们从已经知道答案

$$\frac{1}{(n-1)!}=\frac{1}{2\pi i}\oint\frac{e^z}{z^n}dz \tag{19}$$

的一个例子开始，来熟悉一下马鞍点方法。我们的目标是当n很大时，求右边积分值的一个好的近似。通过把$f(z)=e^z/z^n$ 写成为$e^{g(z)}$，其中$g(z)=z-n\ln z$，将很方便地来处理它。马鞍点出现于$g'(z_0)=1-n/z_0$为零处，即在$z_0=n$处。如果$z=n+it$，我们有

$$\begin{aligned} g(z) &= g(n)+\sum_{k=2}^{\infty}\frac{g^{(k)}(n)}{k!}(it)^k \\ &= n-n\ln n-\frac{t^2}{2n}+\frac{it^3}{3n^2}+\frac{t^4}{4n^3}-\frac{it^5}{5n^4}+\cdots \end{aligned}$$

因为当$k\geqslant 2$时，$g^{(k)}(z)=(-1)^k(k-1)!n/z^k$。让我们在从$n-im$到$n+im$到$-n+im$到$-n-im$到$n-im$的矩形通路上来积分$f(z)$。

$$\begin{aligned} \frac{1}{2\pi i}\oint\frac{e^z}{z^n}dz &= \frac{1}{2\pi}\int_{-m}^{m}f(n+it)dt+\frac{1}{2\pi i}\int_{n}^{-n}f(t+im)dt \\ &\quad+\frac{1}{2\pi}\int_{m}^{-m}f(-n+it)dt+\frac{1}{2\pi i}\int_{-n}^{n}f(t-im)dt \end{aligned}$$ 66

显然，如果我们选择$m=2n$，则在这条通路上的最后三边上$|f(z)|<2^{-n}f(n)$，因为$|e^z|=e^{\Re z}$和$|z|\geqslant\max(\Re z,\Im z)$。所以我们剩下的是

$$\frac{1}{2\pi i}\oint\frac{e^z}{z^n}dz=\frac{1}{2\pi}\int_{-m}^{m}e^{g(n+it)}dt+O\left(\frac{ne^n}{2^n n^n}\right)$$

现在我们又回到以前已用过多次(例如推导等式5.1.4-(53))的一个技术：如果当$t\in A$时，$\hat{f}(t)$是对于$f(t)$的一个好的近似，而且如果两个和$\sum_{t\in B\setminus A}f(t)$和$\sum_{t\in C\setminus A}\hat{f}(t)$都很小，则$\sum_{t\in C}\hat{f}(t)$是$\sum_{t\in B}f(t)$的一个好的近似。同一想法也适用于积分。[由拉

普拉斯于1782年引入的这个一般的方法，通常称为“折衷的尾部”；参见*CMath* §9.4。]如果$|t|\leqslant n^{1/2+\epsilon}$，我们有

$$\begin{aligned}e^{g(n+it)}&=\exp\left(g(n)-\frac{t^2}{2n}+\frac{it^3}{3n^2}+\cdots\right)\\&=\frac{e^n}{n^n}\exp\left(-\frac{t^2}{2n}+\frac{it^3}{3n^2}+\frac{t^4}{4n^3}+O(n^{5\epsilon-3/2})\right)\\&=\frac{e^n}{n^n}e^{-t^2/(2n)}\left(1+\frac{it^3}{3n^2}+\frac{t^4}{4n^3}-\frac{t^6}{18n^4}+O(n^{9\epsilon-3/2})\right)\end{aligned}$$

而且当$|t|>n^{1/2+\epsilon}$时，有

$$|e^{g(n+it)}|<|f(n+in^{1/2+\epsilon})|=\frac{e^n}{n^n}\exp\left(-\frac{n}{2}\ln(1+n^{2\epsilon-1})\right)=O\left(\frac{e^{n-n^{2\epsilon}/2}}{n^n}\right)$$

而且，不完备伽玛函数

$$\int_{n^{1/2+\epsilon}}^{\infty}e^{-t^2/(2n)}t^k dt=2^{(k-1)/2}n^{(k+1)/2}\Gamma\left(\frac{k+1}{2},\frac{n^{2\epsilon}}{2}\right)=O(n^{O(1)}e^{-n^{2\epsilon}/2})$$

是可忽略的，因此我们可以让掉尾部并且得到近似式

$$\begin{aligned}\frac{1}{2\pi i}\oint\frac{e^z}{z^n}dz&=\frac{e^n}{2\pi n^n}\int_{-\infty}^{\infty}e^{-t^2/(2n)}\left(1+\frac{it^3}{3n^2}+\frac{t^4}{4n^3}-\frac{t^6}{18n^4}+O(n^{9\epsilon-3/2})\right)dt\\&=\frac{e^n}{2\pi n^n}\left(I_0+\frac{i}{3n^2}I_3+\frac{1}{4n^3}I_4-\frac{1}{18n^4}I_6+O(n^{9\epsilon-3/2})\right)\end{aligned}$$

其中 $I_k=\int_{-\infty}^{\infty}e^{-t^2/(2n)}t^k dt$ 。当然当k为奇数时，$I_k=0$。否则通过使用以下著名事实，即当$a>0$时

$$\int_{-\infty}^{\infty}e^{-at^2}t^{2l}dt=\frac{\Gamma((2l+1)/2)}{a^{(2l+1)/2}}=\frac{\sqrt{2\pi}}{(2a)^{(2l+1)/2}}\prod_{j=1}^{l}(2j-1)\qquad(20)$$

我们可计算I_k。把每件东西都放到一起，就给出了我们渐近的估计，对所有$\epsilon>0$

$$\frac{1}{(n-1)!}=\frac{e^n}{\sqrt{2\pi}n^{n-1/2}}\left(1+0+\frac{3}{4n}-\frac{15}{18n}+O(n^{9\epsilon-3/2})\right)\qquad(21)$$

这个结果同斯特林的近似很完美地一致。在1.2.11.2-(19)中，我们使用了十分不同
67 的方法推导过它。在$g(n+it)$的展开式中进一步的项允许我们来证明，(21)中的真正误差仅仅是$O(n^{-2})$。因为相同的过程对于所有的m产生一般形式

$$e^n/(\sqrt{2\pi}n^{n-1/2})(1+c_1/n+c_2/n^2+\cdots+c_m/n^m+O(n^{-m-1}))$$

的渐近级数。

我们对这一结果的推导掩饰了一个重要的技术细节：函数$\ln z$沿着积分通路不

是单值的，因为当我们绕着原点循环时，它以$2\pi i$增长。确实，这个事实奠定了基本机理的基础，即它使得残数定理有效。但我们的推理是正确的，因为对数的这个二义性不影响当n是整数时$f(z)=e^z/z^n$的被积函数。而且，如果n真的不是整数的话，我们就可以采用这个论断，并且通过选择沿着在$-\infty$处开始逆时钟地环绕原点并返回$-\infty$的一条通路来进行积分，而使它严格化。而这将给我们提供伽玛函数的汉克尔(Hankel)积分，即等式1.2.5-(17)。由此我们推导出，当$x\to\infty$时对所有实数x正确的渐近公式

$$\frac{1}{\Gamma(x)}=\frac{1}{2\pi i}\oint\frac{e^z}{z^x}dz=\frac{e^x}{\sqrt{2\pi}x^{x-1/2}}\left(1-\frac{1}{12x}+O(x^{-2})\right) \qquad (22)$$

因此马鞍点方法似乎有效——尽管它不是得到这一特定结果的最简单方法。我们现在应用它来推导贝尔数的近似大小:

$$\frac{\varpi_{n-1}}{(n-1)!}=\frac{1}{2\pi ie}\oint e^{g(z)}dz,\quad g(z)=e^z-n\ln z \qquad (23)$$

现在马鞍点出现在点$z_0=\xi>0$处，其中

$$\xi e^\xi=n \qquad (24)$$

(我们实际上应当写$\xi(n)$来指出ξ依赖于n；但这将使下面的一些公式很杂乱。)暂时且假定我们从某处获知了ξ的值，然后我们要沿着使$z=\xi+it$的一条通路来计算积分，而且我们有

$$g(\xi+it)=e^\xi-n\left(\ln\xi-\frac{(it)^2}{2!}\frac{\xi+1}{\xi^2}-\frac{(it)^3}{3!}\frac{\xi^2-2!}{\xi^3}-\frac{(it)^4}{4!}\frac{\xi^3+3!}{\xi^4}+\cdots\right)$$

通过在一个适当的矩形通路上进行积分，我们可以如同在上边那样证明，(23）中的积分可以通过下式很好地近似

$$\int_{-n^{\epsilon-1/2}}^{n^{\epsilon-1/2}}e^{g(\xi)-na_2t^2-nia_3t^3+na_4t^4+\cdots}dt,\quad a_k=\frac{\xi^{k-1}+(-1)^k(k-1)!}{k!\,\xi^k} \qquad (25)$$

参见习题43。注意a_kt^k是在这个积分中的$O(n^{k\epsilon-k/2})$，我们得到形如

$$\varpi_{n-1}=\frac{e^{e^\xi-1}(n-1)!}{\xi^{n-1}\sqrt{2\pi n(\xi+1)}}\left(1+\frac{b_1}{n}+\frac{b_2}{n^2}+\cdots+\frac{b_m}{n^m}+O\left(\left(\frac{\log n}{n}\right)^{m+1}\right)\right) \qquad (26)$$ 68

的渐近展开，其中$(\xi+1)^{3k}b_k$是ξ的次数为$4k$的一个多项式(参见习题44)。例如

$$b_1=-\frac{2\xi^4-3\xi^3-20\xi^2-18\xi+2}{24(\xi+1)^3} \qquad (27)$$

$$b_2=\frac{4\xi^8-156\xi^7-695\xi^6-696\xi^5+1092\xi^4+2916\xi^3+1972\xi^2-72\xi+4}{1152(\xi+1)^6} \qquad (28)$$

在(26)中使用斯特林近似式(21)可证明

$$\varpi_{n-1} = \exp\left(n\left(\xi - 1 + \frac{1}{\xi}\right) - \xi - \frac{1}{2}\ln(\xi+1) - 1 - \frac{\xi}{12n} + O\left(\frac{\log n}{n}\right)^2\right) \tag{29}$$

而且习题45证明类似的公式

$$\varpi_{n} = \exp\left(n\left(\xi - 1 + \frac{1}{\xi}\right) - \frac{1}{2}\ln(\xi+1) - 1 - \frac{\xi}{12n} + O\left(\frac{\log n}{n}\right)^2\right) \tag{30}$$

结果我们有 $\varpi_n/\varpi_{n-1} \approx e^{\xi} = n/\xi$ 。更精确地

$$\frac{\varpi_{n-1}}{\varpi_n} = \frac{\xi}{n}\left(1 + O\left(\frac{1}{n}\right)\right) \tag{31}$$

但是什么是ξ的渐近值呢？定义(24)意味着

$$\begin{aligned}\xi = \ln n - \ln\xi &= \ln n - \ln(\ln n - \ln\xi)\\ &= \ln n - \ln\ln n + O\left(\frac{\log\log n}{\log n}\right)\end{aligned} \tag{32}$$

我们可以像在习题49中所示那样，沿着这一方向继续进行下去。但是对于越来越大的m值，用这个方法建立的ξ的渐近级数绝不给出比$O(1/(\log\ n)^m)$更好的精确性。所以当在关于ϖ_{n-1}的公式(29)和关于ϖ_n的公式(30)中乘以n时，它是极其不精确的。

因此如果我们要使(29)或(30)来计算贝尔数的好的数值近似的话，最好的策略是不使用一个缓慢地收敛的级数，而以计算对于ξ的一个好的数值值开始。在算法4.7N前边的评述中所讨论的牛顿的求根法，产生有效的迭代方案

$$\xi_0 = \ln n, \quad \xi_{k+1} = \frac{\xi_k}{\xi_k + 1}(1 + \xi_0 - \ln\xi_k) \tag{33}$$

它快速地收敛到正确的值。例如，当n=100时，第5个迭代

$$\xi_5 = 3.38563\ 01402\ 90050\ 18488\ 82443\ 64529\ 72686\ 74917\ - \tag{34}$$

已经正确到40位十进制数。使用(29)中的这个值，当考虑到项b_0, b_1, b_2, b_3时，给了我们逐次的近似

$$(1.6176088053\cdots, 1.6187421339\cdots, 1.6187065391\cdots, 1.6187060254\cdots) \times 10^{114}$$

69 ϖ_{99}的实际值是115位整数16187060274460…20741。

既然我们知道了集合分划ϖ_n 的个数，让我们尝试来想出它们中有多少个恰有m个块。结果是{1, …, n}的几乎所有分划都有大约$n/\xi=e^{\xi}$块，而且每块大约有ξ个元素。例如，图34示出当n=100时数 $\left\{ n \atop m \right\}$ 的一个矩方图且$e^{\xi} \approx 29.54$。

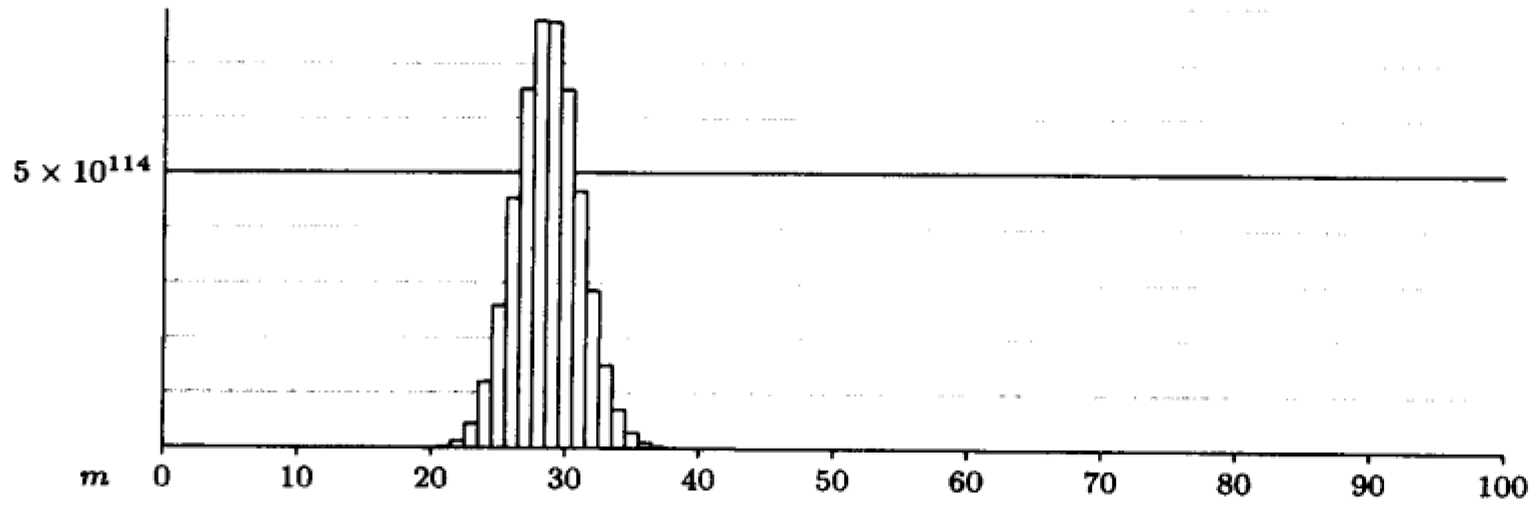


图34　在接近m=28和m=29处，斯特林数$\left\{ {100 \atop m} \right\}$是最大的

通过应用马鞍点方法到公式1.2.9-(23)，我们可以观察$\left\{ {n \atop m} \right\}$的大小，它指出

$$\left\{ {n \atop m} \right\} = \frac{n!}{m!}[z^n](e^z-1)^m = \frac{n!}{m!}\frac{1}{2\pi i}\oint e^{m\ln(e^z-1)-(n+1)\ln z}dz \qquad (35)$$

令$\alpha=(n+1)/m$。当

$$\frac{\sigma}{1-e^{-\sigma}} = \alpha \qquad (36)$$

时，函数$g(z)=\alpha^{-1}\ln(e^z-1)-\ln z$在$\sigma>0$处有一个马鞍点。注意对于$1<m<n$，$\alpha>1$。这特殊的$\sigma$值通过

$$\sigma=\alpha-\beta,\quad \beta=T(\alpha e^{-\alpha}) \qquad (37)$$

给出，其中T是等式2.3.4.4-(30)的树函数。其实，β是0和1之间的值，对于它我们有

$$\beta e^{-\beta}=\alpha e^{-\alpha} \qquad (38)$$

当x从0增到1时，函数xe^{-x}从0增到e^{-1}，然后它再次减少到0。因此β是惟一确定的，而且我们有

$$e^{\sigma}=\frac{\alpha}{\beta} \qquad (39)$$

通过使用反演公式

$$\alpha=\frac{\sigma e^{\sigma}}{e^{\sigma}-1},\quad \beta=\frac{\sigma}{e^{\sigma}-1} \qquad (40)$$

所有这样的对偶α和β可得到，例如值$\alpha=\ln 4$和$\beta=\ln 2$对应于$\sigma=\ln 2$。

我们可以如同上边一样证明，(35)中的积分渐近地等价于在通路$z=\sigma+it$上的$e^{(n+1)g(z)}dz$的一个积分(参见习题58)。习题56证明，关于$z=\sigma$的泰勒级数 70

$$g(\sigma+it)=g(\sigma)-\frac{t^2(1-\beta)}{2\sigma^2}-\sum_{k=3}^{\infty}\frac{(it)^k}{k!}g^{(k)}(\sigma) \qquad (41)$$

有以下性质，即对于所有$k>0$

$$\left|g^{(k)}(\sigma)\right| < 2(k-1)!(1-\beta)/\sigma^k \tag{42}$$

因此我们可以方便地从幂级数$(n+1)g(z)$删去$N=(n+1)(1-\beta)$的一个因式，而且马鞍点方法导致以下公式，当$N\to\infty$时

$$\left\{ {n \atop m} \right\} = \frac{n!}{m!}\frac{1}{(\alpha-\beta)^{n-m}\beta^m\sqrt{2\pi N}}\left(1+\frac{b_1}{N}+\frac{b_2}{N^2}+\cdots+\frac{b_l}{N_l}+O\left(\frac{1}{N^{l+1}}\right)\right) \tag{43}$$

其中$(1-\beta)^{2k}b_k$是α和β的一个多项式。(分母中的量$(\alpha-\beta)^{n-m}\beta^m$来自于这样一个事实，即由（37）和（39），$(e^\sigma-1)^m/\sigma^n=(\alpha/\beta-1)^m/(\alpha-\beta)^n$。)例如

$$b_1=\frac{6-\beta^3-4\alpha\beta^2-\alpha^2\beta}{8(1-\beta)}-\frac{5(2-\beta^2-\alpha\beta)^2}{24(1-\beta)^2} \tag{44}$$

习题57证明$N\to\infty$当且仅当$n-m\to\infty$。$\left\{ {n \atop m} \right\}$的一个渐近展开类似于（43），但稍微更复杂些，它首先是由利奥·莫泽(Leo Moser)和马克·维曼(Max Wyman)得到的[*Duke Math. J.* **25** (1957)，29-43]。

公式（43）看起来有点吓人，因为它被设计来应用到块计数m的整个范围，但当m相对较小或相对较大时，有可能做重大简化(参见习题60和61)。但当$\left\{ {n \atop m} \right\}$是最大时，简化的公式不给出在重要情况下的精确结果。现在来更仔细地考察那些关键的情况，使得我们可以解释图34中示出的陡峭的峰值。

如同在（24）中那样令$\xi e^\xi=n$，并假定$m=\exp(\xi+r/\sqrt{n})=ne^{r/\sqrt{n}}/\xi$；我们将假定$|r|\le n^\epsilon$，使得$m$接近于$e^\xi$。（43）中的前导项可以改写成

$$\frac{n!}{m!}\frac{1}{(\alpha-\beta)^{n-m}\beta^m\sqrt{2\pi(n+1)(1-\beta)}}=$$
$$\frac{m^n}{m!}\frac{(n+1)!}{(n+1)^{n+1}}\frac{e^{n+1}}{\sqrt{2\pi(n+1)}}\left(1-\frac{\beta}{\alpha}\right)^{m-n}\frac{e^{-\beta m}}{\sqrt{1-\beta}} \tag{45}$$

而且$(n+1)!$的斯特林近似式明显地已可在这表达式中间删去。借助于计算机代数的帮助，我们求得

$$\frac{m^n}{m!}=\frac{1}{\sqrt{2\pi}}\exp\left(n\left(\xi-1+\frac{1}{\xi}\right)-\frac{1}{2}\left(\xi+r^2+\frac{r^2}{\xi}\right)\right.$$
$$\left.-\left(\frac{r}{2}+\frac{r^3}{6}+\frac{r^3}{3\xi}\right)\frac{1}{\sqrt{n}}+O(n^{4\epsilon-1})\right)$$

71

而且同α和β相关的量是

$$\frac{\beta}{\alpha}=\frac{\xi}{n}+\frac{r\xi^2}{n\sqrt{n}}+O(\xi^3 n^{2\varepsilon-2})$$

$$e^{-\beta m}=\exp\left(-\xi-\frac{r\xi^2}{\sqrt{n}}+O(\xi^3 n^{2\varepsilon-1})\right)$$

$$\left(1-\frac{\beta}{\alpha}\right)^{m-n}=\exp\left(\xi-1+\frac{r(\xi^2-\xi-1)}{\sqrt{n}}+O(\xi^3 n^{2\varepsilon-1})\right)$$

因此整个结果是

$$\left\{{n \atop e^{\xi+r/\sqrt{n}}}\right\}=\frac{1}{\sqrt{2\pi}}\exp\left(n\left(\xi-1+\frac{1}{\xi}\right)-\frac{\xi}{2}-1-\frac{\xi+1}{2\xi}\left(r+\frac{3\xi(2\xi+3)+(\xi+2)r^2}{6(\xi+1)\sqrt{n}}\right)^2+O(\xi^3 n^{4\varepsilon-1})\right) \tag{46}$$

当

$$r=-\frac{\xi(2\xi+3)}{2(\xi+1)\sqrt{n}}+O(\xi^2 n^{-3/2})$$

时最后一行上的平方表达式为零，因此当块的个数为

$$m=\frac{n}{\xi}-\frac{3+2\xi}{2+2\xi}+O\left(\frac{\xi}{n}\right) \tag{47}$$

时出现极大值。通过把（47）同（30）做比较，我们看到，对于一个给定的n的值，最大的斯特林数$\left\{{n \atop m}\right\}$近似地等于 $\xi\varpi_n/\sqrt{2\pi n}$ 。

马鞍点方法适用于比我们在这里已经考虑的一些问题还要困难得多的问题。在好几本书中都可找到关于高级技术的精彩阐述：尼·戈·德·布鲁因(N. G. de Bruijn)，*Asymptotic Methods in Analysis* (1958)，第5章和第6章；弗·威·约·奥尔弗(F. W. J. Olver)，*Asymptotics and Special Functions* (1974)，第4章；王世全，*Asymptotic Approximations of Integrals* (2001)，第2章和第7章。

***随机的集合分划**。在{1, …, n}的一个分划中，块的大小本身组成数n的一个通常的分划。因此我们可能不知道它们大概是什么类型的分划。7.2.1.4节中的图30示出同25的所有$p(25)=1958$个分划的费尔利斯框图重迭的结果。这些分划趋向于遵循公式7.2.1.4-(49)的对称曲线。相反，图35示出当我们把集合{1, …, 25}的所有$\varpi_{25}\approx 4.6386\times 10^{18}$个分划对应的框图重叠时发生什么情况。显然，一个随机的集合分划的"形状"十分不同于一个随机整数分划的形状。

这个变化来自于如下事实，即某些整数分划作为集合分划的块大小只出现一些次，至于其他的则极为普通。例如，分划$n=1+1+\cdots+1$只是作为一种方式出现， 72
但是如果n是偶数，则$n=2+2+\cdots+2$以$(n-1)(n-3)\cdots(1)$种方式出现。当$n=25$时，整数分划

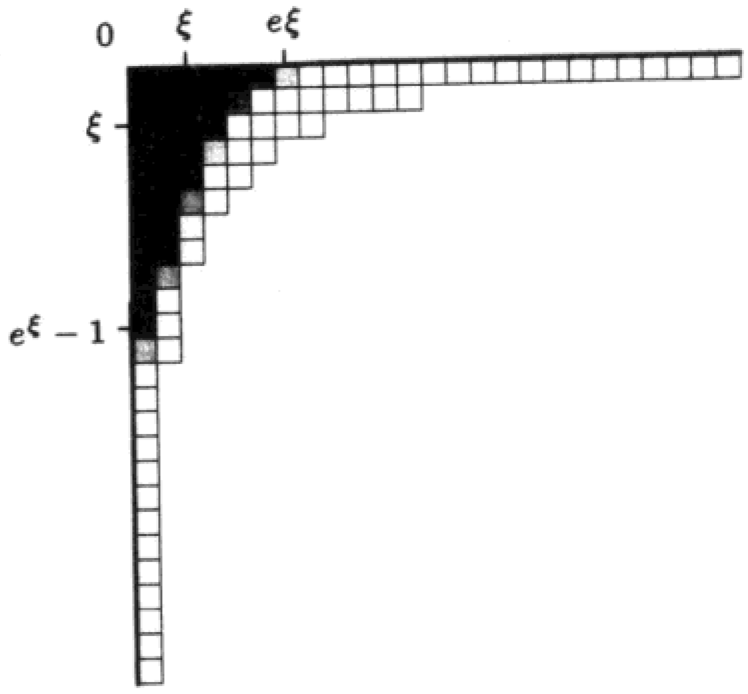


图35 当n=25时，一个随机集合分划的形状

$$25=4+4+3+3+3+2+2+2+1+1$$

实际上在所有可能的集合分划的2%以上中出现。(这个特殊的分划结果在n=25情况中是最普通的。习题1.2.5-21的答案说明恰有

$$\frac{n!}{c_1!1!^{c_1}\,c_2!2!^{c_2}\cdots c_n!n!^{c_n}} \tag{48}$$

个集合分划对应于整数分划$n=c_1\cdot 1+c_2\cdot 2+\cdots+c_n\cdot n$。)

我们能很容易地确定在一个随机的集合分划中k块的平均数，如果我们写出所有ϖ_n的可能性，则每个特定的k元块恰出现ϖ_{n-k}次，因此平均数为

$$\binom{n}{k}\frac{\varpi_{n-k}}{\varpi_n} \tag{49}$$

而且，在习题64中证明的上面等式(31)的一个扩充表明，如果$k\leqslant n^{2/3}$

$$\frac{\varpi_{n-k}}{\varpi_n}=\left(\frac{\xi}{n}\right)^k\left(1+\frac{k\xi(k\xi+k+1)}{2(\xi+1)^2 n}+O\left(\frac{k^3}{n^2}\right)\right) \tag{50}$$

其中ξ在(24)中定义。因此，若$k\leqslant n^{\epsilon}$，公式(49)就简化为

$$\frac{n^k}{k!}\left(\frac{\xi}{n}\right)^k\left(1+O\left(\frac{1}{n}\right)\right)=\frac{\xi^k}{k!}(1+O(n^{2\epsilon-1})) \tag{51}$$

平均来说，大约有大小为1的ξ块，大小为2的$\xi^2/2!$块，等等。

这些量的方差很小(参见习题65)，因而一个随机分划的特性实际上就如同k元块的个数是具有均值$\xi^k/k!$的一个泊松偏离那样。图35中示出的光滑曲线跑遍费尔利斯坐标中的点$(f(k),k)$，其中

73 $$f(k)=\xi^{k+1}/(k+1)!+\xi^{k+2}/(k+2)!+\xi^{k+3}/(k+3)!+\cdots \tag{52}$$

是从对应于块大小$k\geqslant 0$的顶部线的近似距离。(当n变得很大时，这个曲线就更接

近于垂直了。）

最大的块趋向于近似地包含$e\xi$ 个元素。而且，含有元素1的块有小于 $\xi+a\sqrt{\xi}$ 的大小的概率趋近于正态离差小于a的概率。[参见约翰·海格(John Haigh)，*J. Combinatorial Theory* **A13** (1972)，287-295；伏·尼·萨兹科夫(V. N. Sachkov)，从1978年出版的一本俄文书翻译过来的书*Probabilistic Methods in Combinatorial Analysis* (1997)，第4章；尤·雅库波维奇(Yu. Yakubovich)，从1995年发表的一篇俄文论文翻译过来的*J. Mathematical Sciences* **87** (1997)，4124-4137；博·皮梯尔(B. Pittel)，*J. Combinatorial Theory* **A79** (1997)，326-359。]

生成{1, 2, …, n}的随机分划的一个漂亮方法是由阿·约·斯塔姆(A. J. Stam)在[*Journal of Combinatorial Theory* **A35** (1983)，231-240]引进的：令M是以

$$p_m=\frac{m^n}{em!\varpi_n} \tag{53}$$

的概率取值m的一个随机整数，这些概率由于(16)而有和数1。一旦选择了M，就生成一个随机的n元组$X_1X_2\cdots X_n$，其中每个X_j 在0和$M-1$之间一致和独立地分布。然后令在这分划中$i\equiv j$当且仅当$X_i=X_j$。这个过程有效，因为每个k块分划以 $\sum_{m>0}(m^{\underline{k}}/m^n)p_m=1/\varpi_n$ 的概率被得到。

例如，如果n=25，我们有

$p_4\approx0.00000372$	$p_9\approx0.15689865$	$p_{14}\approx0.04093663$	$p_{19}\approx0.00006068$
$p_5\approx0.00019696$	$p_{10}\approx0.21855285$	$p_{15}\approx0.01531445$	$p_{20}\approx0.00001094$
$p_6\approx0.00313161$	$p_{11}\approx0.21526871$	$p_{16}\approx0.00480507$	$p_{21}\approx0.00000176$
$p_7\approx0.02110279$	$p_{12}\approx0.15794784$	$p_{17}\approx0.00128669$	$p_{22}\approx0.00000026$
$p_8\approx0.07431024$	$p_{13}\approx0.08987171$	$p_{18}\approx0.00029839$	$p_{23}\approx0.00000003$

而且其他概率是可忽略的。所以，通常我们通过考察在9, 10, 11或12进制之下的一个随机25位整数，可得到25个元素的一个分划。使用3.4.1-(3)可以生成数M，它倾向近似于$n/\xi=e^{\xi}$(参见习题67)。

***一个多重集合的分划。**一个整数和一个集合的分划仅仅是一个更一般得多的问题，即一个多重集合的分划的极端情况。其实，n的分划实际上和{1, 1, …, 1}的分划一样，这里给出的是n个1。

从这个观点出发，n个元素实际上有$p(n)$个不同的多重集合。例如，当n=4时，就出现5种不同的多重集合分划的情况：

1234, 123|4, 124|3, 12|34, 12|3|4, 134|2, 13|24, 13|2|4,
14|23, 14|2|3, 1|234, 1|23|4, 1|24|3, 1|2|34, 1|2|3|4;
1123, 112|3, 113|2, 11|23, 11|2|3, 123|1, 12|13, 12|1|3, 13|1|2, 1|1|23, 1|1|2|3;
1122, 112|2, 11|22, 11|2|2, 122|1, 12|12, 12|1|2, 1|1|22, 1|1|2|2;
1112, 111|2, 112|1, 11|12, 11|1|2, 12|1|1, 1|1|1|2;
1111, 111|1, 11|11, 11|1|1, 1|1|1|1 (54) 74

当多重集合包含m个不同元素，且一类有n_1，另一类有n_2……最后一类有n_m时，则对于分划的总数我们写$p(n_1, n_2, \cdots, n_m)$，因此(54)中的例子表明

$$p(1,1,1,1)=15,\ p(2,1,1)=11,\ p(2,2)=9,\ p(3,1)=7,\ p(4)=5 \qquad (55)$$

对于m=2的分划，通常叫做"2部分划"。对于m=3的分划叫做"3部分划"。而一般这些组合对象叫做多重分划。对于多重分划的研究是由珀·阿·麦克马洪在很久以前开始的[*Philosophical Transactions* **181** (1890), 481-536; **217** (1917), 81-113; *Proc. Cambridge Philos. Soc.* **22** (1925), 951-963]；但这个话题是如此广泛，因而还有许多未获解的问题。在本节的剩余部分和随后的习题中，我们将一瞥迄今为止已发现的这一理论的最有趣和最有教益的方面。

首先，重要的是要注意到，多重分划实际上是带有非负整分量的向量的分划，即把这样一个向量分解为这样的向量之和的方法。例如，在（54）中列出的{1, 1, 2, 2}的9个分划和双部列向量$\genfrac{(}{)}{0pt}{}{2}{2}$的9个分划一样，即

$$\begin{smallmatrix}2\\2\end{smallmatrix},\ \begin{smallmatrix}20\\11\end{smallmatrix},\ \begin{smallmatrix}20\\02\end{smallmatrix},\ \begin{smallmatrix}200\\011\end{smallmatrix},\ \begin{smallmatrix}11\\20\end{smallmatrix},\ \begin{smallmatrix}11\\11\end{smallmatrix},\ \begin{smallmatrix}110\\101\end{smallmatrix},\ \begin{smallmatrix}110\\002\end{smallmatrix},\ \begin{smallmatrix}1100\\0011\end{smallmatrix} \qquad (56)$$

(为简单起见，如同在一维整数分划中那样，我们省去加号。)如果我们以非递增的词典顺序来列出它的部分，则每个分划可写成一个规范形式。

一个简单的算法足以生成任何给定多重集合的分划。在下列过程中，我们在包含元素三元组(c, u, v)的一个栈上表示分划，其中c表示一个分量的号码，$u>0$表示在分量c中还剩下的未被分划的数量，而$v\le u$表示当前部分的c分量。为方便起见，三元组实际上被保持在三个数组$(c_0, c_1, \cdots)$，$(u_0, u_1, \cdots)$以及$(v_0, v_1, \cdots)$中，而且还维护一个"栈框架"数组$(f_0, f_1, \cdots)$，使得分划的第$(l+1)$个向量由在c、u和v数组中的元素f_l直到$f_{l+1}-1$组成。例如，下列数组将表示双部分划$\begin{smallmatrix}3221100\\1201131\end{smallmatrix}$：

j	0	1	2	3	4	5	6	7	8	9	10	
c_j	1	2	1	2	1	1	2	1	2	2	2	
u_j	9	9	6	8	4	2	6	1	5	4	1	
v_j	3	1	2	2	2	1	1	1	1	3	1	
	$f_0=0$		$f_1=2$		$f_2=4$	$f_3=5$		$f_4=7$		$f_5=9$	$f_6=10$	$f_7=11$

(57)

算法M(*在递减词典顺序下的多重分划。*) 给定一个多重集合$\{n_1\cdot 1, \cdots, n_m\cdot m\}$，使用上面所述的数组$f_0 f_1 \cdots f_n$，$c_0 c_1 \cdots c_n$，$u_0 u_1 \cdots u_n$以及$v_0 v_1 \cdots v_n$，本算法访问它

75 的所有分划，其中$n=n_1+\cdots+n_m$。我们假定$m>0$和$n_1, \cdots, n_m>0$。

M1.[初始化。] 对于$0\le j<m$，置$c_j\leftarrow j+1$和$u_j\leftarrow v_j\leftarrow n_{j+1}$；并置$f_0\leftarrow a\leftarrow l\leftarrow 0$和$f_1\leftarrow b\leftarrow m$。在以下的步骤中，当前栈框架从$a$跑到$b-1$(含)。

M2.[从u减去v。] (此时我们要找在当前的框架中，向量u分成按词典顺序$\le v$的部分的所有分划。首先我们将使用v本身。)置$j\leftarrow a$和$k\leftarrow b$。然后当$j<b$时做下列操作：置$u_k\leftarrow u_j-v_j$，而且如果$u_k\ge v_j$则置$c_k\leftarrow c_j$，$v_k\leftarrow v_j$，$k\leftarrow k+1$，

$j \leftarrow j+1$。但如果在u_k已被减少后，u_k小于v_j，则动作改变：首先置$c_k \leftarrow c_j$，$v_k \leftarrow u_k$，而且如果u_k非零，则$k \leftarrow k+1$，然后置$j \leftarrow j+1$。当$j<b$时，置$u_k \leftarrow u_j - v_j$，$c_k \leftarrow c_j$，$v_k \leftarrow u_k$，而且如果$u_j \neq v_j$则$k \leftarrow k+1$；然后再次地$j \leftarrow j+1$直到$j=b$为止。

M3.[如果非零则压栈。] 如果$k>b$，则置$a \leftarrow b$，$b \leftarrow k$，$l \leftarrow l+1$，$f_{l+1} \leftarrow b$并返回M2。

M4.[访问一个分划。] 访问由当前在栈中的$l+1$个向量表示的分划。(对于$0 \leqslant k \leqslant l$，对于$f_k \leqslant j<f_{k+1}$，这个向量在分量$c_j$中有$v_j$。)

M5.[减少v。] 置$j \leftarrow b-1$，并且如果$v_j=0$则置$j \leftarrow j-1$，直到$v_j>0$为止。然后如果$j=a$且$v_j=1$，则转到M6。否则置$v_j \leftarrow v_j-1$，而且对于$j<k<b$置$v_k \leftarrow u_k$。返回M2。

M6.[回溯。] 如果$l=0$则结束，否则置$l \leftarrow l-1$，$b \leftarrow a$，$a \leftarrow f_l$，并返回M5。 ▌

本算法的关键是步骤M2，它把当前驻留向量u减小最大允许部分v；如果必要，这一步也把v减小成按词典顺序的最大的$\leqslant v$的向量，即小于或等于在每个分量中新的驻留的量。

现在通过讨论多重分划和对于进制排序的最低有效位优先的过程(算法5.2.5R)之间的有趣联系，我们来结束本节。通过考虑一个例子可以最好地理解这个想法。参见表1，其中步骤(0)示出在词典顺序下的9个4部分的列向量。序列号码①~⑨已被加到底下作为标识。步骤(1)实现一个稳定的向量排序，并把它们的第四(最低有效)条目变成递减顺序；类似地，步骤(2)、(3)和(4)对于第3、第2和顶部的行进行稳定的排序。进制排序的理论告诉我们，原来的词典顺序由此被恢复。

表1 进制排序和多重分划

步骤(0)：原来的分划	步骤(1)：排序行4	步骤(2)：排序行3
6 5 5 4 3 2 1 0 0	0 6 4 3 5 0 5 2 1	0 6 5 2 5 1 4 3 0
3 2 1 0 4 5 6 4 2	2 3 0 4 2 4 1 5 6	2 3 2 5 1 6 0 4 4
6 6 3 1 1 5 2 0 7	7 6 1 1 6 0 3 5 2	7 6 6 5∧3 2∧1 1 0
4 2 1 3 3 1 1 2 5	5∧4 3 3∧2 2∧1 1 1	5 4 2 1 1 1 3 3 2
①②③④⑤⑥⑦⑧⑨	⑨①④⑤②⑧③⑥⑦	⑨①②⑥③⑦④⑤⑧
	α_4 = (9∧1 4 5∧2 8∧3 6 7)	α_3 = (1 2 5 8∧7 9∧3 4 6)

步骤(3)：排序行2	步骤(4)：排序行1
1 2 3 0 6 0 5 5 4	6 5 5 4∧3∧2∧1 0 0
6∧5 4 4∧3∧2 2 1 0	3 2 1 0 4 5 6 4 2
2 5 1 0 6 7 6 3 1	6 6 3 1 1 5 2 0 7
1 1 3 2 4 5 2 1 3	4 2 1 3 3 1 1 2 5
⑦⑥⑤⑧①⑨②③④	①②③④⑤⑥⑦⑧⑨
α_2 = (6∧4 8 9∧2∧1 3 5 7)	α_1 = (5 7 8 9∧3∧2∧1 4 6)

假设在这些稳定的排序操作之后，序列号的序列分别是α_4、$\alpha_3\alpha_4$、$\alpha_2\alpha_3\alpha_4$和$\alpha_1\alpha_2\alpha_3\alpha_4$，其中诸$\alpha$是排列。表1示出在括弧中的$\alpha_4$、$\alpha_3$、$\alpha_2$和$\alpha_1$的值。而且现在达到这样一点，无论在哪里如果排列$\alpha_j$有一个下降，则在排序之后在行j的号码也有一个下降，因为排序是稳定的。(这些下降在表中由脱字号(^)来指出。)例如，α_3

有8后面跟着7，我们因而在行3处有5后面跟着3。因此，在步骤(2)之后行3处的条目$a_1 \cdots a_9$不是它们之和的任意分划，它们必须满足

$$a_1 \geqslant a_2 \geqslant a_3 \geqslant a_4 > a_5 \geqslant a_6 > a_7 \geqslant a_8 \geqslant a_9 \tag{58}$$

但是数$(a_1-2, a_2-2, a_3-2, a_4-2, a_5-1, a_6-1, a_7, a_8, a_9)$确实形成原来的和，减去(4+6)的实质上任意的分划。减去的数量(4+6)是出现下降的下标之和。这个数就是我们在5.1.1节所称的ind α_3，即α_3的下标。

因此我们看到，一个m部分的数分成至多r部分的任何给定的分划(且加上额外的零使得列数恰好为r)，可以被编码成$\{1,\cdots,r\}$的排列$\alpha_1,\cdots,\alpha_m$的一个序列，使得乘积$\alpha_1 \cdots \alpha_m$是单位，连同数$(n_1-\text{ind }\alpha_1, \cdots, n_m-\text{ind }\alpha_m)$至多分成$r$部分的通常的一维分划的序列。例如，在表1中的向量表示(26, 27, 31, 22)分成9个部分的一个分划；排列$\alpha_1 \cdots \alpha_4$出现在这个表中，而且我们有$(\text{ind }\alpha_1, \cdots, \text{ind }\alpha_4)=(15, 10, 10, 11)$。诸分划分别为

$$26-15=(3\ 2\ 2\ 1\ 1\ 1\ 1\ 0\ 0),\qquad 27-10=(3\ 3\ 2\ 2\ 2\ 2\ 2\ 1\ 0),$$
$$31-10=(5\ 4\ 4\ 3\ 2\ 1\ 1\ 1\ 0),\qquad 22-11=(2\ 2\ 1\ 1\ 1\ 1\ 1\ 1\ 1)$$

反之，任何这样的排列和分划将产生$(n_1, \cdots, n_m)$的一个多重分划。如果r和m很小，则当列出和推导多重分划时，考虑一维分划的这些$r!^{m-1}$个一维分划序列可能是有帮助的。特别是在双部分的情况下。[这个构造是由巴希尔 · 戈登(Basil Gordon)给出的，*J. London Math. Soc.* **38** (1963)，459-464。]

关于多重分划早期工作的一个不错的小结，包括分成不同部分和/或严格地为正的部分的研究，出现在莫 · 辛 · 契玛(M. S. Cheema)和西 · 萨 · 莫兹金(T.S.Motzkin)的论文[*Proc. Symp. Pure Math.* **19** (Amer. Math. Soc.，1971)，39-70]上。

习 题

76～77 1. [*20*] (乔 · 哈特钦森。)给定n和$r \geqslant 2$，试证明，对于算法H的一个简单修改将生成$\{1, \cdots, n\}$分成至多r个块的所有分划。

▶ 2. [*22*] 当在实践中使用集合分划时，我们经常要把每块的元素链接在一起。因此有一个链接的数组$l_1 \cdots l_n$和标题数组$h_1 \cdots h_t$是方便的，使得一个t元块分划的第j块元素是$i_1 > \cdots > i_k$，其中

$$i_1 = h_j,\ i_2 = l_{i_1}, \cdots, i_k = l_{i_{k-1}} \text{ 以及 } l_{i_k}=0$$

例如，137|25|489|6的表示将有$t=4$，$l_1 \cdots l_9$=001020348，而且$h_1 \cdots h_4$=7596。

试设计使用这个表示来生成分划的算法H的版本。

3. [*M23*] 什么是由算法H生成的{1, ⋯, 12}的第100万个分划?

▶ 4. [*21*] 如果$x_1 \cdots x_n$是任意串，令$\rho(x_1 \cdots x_n)$是对应于等价关系$j \equiv k \Leftrightarrow x_j = x_k$的限制增长串。通过应用这个$\rho$函数，例如$\rho$(tooth)=01102，把在斯坦福图库中的每个5字母英文词分类。这样一来，5个元素的52个集合分划中有多少可以由英文词表示？每种类型最普通的词是什么？

5. [22] 试猜测以下两个序列的下一个元素是什么？(a) 0, 1, 1, 1, 12, 12, 12, 12, 12, 12, 100, 121, 122, 123, 123, …；(b) 0, 1, 12, 100, 112, 121, 122, 123, …。

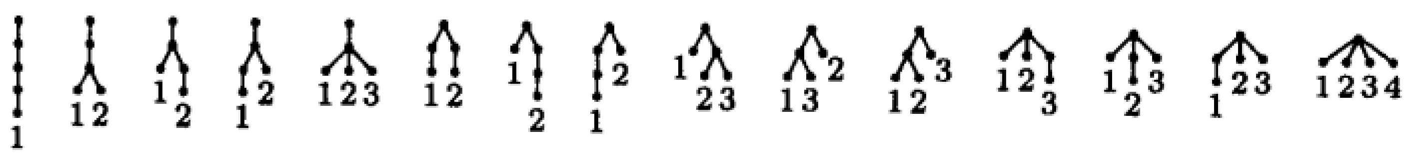
▶ 6. [25] 试提出一个生成{1, …, n}的所有这样分划的算法，即其中恰有大小为1的块c_1个，大小为2的块c_2个，等等。

7. [M20] {1, …, n}的排列$a_1 \cdots a_n$中有多少有这样的性质，即$a_{k-1}>a_k>a_j$意味着$j>k$?

8. [20] 试提出一个生成{1, …, n}所有这样的排列的算法，即这些排列恰有m个自左到右的极小。

9. [M20] 给定整数$k_0, k_1, \cdots, k_{n-1}$，有多少限制增长串$a_1 \cdots a_n$恰含$k_j$个j的出现?

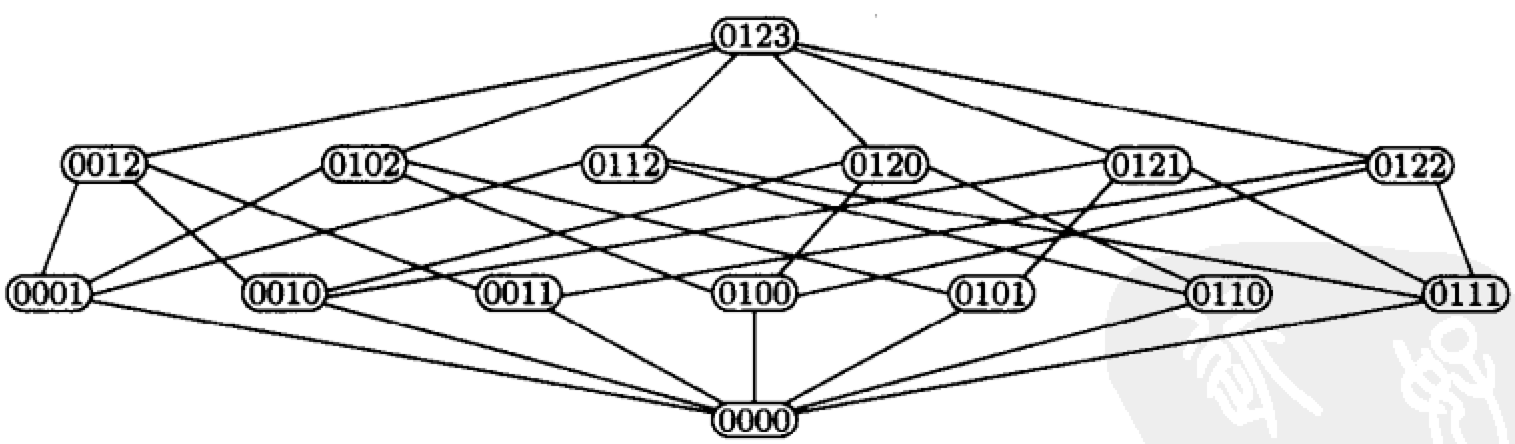
10. [25] 一个半标号树是这样一种有向树，其中的叶以整数{1, …, k}标号，但其他节点则无标号，因此对于5个顶点来说，共有15株半标号的树:



试求{1, …, n}的分划和具有n+1个顶点的半标号树之间的一一对应。

▶ 11. [28] 在7. 2. 1. 2节我们发现达德尼的著名问题send+more=money是一个“纯粹”的字母算术，即具有惟一解的一个字母算术。他的难题对应于13个数字位置的一个集合分划，对于它限制增长串ρ(sendmoremoney)是0123456145217；而且我们可能不知道他要怎样幸运才能得到这样一个构造。有多少长度等于13的限制增长串来定义形如$a_1a_2a_3a_4+a_5a_6a_7a_8=a_9a_{10}a_{11}a_{12}a_{13}$的纯粹字母算术?

12. [M31] (分划格。)如果Π和Π'是相同集合的分划，我们写$\Pi \preceq \Pi'$，如果每当$x \equiv y (\text{modulo}\,\Pi')$时就有$x \equiv y (\text{modulo}\,\Pi)$。换言之，$\Pi \preceq \Pi'$意味着Π'是Π的一个“加细”，通过0个或多个后者的块的分划得到；而且Π是Π'的“粗化”或联合，通过加0个或多个块合并在一起得到。容易看出这个偏序是一个格，而且$\Pi \vee \Pi'$是Π和Π'最大的公共加细，而$\Pi \wedge \Pi'$ 78
是它们最小的公共的联合。例如{1, 2, 3, 4}的分划的格是



如果我们通过限制增长串$a_1a_2a_3a_4$表示分划；在这个框图中的向上通路把每个分划变成为它的加细。有t个块的分划出现在从底部开始的级t上，而且它们的后裔形成{1, …, t}的分划格。

a) 给定$a_1 \cdots a_n$和$a'_1 \cdots a'_n$，试说明如何计算$\Pi \vee \Pi'$。

b) 给定$a_1 \cdots a_n$和$a'_1 \cdots a'_n$，试说明如何计算$\Pi \wedge \Pi'$。

c) 在这个格中什么时候Π'覆盖Π(参见习题7. 2. 1. 4-55)?

d) 如果Π有大小为$s_1, \cdots, s_t$的t个块，它一共覆盖多少个分划?

e) 如果Π有大小为$s_1, \cdots, s_t$的t个块，有多少个分划覆盖它？

f) 真或假：如果$\Pi \vee \Pi'$覆盖Π，则Π'覆盖$\Pi \wedge \Pi'$。

g) 真或假：如果Π'覆盖$\Pi \wedge \Pi'$，则$\Pi \vee \Pi'$覆盖Π。

h) 令$b(\Pi)$表示Π的块的个数。试证明$b(\Pi)+b(\Pi') \leqslant b(\Pi \vee \Pi')+b(\Pi \wedge \Pi')$。

13. [*M28*](斯蒂芬·卡·密尔尼(Stephen C. Milne)，1977。)如果A是$\{1, \cdots, n\}$的分划的一个集合，它的阴影∂A是对于某个$\Pi \in A$，使Π覆盖Π'的所有Π'分划的集合。(我们在7.2.1.3-(54)中考虑过子集合格的类似概念了。)

令$\Pi_1, \Pi_2, \cdots$是在它们的限制增长串的词典顺序下，$\{1, \cdots, n\}$分成为t块的分划，并令$\Pi'_1, \Pi'_2, \cdots$也是在词典顺序下$(t-1)$块的分划。试证明有一个函数$f_{nt}(N)$使得，对于$0 \leqslant N \leqslant \left\{ {n \atop t} \right\}$。

$$\partial\{\Pi_1, \cdots, \Pi_N\}=\{\Pi'_1, \cdots, \Pi'_{f_{nt}(N)}\}$$

提示：习题12中的框图示出$(f_{43}(0), \cdots, f_{43}(6))=(0, 3, 5, 7, 7, 7, 7)$。

14. [*23*] 试设计一个算法，来以类似(7)那样的格雷码顺序生成集合分划。

15. [*M21*] 什么是由习题14的算法生成的最后分划？

16. [*16*] 表(11)是拉斯基的A_{35}，什么是A'_{35}？

17. [*26*] 对于$\{1, \cdots, n\}$的所有m块分划，试实现拉斯基的格雷码(8)。

18. [*M46*] 对于什么n，有可能以这样一种方式来生成所有限制增长串$a_1 \cdots a_n$，即在每步中某个a_j改变± 1？

19. [*28*] 试证明，当(a)我们要生成所有ϖ_n串$a_1 \cdots a_n$时；或(b)我们只要生成满足$\max(a_1 \cdots a_n)=m-1$的$\left\{ {n \atop m} \right\}$情况时，对于限制增长串存在一种格雷码，对于它来说，在每步中，某个a_j
79 或者改变± 1，或者改变± 2。

20. [*17*] 如果Π是$\{1, \cdots, n\}$的一个分划，它的共轭Π^T由下列规则定义：

$$j \equiv k \pmod{\Pi^T} \Leftrightarrow n+1-j \equiv n+1-k \pmod{\Pi}$$

假设Π有限制增长串001010202013；什么是Π^T的限制增长串？

21. [*M27*] $\{1, \cdots, n\}$有多少自共轭的分划？

22. [*M23*] 如果X是具有一个给定分布的随机变量，X^n的期望值称作该分布的**第n个动量**。当X是(a)具有均值1的泊松离差(等式3.4.1-(40))时，(b)当$m \geqslant n$时$\{1, \cdots, m\}$的随机排列的不动点个数(等式1.3.3-(27))时，第n个动量是多少？

23. [*HM30*] 如果$f(x)=\sum a_k x^k$是一个多项式，令$f(\varpi)$代表$\sum a_k \varpi_k$。

a) 试证明符号公式$f(\varpi+1)=\varpi f(\varpi)$。例如，如果$f(x)$是多项式$x^2$，则这个公式指出$\varpi_2+2\varpi_1+\varpi_0=\varpi_3$。

b) 类似地，证明对于所有正整数k，$f(\varpi+k)=\varpi^{\underline{k}} f(\varpi)$。

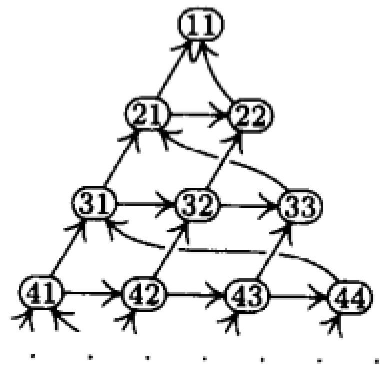
c) 如果p是质数，试证明$\varpi_{n+p} \equiv \varpi_n+\varpi_{n+1} \pmod{p}$。提示：首先证明$x^{\underline{p}} \equiv x^p - x$。

d) 结果当$N=p^{p-1}+p^{p-2}+\cdots+p+1$时，$\varpi_{n+N} \equiv \varpi_n \pmod{p}$。

24. [*HM35*] 继续上一道题，试证明如果p是奇质数，则贝尔数满足周期律$\varpi_{n+p^{e-1}N} \equiv \varpi_n \pmod{p^e}$。提示：证明$x^{\underline{p^e}} \equiv g_e(x)+1(\text{modulo } p^e, p^{e-1}g_1(x), \cdots, \text{and } pg_{e-1}(x))$，其中$g_j(x)=(x^p-x-1)^{p^j}$。

25. [*M27*] 证明$\varpi_n / \varpi_{n-1} \le \varpi_{n+1} / \varpi_n \le \varpi_n / \varpi_{n-1}+1$。

▶ 26. [*M22*] 按照递归等式(13)，在珀西三角中的数ϖ_{nk}计算在无穷有向图



中的从 ⓝⓚ 到 ⑪ 的通路。试说明为什么从 ⓝ① 到 ⑪ 的每条路对应于{1, …, n}的一个分划。

▶ 27. [*M35*] 次序n的一个“摇摆不定的表景循环”是对于$0 \le k \le 2n$，满足$a_{k1} \ge a_{k2} \ge a_{k3} \ge \cdots$的整数分划$\lambda_k = a_{k1} a_{k2} a_{k3} \cdots$的一个序列，使得$\lambda_0 = \lambda_{2n} = e_0$和对于$1 \le k \le 2n$，满足$\lambda_k = \lambda_{k-1} + (-1)^k e_{t_k}$和对于某个$t_k$满足$0 \le t_k < n$。这里$e_t$表示当$0<t \le n$时的单位向量$0^{t-1}10^{n-t}$，而且$e_0$为全零。

a) 列出次序为4的所有摇摆不定表景循环。[提示：总共有15个。]

b) 试证明恰好有ϖ_{nk}个次序n的摇摆不定表景循环，有$t_{2k-1}=0$。

▶ 28. [*M25*] (广义的车多项式。)考虑在行和列中的方格$a_1+\cdots+a_m$的一个安排，其中行k包含在列$1, \cdots, a_k$中的方格。在这些方格中放置零个或多个“车”，而且在每行处至多有一个车，且在每列中也至多有一个车。如果一个方格在其右边和在其下边没有车，则这个空方格称为是自由的。例如，图36中示出两种这样的设置，一个是有4个车在长度(3, 1, 4, 1, 5, 9, 2, 6, 5)的行，而另一个是在9×9的正方形板上有9个车。用黑圆圈表示车，空圆圈被放置 80
在每个车的左边和上边，因此剩下的就是自由的空方格。

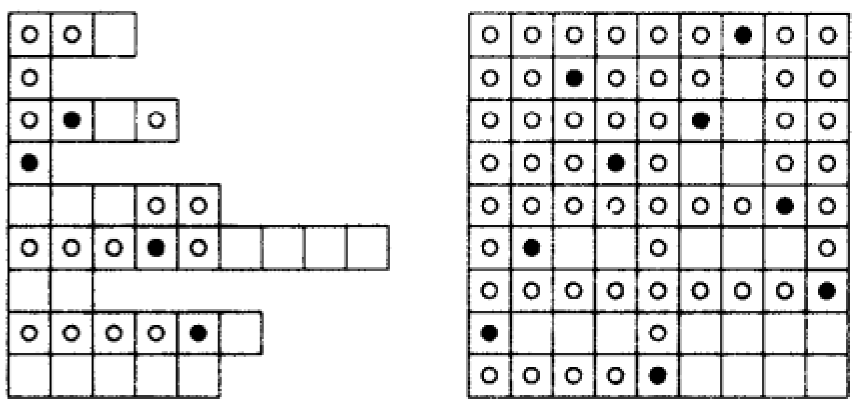


图36 车的放置和自由方格

令$R(a_1, \cdots, a_m)$是在所有合法的车的设置对$x^r y^f$进行求和得到的x和y的多项式，其中r是车的个数，而f是自由空格的个数。例如，图35中的左图对于多项式$R(3, 1, 4, 1, 5, 9, 2, 6, 5)$的贡献为$x^4y^{17}$。

a) 试证明我们有$R(a_1, \cdots, a_m) = R(a_1, \cdots, a_{j-1}, a_{j+1}, a_j, a_{j+2}, \cdots, a_m)$。换言之，行长的顺序无关紧要。因此我们可以假定，像在类似于7.2.1.4-(13)的费尔利斯框图中那样$a_1 \ge \cdots \ge a_m$。

b) 如果$a_1 \ge \cdots \ge a_m$且如果$b_1 \cdots b_n = (a_1 \cdots a_m)^T$是共轭分划，试证明$R(a_1, \cdots, a_m) = R(b_1, \cdots, b_n)$。

c) 试求计算$R(a_1, \cdots, a_m)$的一个递归式，并且用它来计算$R(3, 2, 1)$。

d) 通过把加法规则(13)改变成

$$\varpi_{nk}(x, y) = x\varpi_{(n-1)k}(x, y) + y\varpi_{n(k+1)}(x, y), \quad 1 \le k < n$$

推广珀西三角(12)。因此$\varpi_{21}(x, y)=x+y$, $\varpi_{32}(x, y)=x+xy+y^2$, $\varpi_{31}(x, y)=x^2+2xy+xy^2+y^3$，等等。试证明得到的量$\varpi_{nk}(x, y)$是车多项式$R(a_1, \cdots, a_{n-1})$，其中$a_j=n-j-[j<k]$。

e) 部分(d)中的多项式$\varpi_{n1}(x, y)$可以被当作为一个推广的贝尔数$\varpi_n(x, y)$，表示在习题26的有向图中从ⓝ①到⑪的通路，它们有给定数目的向东北的“x步”和给定数目的向东的“y步”。试证明，对所有长度为n的限制增长串$a_1\cdots a_n$求和，

$$\varpi_n(x, y)=\sum_{a_1\cdots a_n} x^{n-1-\max(a_1,\cdots,a_n)}y^{a_1+\cdots+a_n}$$

29. [*M26*] 继续上一道题，令$R_r(a_1, \cdots, a_m)=[x^r]\,R(a_1, \cdots, a_m)$是关于$y$的多项式，当放置$r$个车时，它枚举自由空格的个数。

a) 试证明在$n\times n$的板上放置n个车的方法数(并保留有f个自由空格)是有f个反演的$\{1, \cdots, n\}$的排列个数。因此由等式5.1.1-(8)和习题5.1.2-16，我们有

$$R_n(\overbrace{n,\cdots,n}^{n})=n!_y=\prod_{k=1}^{n}(1+y+\cdots+y^{k-1})$$

b) 什么是$R_r(\overbrace{n,\cdots,n}^{m})$即在$m\times n$的板上$r$个车的生成函数?

c) 如果$a_1\geqslant\cdots\geqslant a_m$且$t$是非负整数，试证明一般公式

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$$\prod_{j=1}^{m}\frac{1-y^{a_j+m-j+t}}{1-y}=\sum_{k=0}^{m}\frac{t!_y}{(t-k)!_y}R_{m-k}(a_1,\cdots,a_m)$$

[注：当$k>t\geqslant 0$时，量$t!_y/(t-k)!_y=\prod_{j=0}^{k-1}((1-y^{t-j})/(1-y))$为零。因此，例如，当$t=0$时，右边归结为$R_m(a_1, \cdots, a_m)$。通过逐次地设置$t=0, 1, \cdots, m$, 我们可计算$R_m, R_{m-1}, \cdots, R_0$。]

d) 如果$a_1\geqslant a_2\geqslant\cdots\geqslant a_m\geqslant 0$且$a'_1\geqslant a'_2\geqslant\cdots\geqslant a'_m\geqslant 0$，试证明我们有$R(a_1, a_2, \cdots, a_m)=R(a'_1,a'_2,\cdots,a'_m)$，当且仅当相关联的多重集合$\{a_1+m, a_2+m-1, \cdots, a_m+1\}$和$\{a'_1+m, a'_2+m-1,\cdots,a'_m+1\}$相同。

30. [*HM30*] 广义的斯特林数$\left\{{n \atop m}\right\}_q$是由下列递归式定义的

$$\left\{{n+1 \atop m}\right\}_q=(1+q+\cdots+q^{m-1})\left\{{n \atop m}\right\}_q+\left({n \atop m-1}\right)_q;\quad\left({0 \atop m}\right)_q=\delta_{m0}$$

因此$\left\{{n \atop m}\right\}_q$是关于$q$的多项式，而且$\left\{{n \atop m}\right\}_1$就是通常的斯特林数$\left\{{n \atop m}\right\}$，因为它满足在等式1.2.6-(46)中的递归关系。

a) 试证明习题28(e)的广义贝尔数$\varpi_n(x, y)=R(n-1, \cdots, 1)$有显式

$$\varpi_n(x,y)=\sum_{m=0}^{n}x^{n-m}y^{\binom{m}{2}}\left\{{n \atop m}\right\}_y$$

b) 试证明广义的斯特林数也遵守递归式

$$q^m\left\{{n+1 \atop m+1}\right\}_q=q^n\left\{{n \atop m}\right\}_q+\binom{n}{1}q^{n-1}\left\{{n-1 \atop m}\right\}_q+\cdots=\sum_k\binom{n}{k}q^k\left\{{k \atop m}\right\}_q$$

c) 试求对于 $\left\{ {n \atop m} \right\}_q$ 的生成函数，并推广1.2.9-(23)和1.2.9-(28)。

31. [*HM23*] 推广(15)，试证明，如果我们计算和

$$\sum_{nk} \varpi_{nk} \frac{w^{n-k}}{(n-k)!} \frac{z^{k-1}}{(k-1)!}$$

则珀西三角的元素有一个简单的生成函数。

32. [*M22*] 令 δ_n 是这样的限制增长串 $a_1 \cdots a_n$ 的个数，对于它们，和 $a_1 + \cdots + a_n$ 是偶数的减去和 $a_1 + \cdots + a_n$ 是奇数那些的个数。试证明，当 $n \bmod 6 = (1, 2, 3, 4, 5, 0)$ 时

$$\delta_n = (1, 0, -1, -1, 0, 1)$$

提示：参见习题28(e)。

33. [*M21*] $\{1, 2, \cdots, n\}$ 的排列中，有多少有 $1 \not\equiv 2, 2 \not\equiv 3, \ldots, k-1 \not\equiv k$?

34. [*14*] 许多诗的形式涉及节奏方案，它们是具有如下性质的一个诗节的行的分划，即 $j \equiv k$ 当且仅当行 j 同行 k 押韵时。例如，一个"五行打油诗"一般的是具有某个音节限制的一个五行诗，而且具有通过限制增长串00110所描述的节奏方案。

在由(a)圭特顿·德·阿里佐(约公元1270)，(b)彼得拉奇(约公元1350)，(c)斯宾瑟(1595)，(d)莎士比亚(1609)，(e)伊丽莎白·巴利特·布朗宁(1850)所作的经典十四行诗(sonnet)中，使用的是什么节奏方案?

35. [*M21*] 令 ϖ'_n 是在下列意义下"完全押韵"的 n 行诗的方案数，即每行至少同其他一行押韵。于是，我们有 $\langle \varpi'_0, \varpi'_1, \varpi'_2, \cdots \rangle = \langle 1,0,1,1,4,11,41,\cdots \rangle$。试给出对于 $\varpi'_n + \varpi'_{n+1} = \varpi_n$ 这一事实的一个组合证明。

36. [*M22*] 继续习题35，什么是生成函数 $\sum_n \varpi'_n z^n / n!$? 82

37. [*M18*] 亚历山大·普希金不仅基于"阳性节奏"，即其中重音的最后音节彼此一致(pain-gain, form-warm, pun-fun, bucks-crux)，而且基于"阴性节奏"，即一个或两个非重音的音节也参与(humor-tumor, tetrameter-pentameter, lecture- conjecture, iguana-piranha)，在他的诗体小说《尤金·奥涅金》(1833)中，采用了精心设计的结构。《尤金·奥涅金》的每一个诗节都是有严格方案01012233455477的一首十四行诗，其中按照数字是偶数还是奇数，决定节奏是阴性的或者是阳性的。普希金小说的现代翻译者中有许多人已成功地在英文和德文中保留相同的形式。

我如何认为这诗节是正确的? /这些是阴性的韵律吗? 我皱起眉头的沉思? 这整个陈旧过时奢侈? /我怎能(不顾时间地)使用奥涅金的积满灰尘的面包模子! /在里根盛装的面包烘室里? 这面包肯定不能发起，/或不然就在我眼前走味。事实就是，我不能对它作出判断。/但由于没有关键时期的裹尸布能使我的尸体防止令人生厌的蛆虫，/我也有兴趣并愿尝试它/如果它有效，那很好；不然，那也行。/一个理论不会去搁置它的音律。

——维克拉姆·塞特(Vikram Seth)，*The Golden Gate* (1986)

按照习题35，一首十四行诗可以有 ϖ'_{14}=24 011 157个完全的节奏方案。但是如果允许我们对每个块，指定它的节奏是阴性的还是阳性的，那可能有多少个方案?

▶ 38. [*M30*] 令σ_k是循环排列$(1,2,\cdots,k)$。本题的目的是来研究称作σ循环的序列$k_1k_2\cdots k_n$，对于它来说，$\sigma_{k_1}\sigma_{k_2}\cdots\sigma_{k_n}$是恒等排列。例如，当$n$=4时恰有15个$\sigma$循环，即：

1111, 1122, 1212, 1221, 1333, 2112, 2121, 2211, 2222, 2323, 3133, 3232, 3313, 3331, 4444

a) 求出$\{1,2,\cdots,n\}$的分划和长度为n的σ循环之间的一一对应。

b) 给定m和n，有多少长度为n的σ循环有$1\leqslant k_1,\cdots,k_n\leqslant m$?

c) 给定i、j和n，有多少长度为n的σ循环有$k_i=j$?

d) 有多少长度为n的σ循环有$k_1,\cdots,k_n\geqslant 2$?

e) 有多少$\{1,\cdots,n\}$的分划有$1\not\equiv 2, 2\not\equiv 3, \ldots, n-1\not\equiv n$和$n\not\equiv 1$?

39. [*HM16*] 当p和q是非负整数时，试计算$\int_0^\infty e^{-t^{p+1}}t^q\,dt$。提示：参见习题1.2.5-20。

40. [*HM20*] 假设使用马鞍点方法来估计$[z^{n-1}]e^{cz}$。正文中从(19)开始的(21)的推导处理c=1的情况；如果c是任何正常数，该推导应如何改变?

41. [*HM21*] 当$c=-1$时，试求解上题。

42. [*HM23*] 试使用马鞍点方法来估计$[z^{n-1}]e^{z^2}$且有相对误差$O(1/n^2)$。

43. [*HM22*] 论证以(25)代替(23)中积分的正确性。

44. [*HM22*] 试说明如何从(25)中的$a_2,a_3,\cdots$来计算(26)中的$b_1,b_2,\cdots$。

▶ 45. [*HM23*] 除了(26)外，试说明我们也还有展开式

$$\varpi_n=\frac{e^{e^{\xi}-1}n!}{\xi^n\sqrt{2\pi n(\xi+1)}}\left(1+\frac{b_1'}{n}+\frac{b_2'}{n^2}+\cdots+\frac{b_m'}{n^m}+O\left(\frac{1}{n^{m+1}}\right)\right)$$

83 其中$b_1'=-(2\xi^4+9\xi^3+16\xi^2+6\xi+2)/(24(\xi+1)^3)$。

46. [*HM25*] 当$n\to\infty$时，试估计在珀西三角中ϖ_{nk}的值。

47. [*M21*] 试分析算法H的运行时间。

48. [*HM25*] 如果n不是一个整数，则(23)中的积分可以对于整个一个汉克尔等高线来进行，(来对于)所有实数x>0定义一个广义的贝尔数ϖ_x。试证明，如同在(16)中那样

$$\varpi_x=\frac{1}{e}\sum_{k=0}^{\infty}\frac{k^x}{k!}$$

▶ 49. [*HM35*] 试证明，对于很大的n，在等式(24)中定义的数ξ等于

$$\ln n-\ln\ln n+\sum_{j,k>0}\left[{j+k \atop j+1}\right]\alpha^j\frac{\beta^k}{k!},\quad \alpha=-\frac{1}{\ln n},\quad \beta=\frac{\ln\ln n}{\ln n}$$

▶ 50. [*HM21*] 如果$\xi(n)e^{\xi(n)}=n$且$\xi(n)>0$，则$\xi(n+k)$如何同$\xi(n)$相关联?

51. [*HM27*] 使用马鞍点方法来估计$t_n=n![z^n]e^{z+z^2/2}$，即n个元素的卷积的个数(也叫做分成大小$\leqslant 2$块的$\{1,\cdots,n\}$的分划)。

52. [*HM22*] 一个概率分布的累积在等式1.2.10-(23)中定义，当一个随机整数等于k的概率是(a) $e^{1-e^{\xi}}\varpi_k\xi^k/k!$ (b) $\sum_j\left\{{k \atop j}\right\}e^{e^{-1}-1-j}/k!$ 时，累积分别是什么?

▶ 53. [*HM30*] 令$G(z)=\sum_{k=0}^{\infty}p_kz^k$是覆盖$|z|<1+\delta$的一个离散概率分布的生成函数；于是系数$p_k$是非负的，$G(1)=1$，而且均值和方差分别是$\mu=G'(1)$和$\sigma^2=G''(1)+G'(1)-G'(1)^2$。如

果X_1, …, X_n是有这个分布的独立随机变量，则$X_1+\cdots+X_n=m$的概率是$[z^m]G(z)^n$，而且当m接近于均值μn时，我们通常要来估计这个概率。

假定$p_0\neq 0$和没有整数$d>1$是使$p_k\neq 0$的所有下标k的一个共同的因式；这个假定意味着当n很大时，m不需要满足mod d的任何特殊的同余条件。试证明当$\mu n+r$是一个整数时，当$n\to\infty$时

$$[z^{\mu n+r}]\,G(z)^n=\frac{e^{-r^2/(2\sigma^2 n)}}{\sigma\sqrt{2\pi n}}+O\left(\frac{1}{n}\right)$$

提示：在圆圈$|z|=1$上对$G(z)^n/z^{\mu n+r}$进行积分。

54. [*HM20*] 如果α和β由(40)定义，试证明它们的算术和几何均值分别是$\dfrac{\alpha+\beta}{2}=s\coth s$和$\sqrt{\alpha\beta}=s\,\mathrm{csch}\,s$，其中$s=\sigma/2$。

55. [*HM20*] 提出一个好方法来计算在(43)中需要的数β。

▶ 56. [*HM26*] 如同在(37)中那样，令$g(z)=\alpha^{-1}\ln(e^z-1)-\ln z$和$\sigma=\alpha-\beta$。

a) 试证明 $(-\sigma)^{n+1}g^{(n+1)}(\sigma)=n!-\sum_{k=0}^{n}\left\langle{n \atop k}\right\rangle\alpha^k\beta^{n-k}$，其中欧拉数$\left\langle{n \atop k}\right\rangle$在5.1.3节中定义。

b) 试证明对于所有$\sigma>0$，$\dfrac{\beta}{\alpha}n!<\sum_{k=0}^{n}\left\langle{n \atop k}\right\rangle\alpha^k\beta^{n-k}<n!$。提示：参见习题5.1.3-25。

c) 现在验证不等式(42)。

57. [*HM22*] 在(43)的记号下，试证明(a)$n+1-m<2N$，(b)$N<2(n+1-m)$。

58. [*HM31*] 试完成(43)的证明如下：

a) 对于所有$\sigma>0$，证明有一个数$\tau>2\sigma$使得τ是2π的一个倍数，而且对于$0<t<\tau$，$|e^{\sigma+it}-1|/|\sigma+it|$是单调递减的。

b) 证明$\int_{-\tau}^{\tau}\exp((n+1)g(\sigma+it))dt$ 导致(43)。

c) 试证明，对应的积分对于$-n<t<\sigma$在直线通路$z=t\pm i\tau$上进行，而对于$-\tau<t<\tau$的通路$z=-n\pm it$是可忽略的。 84

▶ 59. [*HM23*] (43)预测$\left\{{n \atop n}\right\}$的近似值是什么？

60. [*HM25*] (a) 试证明在恒等式

$$\left\{{n \atop m}\right\}=\frac{m^n}{m!}-\frac{(m-1)^n}{1!(m-1)!}+\frac{(m-2)^n}{2!(m-2)!}-\cdots+(-1)^m\frac{0^n}{m!0!}$$

中的部分和交替地高估和低估最后的值；(b)结论，当$m<n^{1-\epsilon}$时

$$\left\{{n \atop m}\right\}=\frac{m^n}{m!}(1-O(ne^{-n^\epsilon}))$$

(c)从(43)推出一个类似的结果。

61. [*HM26*] 试证明，如果$m=n-r$，其中$r<n^\epsilon$和$\epsilon<n^{1/2}$，则等式（43）产生

$$\left\{{n \atop n-r}\right\}=\frac{n^{2r}}{2^r r!}\left(1+O(n^{2\epsilon-1})+O\left(\frac{1}{r}\right)\right)$$

62. [*HM40*] 严格地证明，如果$\xi e^{\xi}=n$，则当$m=\lfloor e^{\xi}-1\rfloor$或当$m=\lceil e^{\xi}-1\rceil$时，出现极大

值 $\left\{ {n \atop m} \right\}$。

▶ 63. [*M35*] (詹 · 皮特曼(J. Pitman)。)试证明有一个初等方法来确定极大斯特林数的位置和许多类似的量，如下：假设$0 \le p_j \le 1$

a) 令$f(z)=(1+p_1(z-1))\cdots(1+p_n(1-z))$且$a_k=[z^k]f(z)$；因此$a_k$是在$n$个独立的硬币投掷分别有概率$p_1, \cdots, p_n$时，$k$次面朝上的概率。试证明每当$k \le \mu = p_1+\cdots+p_n$，$a_k \ne 0$时，$a_{k-1}<a_k$。

b) 类似地，证明每当$k \ge \mu$和$a_k \ne 0$，$a_{k+1}<a_k$。

c) 如果$f(x)=a_0+a_1x+\cdots+a_nx^n$是带有非负系数和$n$个实根的任何非零多项式，试证明当$k \le \mu$时$a_{k-1}<a_k$，当$k \ge \mu$时$a_{k+1}<a_k$，其中$\mu=f'(1)/f(1)$。因此如果$a_m=\max(a_0, \cdots, a_n)$，我们必定有$m=\lfloor \mu \rfloor$或$m=\lceil \mu \rceil$。

d) 在(c)的假设下，以及当$j<0$或$j>n$时$a_j=0$。试证明存在下标$s \le t$，使得$a_{k+1}-a_k<a_k-a_{k-1}$，当且仅当$s \le k \le t$。[因此，序列$(a_0, a_1, \cdots, a_n)$的矩方图总是“钟形”的。]

e) 关于斯特林数，这些结果能告诉我们什么？

64. [*HM21*] 使用(30)和习题50，证明近似比(50)。

▶ 65. [*HM22*] 在一个$\{1, \cdots, n\}$的随机分划中，大小为k的块的个数的方差是多少？

66. [*M46*] n的什么分划导致$\{1, \cdots, n\}$的最多的分划？

67. [*HM20*] 在斯塔姆方法(53)中M的均值和方差是多少？

68. [*20*] 当算法M生成$\{n_1 \cdot 1, \cdots, n_m \cdot m\}$的所有$p(n_1, \cdots, n_m)$分划时，它得到的栈能有多大？

▶ 69. [*21*] 试修改算法M，使得它只生成至多r部分的分划。

▶ 70. [*M22*] 试分析在n元素多重集合(a) $\{0, \cdots, 0, 1\}$；(b) $\{1, 2, \cdots, n-1, n-1\}$中可能的$r$块分划的个数；对于$r$求和，总数为多少？

71. [*M20*] $\{n_1 \cdot 1, \cdots, n_m \cdot m\}$中有多少分划恰好有2个部分？

72. [*M26*] 能否在多项式时间中计算出$p(n, n)$？

▶ 73. [*M32*] 当有n个2时，能否在多项式时间中计算出$p(2, \cdots, 2)$？

74. [*M46*] 当有n个n时，能否在多项式时间中计算出$p(n, \cdots, n)$？

75. [*HM41*] 试求$p(n, n)$的渐近值。

76. [*HM36*] 当有n个2时，试求$p(2, \cdots, 2)$的渐近值。

85 77. [*HM46*] 当有n个n时，试求$p(n, \cdots, n)$的渐近值。

78. [*20*] (15, 10, 10, 11)的什么分划导致在表1中示出的α_1、α_2、α_3和α_4的排列？

79. [22] 一个序列$u_1, u_2, u_3, \cdots$称为$\{1, \cdots, n\}$的分划的*万有序列*——如果对于$0 \le m<\varpi_n$，它的子序列$(u_{m+1}, u_{m+2}, \cdots, u_{m+n})$表示在约定“$j \equiv k$当且仅当$u_{m+j}=u_{m+k}$”之下所有可能的集合分划。例如，(0, 0, 0, 1, 0, 2, 2)是$\{1, 2, 3\}$的分划的一个万有序列。

试写出一个程序，求出$\{1, 2, 3, 4\}$的分划的所有万有序列且满足下列性质：(i) $u_1=u_2=u_3=u_4=0$；(ii) 这个序列有限制增长；(iii) $0 \le u_j \le 3$；(iv) $u_{16}=u_{17}=u_{18}=0$(因此这个序列实际上是循环的)。

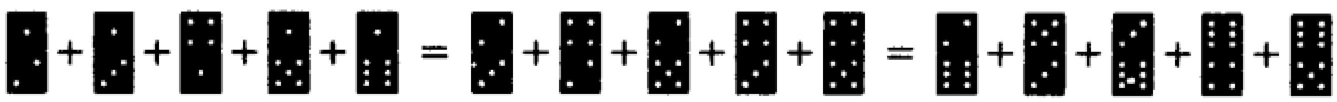
80. [*M28*] 试证明当$n \ge 4$时，在上一道题的意义下，$\{1, 2, \cdots, n\}$的分划的万有循环存在。

81. [29] 试求出一种方法来安排通常的52张扑克牌，使得下列技巧是可能的：5个玩牌者把一副牌打开(应用一个循环排列)，如同他们通常做的那样，然后每一个玩牌者从顶上取

一张牌。一个魔术师让他们考察自己的牌并形成类缘组，即同其他也拥有同样花色的玩牌者联合在一起，有黑梅花的每个玩牌者连在一起，有方片的形成另一个组，等等。(然而，草花J被认为是一个最大王牌，如果有任何一个人拥有它，则应当保持不参与。)

试观察类缘组，但不说出任何花色，魔术师可以说出所有5张牌的花色，如果首先这些牌被适当地安排的话。

82. [22] 在下列15张骨牌中，任意地转动，当把每张牌当作分数时，有多少种方法把它们分划成有相同和的5张牌的3个集合。

 86

习 题 答 案

7.2.1.3节

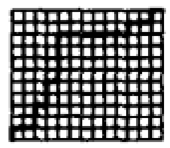
1. 给定一个多重集合，从右到左通过首先列出不同元素，然后是出现两次的那些，然后是出现三次的那些，等等，形成序列$e_t \cdots e_2 e_1$。对于$0 \leqslant j \leqslant s = n - t$，且让我们置$e_{-j} \leftarrow s - j$，使得对于$1 \leqslant j \leqslant t$的每一元素$e_j$等于在序列$e_t \cdots e_1 e_0 \cdots e_{-s}$中它右边的某个元素。如果头一个这样的元素是$e_{c_j - s}$，我们得到(3)的一个解。反之，(3)的每个解产生一个惟一的多重集合$\{e_1, \cdots, e_t\}$，因为对于$1 \leqslant j \leqslant t$，$c_j < s + j$。

[一个类似的对应由尤金 · 卡塔兰(E. Catalan)提出：如果$0 \leqslant e_1 \leqslant \cdots \leqslant e_t \leqslant s$，令

$$\{c_1, \cdots, c_t\} = \{e_1, \cdots, e_t\} \cup \{s + j \mid 1 \leqslant j < t \text{和} e_j = e_{j+1}\}$$

参见*Mémoires de la Soc. roy. des Sciences de Liège* (2) **12** (1885)，*Mélanges Math.* 3。]

2. 在左下角开始；然后对于每个0向上，对于每个1向右动。结果是



3. 在这个算法中，变量r是使得$q_r > 0$的最小正下标。

F1. [初始化。] 对于$1 \leqslant j \leqslant t$，置$q_j \leftarrow 0$且$q_0 \leftarrow s$。(我们假定$st > 0$。)

F2. [访问。] 访问合成$q_t \cdots q_0$。如果$q_0 = 0$则转到F4。

F3. [容易情况。] 置$q_0 \leftarrow q_0 - 1$，$r \leftarrow 1$，并转到F5。

F4. [不易处理的情况。] 如果$r = t$则结束。否则置$q_0 \leftarrow q_r - 1$，$q_r \leftarrow 0$，$r \leftarrow r + 1$。

F5. [增加q_r。] 置$q_r \leftarrow q_r + 1$，并返回F2。 ▮

[参见*CACM* **11** (1968)，430；**12** (1969)，187。以递减的词典顺序生成这样的合成的任务是更困难的。]

4. 我们可以颠倒(14)中0和1的作用，使得 $0^{q_t}10^{q_{t-1}}1\ldots10^{q_1}10^{q_0} = 1^{r_s}01^{r_{s-1}}0\ldots01^{r_1}01^{r_0}$。这给出 $0^1 1 0^0 1 0^2 1 0^2 1 0^4 1 0^0 1 0^0 1 0^0 1 0^0 1 0^0 1 0^1 1 0^0 1 0^1 1 0^0 = 1^0 0 1^2 0 1^0 0 1^1 0 1^0 0 1^1 0 1^0 0 1^0 0 1^0 0 1^6 0 1^2 0 1^1$。$a_{n-1} \cdots a_1 a_0$的词典顺序对应于$r_s \cdots r_1 r_0$的词典顺序。

顺便指出，也有一个多重集合的联系：$\{d_t, \cdots, d_1\} = \{r_s \cdot s, \cdots, r_0 \cdot 0\}$。例如，$\{10, 10, 8, 6, 2, 2, 2, 2, 2, 2, 1, 1, 0\} = \{0 \cdot 11, 2 \cdot 10, 0 \cdot 9, 1 \cdot 8, 0 \cdot 7, 1 \cdot 6, 0 \cdot 5, 0 \cdot 4, 0 \cdot 3, 6 \cdot 2, 2 \cdot 1, 1 \cdot 0\}$。

5. (a)置$x_j = c_j - \lfloor (j-1)/2 \rfloor$在$n + \lfloor t/2 \rfloor$的每$t$个组合中。(b)置$x_j = c_j + j + 1$于$n - t - 2$的每$t$个组合中。

(给定x_{t+1}，$(\delta_t, \cdots, \delta_1)$和$x_0$的值，一个类似的方法求对于$0 \leqslant j \leqslant t$，不等式$x_{j+1} \geqslant x_j + \delta_j$的所有解$(x_t, \cdots, x_1)$。)

6. 假设$t > 0$。当$c_1 > 0$时我们转到T3；当$c_2 = c_1 + 1 > 1$时转到T5；当$c_j = c_1 + j - 1 \geqslant j$时，对于$2 \leqslant j \leqslant t + 1$转到T4。所以计数是：T1，1；T2，$\binom{n}{t}$；T3，$\binom{n-1}{t}$；

T4，$\binom{n-2}{t-1}+\binom{n-2}{t-2}+\cdots+\binom{n-t-1}{0}=\binom{n-1}{t-1}$；T5，$\binom{n-2}{t-1}$；T6，$\binom{n-1}{t-1}+\binom{n-2}{t-1}-1$。

7. 一个稍微比算法T更简单的过程就足够了：假设$s<n$。

S1. [初始化。] 对于$1\leqslant j\leqslant s$，置$b_j\leftarrow j+n-s-1$，然后置$j\leftarrow 1$。

S2. [访问。] 访问组合$b_s\cdots b_2b_1$。如果$j>s$，则结束。

S3. [减少b_j。] 置$b_j\leftarrow b_j-1$。如果$b_j<j$，置$j\leftarrow j+1$并返回S2。

S4. [重置$b_{j-1}\cdots b_1$。]当$j>1$时，置$b_{j-1}\leftarrow b_j-1$，$j\leftarrow j-1$，并重复到$j=1$为止。转到S2。 ▮ 87

(参见斯 · 德沃拉克(S. Dvořák) *Comp. J.* 33 (1990)，188。注意如果对于$1\leqslant k\leqslant s$，$x_k=n-b_k$，则此算法以递增词典顺序跑遍$\{1, 2, \cdots, n\}$的所有组合$x_s\cdots x_2x_1$，且有$1\leqslant x_s<\cdots<x_2<x_1\leqslant n$。)

8. A1. [初始化。] 置$a_n\cdots a_0\leftarrow 0^{s+1}1^t$，$q\leftarrow t$，$r\leftarrow 0$。(我们假定$0<t<n$。)

A2. [访问。] 访问组合$a_{n-1}\cdots a_1a_0$。如果$q=0$则转到A4。

A3. [以...101^{q-1}代替...01^q。] 置$a_q\leftarrow 1$，$a_{q-1}\leftarrow 0$，$q\leftarrow q-1$；然后如果$q=0$则置$r\leftarrow 1$。返回到A2。

A4. [移动1的块。] 置$a_r\leftarrow 0$以及$r\leftarrow r+1$。然后如果$a_r=1$，则置$a_q\leftarrow 1$，$q\leftarrow q+1$，并重复步骤A4。

A5. [向左进行。] 如果$r=n$则结束，否则置$a_r\leftarrow 1$。

A6. [奇数的？] 如果$q>0$，则置$r\leftarrow 0$。返回A2。 ▮

在步骤A2中，q和r分别指向$a_{n-1}\cdots a_0$中最右的0和1。步骤A1，⋯，A6分别以1，$\binom{n}{t}$，$\binom{n-1}{t-1}$，$\binom{n}{t}-1$，$\binom{n-1}{t}$，$\binom{n-1}{t}-1$的频率被执行。

9. (a) 头$\binom{n-1}{t}$个串以0开始而且有$2A_{(s-1)t}$个二进位的变化；其他$\binom{n-1}{t-1}$个串以1开头而且有$2A_{s(t-1)}$个二进位的变化。而且$\nu(01^t0^{s-1}\oplus 10^s1^{t-1})=2\min(s,t)$。

(b) 解1(直接的)：令$B_{st}=A_{st}+\min(s, t)+1$，则当$st>0$时

$$B_{st}=B_{(s-1)t}+B_{s(t-1)}+[s=t]$$

而当$st=0$时$B_{st}=1$。结果，$B_{st}=\sum_{k=0}^{\min(s,t)}\binom{s+t-2k}{s-k}$。如果$s\leqslant t$，则这是$\leqslant\sum_{k=0}^{s}\binom{s+t-k}{s-k}=\binom{s+t+1}{s}=\binom{s+t}{s}\frac{s+t+1}{t+1}<2\binom{s+t}{t}$。

解2(间接的)：答案8的算法使得当步骤(A3，A4)被执行(x, y)次时，有$2(x+y)$个二进位发生变化。因此$A_{st}\leqslant\binom{n-1}{t-1}+\binom{n}{t}-1<2\binom{n}{t}$。

(因此答案7.2.1.1-3中的注释也适用于组合。)

10. 每个场景对应于一个(4, 4)组合$b_4b_3b_2b_1$或$c_4c_3c_2c_1$，其中A赢得比赛$\{8-b_4, 8-b_3, 8-b_2, 8-b_1\}$而N赢得比赛$\{8-c_4, 8-c_3, 8-c_2, 8-c_1\}$，因为我们可假设，输的队在一个8的级数中赢得剩下的比赛。(同样，我们可以生成{A, A, A, A, N, N,

N, N}的所有排列并省略诸A和N尾部的运程。)美国队赢当且仅当$b_1\neq 0$，当且仅当$c_1=0$。公式$\binom{c_4}{4}+\binom{c_3}{3}+\binom{c_2}{2}+\binom{c_1}{1}$对每个场景赋予0和69之间惟一的整数。

例如，ANANAA $\Leftrightarrow a_7\cdots a_1a_0=01010011\Leftrightarrow b_4b_3b_2b_1=7532\Leftrightarrow c_4c_3c_2c_1=6410$，而这是在词典顺序下阶$\binom{6}{4}+\binom{4}{3}+\binom{1}{2}+\binom{0}{1}=19$的场景。(注意项$\binom{c_j}{j}$将为零当且仅当它对应于一个尾部N。)

11. AAAA(9次)、NNNN(8次)及ANAAA(7次)是最普通的。70个中恰有27个不出现，包括以NNNA开始的所有4个。(我们不理会由于黑暗年代的1907年、1912年和1922年的打平局的比赛。ANNAAA的情况或许也应排除掉，因为它仅在1920年作为在9个中最好的一次ANNAAAA的一部分出现。场景NNAAANN于2001年头一次出现。)

12. (a)令V_j是子空间$\{a_{n-1}\cdots a_0\in V\mid$对于$0\leqslant k<j$，$a_k=0\}$，使得$\{0\cdots0\}=V_n\subseteq V_{n-1}\subseteq\cdots\subseteq V_0=V$。则$\{c_1,\cdots,c_t\}=\{c\mid V_c\neq V_{c+1}\}$，而且$\alpha_k$是使得对于$1\leqslant j\leqslant t$满足$a_{c_j}=[j=k]$的$V$的惟一元素$a_{n-1}\cdots a_0$。

顺便指出，对应于一个典范基底$t\times n$的矩阵是在归约的行梯状形式下，通过一
88 个标准的“三角化”算法，可以找到它(参见习题4.6.1－19和算法4.6.2N)。

(b) 习题1.2.6－58的2－多项式系数$\binom{n}{t}_2=2^t\binom{n-1}{t}_2+\binom{n-1}{t-1}_2$有正确的性质，因为$2^t\binom{n-1}{t}_2$二进制向量空间有$c_t<n-1$和$\binom{n-1}{t-1}_2$有$c_t=n-1$。[一般地说，有$r$个星号的一个典范基底的个数是$r$分成至多$t$个部分的分划数，而且没有任何部分超过$n-t$，而由等式7.2.1.4－(51)，这是$[z^r]\binom{n}{t}_z$。参见唐·欧·克努特，*J. Combinatorial Theory* 10 (1971)，178-180。]

(c) 以下算法假设$n>t>0$和对于$t\leqslant j\leqslant n$，$a_{(t+1)j}=0$。

V1. [初始化。] 对于$1\leqslant k\leqslant t$和$0\leqslant j<n$，置$a_{kj}\leftarrow[j=k-1]$。并置$q\leftarrow t$，$r\leftarrow 0$。

V2. [访问。] (这时，对于$1\leqslant k\leqslant q$，我们有$a_{k(k-1)}=1$，$a_{(q+1)q}=0$而且$a_{1r}=1$。)访问典范基底$(a_{1(n-1)}\cdots a_{11}a_{10},\cdots,a_{t(n-1)}\cdots a_{t1}a_{t0})$。如果$q>0$则转向V4。

V3. [找诸1的块。] 置$q\leftarrow 1, 2, \cdots$，直到$a_{(q+1)(q+r)}=0$为止。如果$q+r=n$则结束。

V4. [加1到$q+r$列。] 置$k\leftarrow 1$。如果$a_{k(q+r)}=1$，则置$a_{k(q+r)}\leftarrow 0$，$k\leftarrow k+1$，并且重复直到$a_{k(q+r)}=0$为止。然后如果$k\leqslant q$，则置$a_{k(q+r)}\leftarrow 1$；否则置$a_{q(q+r)}\leftarrow 1$，$a_{q(q+r-1)}\leftarrow 0$，$q\leftarrow q-1$。

V5. [把块右移。] 如果$q=0$，则置$r\leftarrow r+1$。否则如果$r>0$，则对于$1\leqslant k\leqslant q$置$a_{k(k-1)}\leftarrow 1$和$a_{k(r+k-1)}\leftarrow 0$，然后置$r\leftarrow 0$。转向V2。 ▌

步骤V2以$1-(2^{n-t}-1)/(2^n-1)\approx 1-2^{-t}$的概率发现$q>0$，所以我们通过单独处理这个情况可节省时间。

(d) 由于 $999999=4\binom{8}{4}_2+16\binom{7}{4}_2+5\binom{6}{3}_2+5\binom{5}{3}_2+8\binom{4}{3}_2+0\binom{3}{2}_2+4\binom{2}{2}_2+1\binom{1}{1}_2+2\binom{0}{1}_2$，所以第100万个输出有二进制的列4，16/2，5，5，8/2，0，4/2，1，2/2，即

$$\alpha_1 = 001100011$$
$$\alpha_2 = 000000100$$
$$\alpha_3 = 101110000$$
$$\alpha_4 = 010000000$$

[参考文献：尤·卡拉比(E. Calabi)和希·索·威尔弗，*J. Combinatorial Theory* A22 (1977)，107-109。]

13. 令$n=s+t$，共有 $\binom{s-1}{\lceil (r-1)/2\rceil}\binom{t-1}{\lfloor (r-1)/2\rfloor}$ 个以0开始的配置和 $\binom{s-1}{\lfloor (r-1)/2\rfloor}\binom{t-1}{\lceil (r-1)/2\rceil}$ 个以1开始的配置，因为以0开始的艾辛配置对应于s个0分成为$\lceil (r+1)/2\rceil$个部分的合成和t个1分成为$\lfloor (r+1)/2\rfloor$个部分的合成。我们可以生成所有这样的合成对并把它们编入配置中。[参见欧·艾辛，*Zeitschrift für Physik* 31 (1925)，253-258；约·曼·桑·西莫斯·彼莱拉(J. M. S. Simões Pereira) *CACM* 12 (1969)，562。]

14. 对于$1\leqslant j\leqslant n$，以$l[j]\leftarrow j-1$和$r[j-1]\leftarrow j$开始；$l[0]\leftarrow n$，$r[n]\leftarrow 0$。为获得下一个组合，假设$t>0$，如果$l[0]>s$，置$p\leftarrow s$，否则置$p\leftarrow r[n]-1$。如果$p\leqslant 0$则结束；否则置$q\leftarrow r[p]$，$l[q]\leftarrow l[p]$和$r[l[p]]\leftarrow q$。然后如果$r[q]>s$和$p<s$，置$r[p]\leftarrow r[n]$，$l[r[n]]\leftarrow p$，$r[s]\leftarrow r[q]$，$l[r[q]]\leftarrow s$，$r[n]\leftarrow 0$，$l[0]\leftarrow n$；否则置$r[p]\leftarrow r[q]$，$l[r[q]]\leftarrow p$。最后置$r[q]\leftarrow p$和$l[p]\leftarrow q$。

[参见科什(Korsh)和里普舒特兹(Lipschutz)，*J. Algorithms* 25 (1997)，321-335；其中这个想法被推广到多重集合的排列的无循环算法。告诫：像习题7.2.1.1-16一样，这个习题相对于实践性来说，是更学术性的，因为访问链接表的例程可能需要消除无循环生成任何优点的一个循环。] 89

15. (所述事实为真，因为$c_t\cdots c_1$的词典顺序对应于$a_{n-1}\cdots a_0$的词典顺序，它是反序列$1\cdots1\oplus a_{n-1}\cdots a_0$的颠倒的词典顺序。)由定理L，组合$c_t\cdots c_1$在恰好访问其他的 $\binom{b_s}{s}+\cdots+\binom{b_2}{2}+\binom{b_1}{1}$ 之前被访问，因此我们必定有

$$\binom{b_s}{s}+\cdots+\binom{b_1}{1}+\binom{c_t}{t}+\cdots+\binom{c_1}{1}=\binom{s+t}{t}-1$$

当每个x_j是0或1，且$\bar{x}_j=1-x_j$时，这个一般的恒等式可以写成为

$$\sum_{j=0}^{n-1}x_j\binom{j}{x_0+\cdots+x_j}+\sum_{j=0}^{n-1}\bar{x}_j\binom{j}{\bar{x}_0+\cdots+\bar{x}_j}=\binom{n}{x_0+\cdots+x_{n-1}}-1$$

它也由以下等式

$$x_n\binom{n}{x_0+\cdots+x_n}+\bar{x}_n\binom{n}{\bar{x}_0+\cdots+\bar{x}_n}=\binom{n+1}{x_0+\cdots+x_n}-\binom{n}{x_0+\cdots+x_{n-1}}$$

得出。

16. 由于 $999999=\binom{1414}{2}+\binom{1008}{1}=\binom{182}{3}+\binom{153}{2}+\binom{111}{1}=\binom{71}{4}+\binom{56}{3}+\binom{36}{2}+\binom{14}{1}=\binom{43}{5}+$

$\binom{32}{4}+\binom{21}{3}+\binom{15}{2}+\binom{6}{1}$，因此答案是(a)1414 1008; (b) 182 153 111; (c) 71 56 36 14; (d) 43 32 21 15 6; (e)1000000 999999 …2 0。

17. 由定理L，n_t 是使得$N \geqslant \binom{n_t}{t}$的最大整数；余下的项是$N-\binom{n_t}{t}$的$t-1$次表示。

对于$t>1$，一个简单的顺序方法是以$x=1$，$c=t$开始并置$c \leftarrow c+1$，$x \leftarrow xc/(c-t)$零次或多次，直到$x>N$为止；然后我们通过置$x \leftarrow x(c-t)/c$，$c \leftarrow c-1$完成头一阶段，这时我们有 $x=\binom{c}{t} \leqslant N<\binom{c+1}{t}$。置$n_t \leftarrow c$，$N \leftarrow N-x$；如果$t=2$则以$n_1 \leftarrow N$结束。否则置$x \leftarrow xt/c$，$t \leftarrow t-1$，$c \leftarrow c-1$；而当$x>N$时，置$x \leftarrow x(c-t)/c$，$c \leftarrow c-1$，重复进行。如果$N<\binom{n}{t}$，这个方法要求$O(n)$次算术运算，所以除非$t$很小而$N$很大，否则它是合适的。

当$t=2$时，习题1.2.4-41告诉我们 $n_2=\left\lfloor\sqrt{2N+2}+\frac{1}{2}\right\rfloor$ 。一般地说，n_t是$\lfloor x\rfloor$，其中x是 $x^{\underline{t}}=t!N$ 最大的根，把这个根转换成级数 $y=(x^{\underline{t}})^{1/t}=x-\frac{1}{2}(t-1)+\frac{1}{24}(t^2-1)x^{-1}+\cdots$，可以获得其近似值来得到$x=$ $y+\frac{1}{2}(t-1)+\frac{1}{24}(t^2-1)/y+O(y^{-3})$ 。在这个公式中置$y=(t!N)^{1/t}$给出一个好的近似，在它之后我们可以检查$\binom{\lfloor x\rfloor}{t}\leqslant N<\binom{\lfloor x\rfloor+1}{t}$或作一次最后的调整。[参见阿·S·弗连克尔(A. S. Fraenkel)和莫·莫尔(M. Mor), *Comp. J.* **26** (1983), 336-343。]

18. 得到2^n-1个节点的一个完备二叉树，而且在顶层有一个额外的节点，好像在替代选择排序中的"失败者树"(5.4.1节中的图63)那样。因此不需要显式的链接；对于$1\leqslant k<2^{n-1}$，节点k的右儿子是节点$2k+1$，而且左兄弟是$2k$。

二项式树的这个表示有如下奇怪性质，即节点$k=(0^a1\alpha)_2$对应于其二进制串为$0^a1\alpha^R$的组合。

19. 它是post (1000000)，其中如果$2^k\leqslant n<2^{k+1}$，post $(n)=2^k+\text{post}(n-2^k+1)$，而且post(0)=0，所以它是11110100001001000100。

20. $f(z)=(1+z^{\omega_{n-1}})\cdots(1+z^{\omega_1})/(1-z), g(z)=(1+z^{\omega_0})f(z), h(z)=z^{\omega_0}f(z)$ 。

21. $c_t\cdots c_2c_1$的阶是$\binom{c_t+1}{t}-1$减去$c_{t-1}\cdots c_2c_1$的阶。[参见海·伦尼伯格(H. Lüneburg), *Abh. Math. Sem. Hamburg* **52** (1982), 208－227。]

22. 由于 $999999=\binom{1415}{2}-\binom{406}{1}=\binom{183}{3}-\binom{98}{2}+\binom{21}{1}=\binom{72}{4}-\binom{57}{3}+\binom{32}{2}-\binom{27}{1}=\binom{44}{5}-\binom{40}{4}+\binom{33}{3}-\binom{13}{2}+\binom{3}{1}$，因此答案是(a) 1414 405; (b) 182 97 21; (c)71 56 31 26; (d)43
90 39 32 12 3; (e) 1000000 999999 999998 999996…0。

23. 对于$r=1, 2, \cdots, t$，有$\binom{n-r}{t-r}$个使$j>r$的组合。(如果$r=1$，我们有$c_2=c_1+1$；如果$r=2$，我们有$c_1=0$，$c_2=1$。如果$r=3$，我们有$c_1=0$，$c_2=1$，$c_4=c_3+1$，等等。)因此均

值是 $(\binom{n}{t}+\binom{n-1}{t-1}+\cdots+\binom{n-t}{0})/\binom{n}{t}=\binom{n+1}{t}/\binom{n}{t}=(n+1)/(n+1-t)$ 。每一步的平均运行时间近似地与这个量成正比，因此当t很小时这个算法是十分快的，但当t接近n时很慢。

24. 事实上，当$j_k\equiv t$ (modulo 2)时，$j_k-2\leqslant j_{k+1}\leqslant j_k+1$，而当$j_k\not\equiv t$时，$j_k-1\leqslant j_{k+1}\leqslant j_k+2$，因为对于$1\leqslant i<j$，当$c_i=i-1$时R5才被执行。

因此我们可以说，在R2结束时，如果t为奇数，"如果$j\geqslant 4$，置$j\leftarrow j-1-[j$为奇数]，并转到R5"。如果t为偶数，"如果$j\geqslant 3$，则置$j\leftarrow j-1-[j$为偶数]，并转到R5"，那么算法将是无循环的，因为每次访问R4和R5至多被执行两次。

25. 假设$N>N'$且$N-N'$为极小，而且设t和c_t在那些假设之下为极小，则$c_t>c_t'$。

如果有一个元素 $x\notin C\cup C'$ 且$0\leqslant x<c_t$，则对于$j>x$，通过改变$j\mapsto j-1$映射$C\cup C'$ 的每个t组合；或者如果有一个元素$x\in C\cap C'$，则把包含x的每个t组合映射到一个$(t-1)$组合，办法是省去x和对于$j<x$，改变$j\mapsto x-j$。在任一种情况下，这个映射保持交替的词典顺序。因此$N-N'$ 必定超过C和C' 的映像之间的组合个数。但是c_t是极小的，所以不存在这样的x。结果$t=m$且$c_t=2m-1$。

现在如果$c_m'<c_m-1$，我们通过增加c'_m可以减少$N-N'$。因此$c'_m=2m-2$，因而问题就归结为寻找$\mathrm{rank}(c_{m-1}\cdots c_1)-\mathrm{rank}(c'_{m-1}\cdots c'_1)$的极大值，其中rank可像在(30)中那样计算。

在所有$\{b_s,\cdots,b_1,c_t,\cdots,c_1\}=\{0,\cdots,s+t-1\}$上，令$f(s,t)=\max\ (\mathrm{rank}(b_s\cdots b_1)-\mathrm{rank}(c_t\cdots c_1))$，则$f(s,t)$满足奇怪的递归

$$f(s,0)=f(0,t)=0;\qquad f(1,t)=t;$$
$$f(s,t)=\binom{s+t-1}{s}+\max(f(t-1,s-1),f(s-2,t))\text{ ，如果}st>0\text{且}s>1$$

当$s+t=2u+2$时，结果解为

$$f(s,t)=\binom{2u+1}{t-1}+\sum_{j=1}^{u-r}\binom{2u+1-2j}{r}+\sum_{j=0}^{r-1}\binom{2j+1}{j},\qquad r=\min(s-2,t-1)$$

而且当$s\leqslant t$时，极大值出现于$f(t-1,s-1)$处，当$s\geqslant t+2$时在$f(s-2,t)$处。

因此对于

$$C=\{2m-1\}\cup\{2m-2-x\mid 1\leqslant x\leqslant 2m-2,\ x \bmod 4\leqslant 1\}$$
$$C'=\{2m-2\}\cup\{2m-2-x\mid 1\leqslant x\leqslant 2m-2,\ x \bmod 4\geqslant 2\}$$

极小值$N-N'$ 出现，而且它等于 $\binom{2m-1}{m-1}-\sum_{k=0}^{m-2}\binom{2k+1}{k}=1+\sum_{k=1}^{m-1}\binom{2k}{k-1}$。[参见阿·简·范·赞廷(A. J. van Zanten)，*IEEE Trans.* IT-37 (1991)，1229-1233。]

26. (a) 是的。头一个是 $0^{n-\lceil t/2\rceil}1^{t \bmod 2}2^{\lfloor t/2\rfloor}$ ，而最后一个是 $2^{\lfloor t/2\rfloor}1^{t \bmod 2}0^{n-\lceil t/2\rceil}$；转换是形如$02^a1\leftrightarrow 12^a0$，$02^a2\leftrightarrow 12^a1$，$10^a1\leftrightarrow 20^a0$，$10^a2\leftrightarrow 20^a1$的子串。

(b) 否：如果$s=0$，则从02^t0^{r-1}到20^r2^{t-1}有一个大的跳跃。

27. 下列过程抽取权$\leqslant t$的Γn的所有组合$c_1\cdots c_k$；以$k\leftarrow 0$和$c_0\leftarrow n$开始，访问$c_1\cdots c_k$。如果k为偶数而且$c_k=0$，则置$k\leftarrow k-1$；如果k为偶数且$c_k>0$，如果$k=t$则置$c_k\leftarrow$

91 c_k-1，否则$k\leftarrow k+1$而且$c_k\leftarrow 0$。另一方面，如果k为奇数，且$c_k+1=c_{k-1}$，则置$k\leftarrow k-1$，而且$c_k\leftarrow c_{k+1}$（但如果$k=0$则结束）。如果k为奇数，而且$c_k+1<c_{k-1}$，则如果$k=t$置$c_k\leftarrow c_k+1$，否则$k\leftarrow k+1$，$c_k\leftarrow c_{k-1}$，$c_{k-1}\leftarrow c_k+1$。重复。

（当$t=n$时，这个无循环算法归结成习题7.2.1.1-12(b)那样，但记号有小变化。）

28. 真的。二进位串$a_{n-1}\cdots a_0=\alpha\beta$和$a'_{n-1}\ldots a'_0=\alpha\beta'$对应于下标表($b_s\cdots b_1=\theta\chi$，$c_t\cdots c_1=\phi\psi$)和($b'_s\ldots b'_1=\theta\chi', c'_t\ldots c'_1=\phi\psi'$)，使得$\alpha\beta$和$\alpha\beta'$之间的每个东西以$\alpha$开始当且仅当$\theta\chi$和$\theta\chi'$之间的每个东西以$\theta$开始且$\phi\psi$和$\phi\psi'$之间的每个东西以$\phi$开始。例如，如果$n=10$，则前缀$\alpha=01101$对应于前缀$\theta=96$和$\phi=875$。

（但仅仅存在广义词典顺序下的$c_t\cdots c_1$是一个弱得多的条件。例如，当$t=1$时，每一个这样的序列是广义词典顺序。）

29. (a) 对于$k,l,m\geqslant 0$，$-^k0^{l+1}$或$-^k0^{l+1}+\pm^m$或$\pm^k$。

(b) 否；在平衡的三进制记号下后继者总是较小的。

(c) 对于所有α和所有k，l，$m\geqslant 0$，我们有$\alpha 0-^{k+1}0^l+\pm^m\rightarrow\alpha-+^k0^{l+1}-\pm^m$和$\alpha+-^k0^{l+1}+\pm^m\rightarrow\alpha 0+^{k+1}0^l-\pm^m$；而且$a0-^{k+1}0^l\rightarrow\alpha-+^k0^{l+1}$和$\alpha+-^k0^{l+1}\rightarrow\alpha 0+^{k+1}0^l$。

(d) 令α_i的第j个符号是$(-1)^{a_{ij}}$，且令它在位置b_{ij}处。那么，对于$0\leqslant i<k$和$1\leqslant j\leqslant t$，如果我们令$b_{i0}=0$，则有$(-1)^{a_{ij}+b_{i(j-1)}}=(-1)^{a_{(i+1)j}+b_{(i+1)(j-1)}}$。

(e) 由部分(a)、(b)和(c)，α属于某个链$\alpha_0\rightarrow\cdots\rightarrow\alpha_k$，其中$\alpha_k$是最后的(无后继)，而$\alpha_0$是初始的(无前驱)。由部分(d)，每个链至多有$\binom{s+t}{t}$个元素。但由(a)有$2^s$个最后的串，而且有$2^s\binom{s+t}{t}$个串有$s$个符号和$t$个零。所以$k$必须是$\binom{s+t}{t}-1$。

参考文献：*SICOMP* 2 (1973)，128-133。

30. 假设$t>0$，初始串是最后串的否定。对于$0\leqslant j<2^{s-1}$，令σ_j是初始的串$0^t-\tau_j$，其中对于$1\leqslant k<s$，τ_j的第k个字符是当j是二进数$(a_{s-1}\cdots a_1)_2$时$(-1)^{a_k}$的符号。因此$\sigma_0=0^t-++\cdots+$，$\sigma_1=0^t--+\cdots+$，…，$\sigma_{2^{s-1}-1}=0^t---\ldots-$。令$\rho_j$是通过在$\tau_j$的减符号的头一个(可能为空的)运程之后插入$-0^t$所得到的最后的串。于是，$\rho_0=-0^t++\cdots+$，$\rho_1=--0^t+\cdots+$，…，$\rho_{2^{s-1}-1}=--\ldots-0^t$。我们还令$\sigma_{2^{s-1}}=\sigma_0$和$\rho_{2^{s-1}}=\rho_0$，那么我们可以通过归纳法证明，对于$1\leqslant j\leqslant 2^{s-1}$，当$t$为偶数时，这个链以$\sigma_j$开始而且以$\rho_j$结尾，而当$t$为奇数时，则以$\rho_{j-1}$结尾。因此以$-\rho_j$开始的链以$-\sigma_j$或$-\sigma_{j+1}$结尾。

令$A_j(s,t)$是由把以σ_j开始的链映射所导出的(s,t)组合的序列，而$B_j(s,t)$是由$-\rho_j$导出的类似的序列。于是，对于$1\leqslant j\leqslant 2^{s-1}$，当$t$为偶数时，颠倒的序列$A_j(s,t)^R$是$B_j(s,t)$；当$t$是奇数时，它是$B_{j-1}(s,t)$。当$st>0$时对应的递归是

$$A_j(s,t)=\begin{cases}1A_j(s,t-1),0A_{\lfloor(2^{s-1}-1-j)/2\rfloor}(s-1,t)^R, & \text{如果}j+t\text{为偶数}\\ 1A_j(s,t-1),0A_{\lfloor j/2\rfloor}(s-1,t), & \text{如果}j+t\text{为奇数}\end{cases}$$

而且当$st>0$时，这些序列中的所有2^{s-1}个全不同。

蔡斯序列C_{st} 为 $A_{\lfloor 2^s/3\rfloor}(s,t)$， $\hat{C}_{st}$ 是 $A_{\lfloor 2^{s-1}/3\rfloor}(s,t)$ 。顺便指出，(31)的同态序列K_{st}是 $A_{2^{s-1}-[t\text{为偶数}]}(s,t)^R$。

31. (a) 当$f(s,0)=f(0,t)=1$时， $2^{\binom{s+t}{t}-1}$ 解决递归$f(s,t)=2f(s-1,t)f(s,t-1)$;
(b) 现在$f(s,t)=(s+1)!\,f(s,t-1)\ldots f(0,t-1)$有解

$$(s+1)!^t\, s!^{\binom{t}{2}}(s-1)!^{\binom{t+1}{3}}\ldots 2!^{\binom{s+t-2}{s}}=\prod_{r=1}^{s}(r+1)!^{\binom{s+t-1-r}{t-2}+[r=s]}$$

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32. (a) 似乎无简单的公式存在，但通过系统地计算经由转动门移动，跑遍从一个给定的起始点开始到一个给定的结束点为止所有权t的串的广义词典顺序的通路个数，对于小的s和t，可以统计表的数目。对于$s+t\leqslant 6$总数为

1
1 1
1 2 1
1 4 4 1
1 8 20 8 1
1 16 160 160 16 1
1 32 2264 17152 2264 32 1

而且$f(4,4)=95,304,112,865,280$; $f(5,5)\approx 5.92646\times 10^{48}$。[这类组合生成元首先是由吉 · 厄尔里兹研究的(*JACM* 20 (1973)，500－513)，但他并不试图枚举它们。]

(b) 通过扩充定理N的证明，人们可以证明，所有这样的表或它们的颠倒必定是对于某个a，$1\leqslant a\leqslant s$，从$1^t\,0^s$运行到$0^a\,1^t\,0^{s-a}$。而且，给定s、t和a，且$st>0$，则可能性的数目n_{sta}满足$n_{1t1}=1$以及

$$n_{sta}=\begin{cases} n_{s(t-1)1}n_{(s-1)t(a-1)}, & \text{如果}a>1\\ n_{s(t-1)2}n_{(s-1)t1}+\cdots+n_{s(t-1)s}n_{(s-1)t(s-1)}, & \text{如果}a=1<s\end{cases}$$

这个递归有令人注目的解$n_{sta}=2^{m(s,t,a)}$，其中

$$m(s,t,a)=\begin{cases}\binom{s+t-3}{t}+\binom{s+t-5}{t-2}+\cdots+\binom{s-1}{2}, & \text{如果}t\text{为偶数}\\ \binom{s+t-3}{t}+\binom{s+t-5}{t-2}+\cdots+\binom{s}{3}+s-a-[a<s], & \text{如果}t\text{为奇数}\end{cases}$$

33. 首先考虑$t=1$的情况：从i到$j>i$，接近完美的通路的个数是$f(j-i-[i>0]-[j<n-1])$，其中$\sum_j f(j)z^j=1/(1-z-z^3)$。(碰巧，相同的序列$f(j)$出现在6条带上的卡伦(Caron)多阶段合并，见表5.4.2－2。)在$0\leqslant i<j<n$上的和是$3f(n)+f(n-1)+f(n-2)+2-n$；因此我们必须把这加倍，来覆盖$j>i$的情况。

当$t>1$时，我们可以构造 $\binom{n}{t}\times\binom{n}{t}$ 的矩阵，它们告知有多少广义词典顺序的表以特殊组合开始和结束。这些矩阵的元素是对于$t=1$所考虑的所有类型通路求和，

对于$t-1$情况矩阵乘积的和。对于$s+t\leqslant 6$的总数结果是

1	1
1 1	1 1
1 2 1	1 2 1
1 6 2 1	1 2 0 1
1 12 10 2 1	1 2 2 0 1
1 20 44 10 2 1	1 2 0 0 0 1
1 34 238 68 10 2 1	1 2 6 0 0 0 1

其中右边的三角形求出循环$g(s,t)$的个数。进一步的值包括$f(4,4)=17736$；$f(5,5)=9\,900\,888\,879\,984$；$g(4,4)=96$；$g(5,5)=30\,961\,456\,320$。

当$s=2$和$n\geqslant 4$时，恰好有10个这样的方案。例如，当$n=7$时，它们从43210跳到
93 65431或65432，或者从54321跳到65420或65430或65432，或者颠倒过来。

34. 这个极小值可以像在上道题中那样计算，但使用极小加的矩阵乘法$c_{ij}=\min_k(a_{ik}+b_{kj})$而不是通常的矩阵乘法$c_{ij}=\sum_k a_{ik}b_{kj}$。(当$s=t=5$时，仅带有49个不完备转换的图26(e)中的广义词典顺序的通路实际上是惟一的。对于$s=t=5$，有一个广义词典顺序的循环，它仅有55个不完备性。)

35. 从递归式(35)，我们有$a_{st}=b_{s(t-1)}+[s>1][t>0]+a_{(s-1)t}$，$b_{st}=a_{s(t-1)}+a_{(s-1)t}$；结果$a_{st}=b_{st}+[s>1][t\text{奇数}]$和$a_{st}=a_{s(t-1)}+a_{(s-1)t}+[s>1][t\text{奇数}]$。解为

$$a_{st}=\sum_{k=0}^{t/2}\binom{s+t-2-2k}{s-2}-[s>1][t\text{ 为偶数}]$$

这个和近似地为$s/(s+2t)$乘以$\binom{s+t}{t}$。

36. 考虑具有根节点(s,t)的二叉树，而且每当$st>0$时，该树有根为$(s-1,t)$和$(s,t-1)$的递归定义的子树；如果$st=0$节点(s,t)是一个叶。于是，在(s,t)处为根的子树有$\binom{s+t}{t}$个叶，并且对应于所有(s,t)组合$a_{n-1}\cdots a_1a_0$。在l级上的节点对应于前缀$a_{n-1}\cdots a_{n-l}$，而且在l级上的叶是对于$r=n-l$的组合。

对于组合$a_{n-1}\cdots a_1a_0$的任何广义词典顺序算法，在$\binom{s+t}{t}-1$个分支节点的孩子已经以任何合意的顺序排序之后，对应于这样一棵树的前缀遍历。事实上，这即是为什么有$2^{\binom{s+t}{t}-1}$个这样的广义词典顺序方案(习题31(a))。而且，对于每一个节点操作$j\leftarrow j+1$恰被执行一次，即在两个孩子都被处理之后。

顺便指出，习题7.2.1.2－6(a)意味着，r的平均值是$s/(t+1)+t/(s+1)$，它可能是$\Omega(n)$，因此为记住r所需要的额外时间是值得的。

37. (a) 在词典顺序下，我们不必维护w_j表，因为对于$j\geqslant r$，a_j是活动的当且仅当$a_j=0$。在设置$a_j\leftarrow 1$和$a_{j-1}\leftarrow 0$之后，如果$j>1$有两种情况要考虑。如果$r=j$，则置$r\leftarrow j-1$；否则置$a_{j-2}\cdots a_0\leftarrow 0^r1^{j-1-r}$以及$r\leftarrow j-1-r$(或如果$r$是$j-1$，则$r\leftarrow j$)。

(b) 现在当$j>1$时，要加以处理的转换是改变$a_j\cdots a_0$如下：$01^r\rightarrow 1101^{r-2}$，$010^r\rightarrow 10^{r+1}$，$010^a1^r\rightarrow 110^{a+1}1^{r-1}$，$10^r\rightarrow 010^{r-1}$，$110^r\rightarrow 010^{r-1}1$，$10^a1^r\rightarrow 0^a1^{r+1}$；这6种情况很

容易区分，r的值应适当改变。

(c) 再次，j=1的情况是平凡的。否则$01^a0^r \to 101^{a-1}0^r$；$0^a1^r \to 10^a1^{r-1}$；$101^a0^r \to 01^{a+1}0^r$；$10^a1^r \to 0^a1^{r+1}$；而且也有一个二义性的情况，仅当$a_{n-1}\cdots a_{j+1}$至少含有一个0时它才能出现。设$k>j$是满足$a_k=0$的极小值。则如果k为奇数，$10^r \to 010^{r-1}$，如果k为偶数则$10^r \to 0^r1$。

38. 除开下面三种情况，同样的算法有效：(i) 如果n为奇数，步骤C1置$a_{n-1}\cdots a_0 \leftarrow 01^t0^{s-1}$，或者如果$n$为偶数且$s>1$，并有一个适当的$r$的值，置$s$=1，$a_{n-1}\cdots a_0 \leftarrow 001^t0^{s-2}$；(ii)步骤C3交换偶和奇的作用；(iii)如果j=1步骤C5也转到C4。

39. 一般地说，以$r\leftarrow 0$，$j\leftarrow s+t-1$开始，并重复以下步骤直到st=0为止。

$$r \leftarrow r+[w_j=0]\binom{j}{s-a_j},\ s\leftarrow s-[a_j=0],\quad t\leftarrow t-[a_j=1],\quad j\leftarrow j-1$$

然后，r是$a_{n-1}\cdots a_1a_0$的阶。所以11001001000011111101101010的阶是$\binom{23}{12}+\binom{22}{11}+\binom{21}{9}+\binom{17}{8}+\binom{16}{7}+\binom{14}{5}+\binom{13}{3}+\binom{12}{3}+\binom{11}{3}+\binom{10}{3}+\binom{9}{3}+\binom{8}{3}+\binom{4}{3}+\binom{3}{1}+\binom{1}{0}=2390131$。 94

40. 我们以$N\leftarrow 999999$，$v\leftarrow 0$开始，并且重复以下步骤直到st=0为止：如果v=0则置$t\leftarrow t-1$且如果$N<\binom{s+t-1}{s}$则置$a_{s+t}\leftarrow 1$。否则置$N\leftarrow N-\binom{s+t-1}{s}$，$v\leftarrow(s+t) \bmod 2$，$s\leftarrow s-1$，$a_{s+t}\leftarrow 0$。如果$v$=1，则置$v\leftarrow(s+t) \bmod 2$，$s\leftarrow s-1$，以及如果$N<\binom{s+t-1}{s}$则$a_{s+t}\leftarrow 0$，否则置$N\leftarrow N-\binom{s+t-1}{s}$，$t\leftarrow t-1$，$a_{s+t}\leftarrow 1$。最后如果$s$=0，则置$a_{t-1}\cdots a_0\leftarrow 1^t$；如果$t$=0，则置$a_{s-1}\cdots a_0\leftarrow 0^s$。答案是$a_{25}\cdots a_0$=11101001111110101001000001。

41. 令$c(0),\cdots,c(2^n-1)=C_n$，其中$C_{2n}=0C_{2n-1}$，$1C_{2n-1}$；$C_{2n+1}=0C_{2n}$，$1\hat{C}_{2n}$；$\hat{C}_{2n}=1C_{2n-1},0\hat{C}_{2n-1};\hat{C}_{2n+1}=1\hat{C}_{2n},0\hat{C}_{2n};C_0=\hat{C}_0=\varepsilon$。然后$a_j\oplus b_j=b_{j+1}\&(b_{j+2}|(b_{j+3}\&(b_{j+4}|\cdots)))$，如果$j$为偶数，$b_{j+1}|(b_{j+2}\&(b_{j+3}|(b_{j+4}\&\cdots)))$，如果$j$为奇数。奇怪，我们也有逆关系$e((\ldots a_4\bar{a}_3a_2\bar{a}_1a_0)_2)=(\ldots b_4\bar{b}_3b_2\bar{b}_1b_0)_2$。

42. 等式(40)表明，如果a_l=0和$l>r$，则左上下文$a_{n-1}\cdots a_{l+1}$不影响算法在$a_{l-1}\cdots a_0$上的行为。因此，我们可以通过计算以某个二进位模式结尾的组合个数来分析算法C，因而得出，对于一个适当的多项式$p(w,z)$，每个操作被执行的次数可表示为$[w^sz^t]p(w,z)/(1-w^2)^2(1-z^2)^2(1-w-z)$。

例如，对于整数a，$b\geqslant 0$，对于以$01^{2a+1}01^{2b+1}$结尾或有形式$1^{a+1}01^{2b+1}$的每一组合，算法从C5转到C4一次。对应的生成函数是$w^2z^2/(1-z^2)^2(1-w-z)$和$w(z^2+z^3)/(1-z^2)^2$。

以下是关键操作的多项式$p(w,z)$。令$W=1-w^2$，$Z=1-z^2$。

$C3\to C4$:	$wzW(1+wz)(1-w-z^2)$;	$C5(r\leftarrow 1)$:	$w^2zW^2Z(1-wz-z^2)$;
$C3\to C5$:	$wzW(w+z)(1-wz-z^2)$;	$C5(r\leftarrow j-1)$:	$w^2z^3W^2(1-wz-z^2)$;
$C3\to C6$:	$w^2z^2W(w+z)$;	$C6(j=1)$:	w^2zW^2Z;

$C3 \to C7$: $w^2 z W(1+wz)$;

$C4(j=1)$: $wzW^2 Z(1-w-z^2)$;

$C4(r \leftarrow j-1)$: $w^3 zWZ(1-w-z^2)$;

$C4(r \leftarrow j)$: $wz^2 W^2(1+z-2wz-z^2-z^3)$;

$C5 \to C4$: $wz^2 W^2(1-wz-z^2)$;

$C5(r \leftarrow j-2)$: $w^4 zWZ(1-wz-z^2)$;

$C6(r \leftarrow j-1)$: $w^2 z^3 W^2$;

$C6(r \leftarrow j)$: $w^3 z^2 WZ$;

$C7 \to C6$: $w^2 zW^2$;

$C7(r \leftarrow j)$: $w^4 z\, WZ$;

$C7(r \leftarrow j-2)$: $w^3 z^2 W^2$。

对于固定的$0<x<1$，当$n \to \infty$时，如果$t=xn+O(1)$，则渐近值为$\binom{s+t}{t}(p(1-x,x)/(2x-x^2)^2(1-x^2)^2+O(n^{-1}))$。因此，例如，我们发现，在步骤C3中的四路转移以$x+x^2-x^3$: 1: x: $1+x-x^2$的相对频率发生。

顺便指出，j为奇数的情况数超过j为偶数的情况数，在任何使用(39)的广义词典顺序的方案下，这个数为

$$\sum \binom{s+t-2k-2l}{s-2k} [2k+2l \leqslant s+t]+[s\text{奇数}][t\text{奇数}]$$

这个量有有趣的生成函数$wz/(1+w)(1+z)(1-w-z)$。

43. 对于所有非负整数x，此恒等式为真，除非当$x=1$时。

44. 事实上，$C_t(n)-1=\hat{C}_t(n-1)^R$，而且$\hat{C}_t(n)-1=C_t(n-1)^R$。(因此$C_t(n)-2=C_t(n-2)$，等等。)

45. 在下列算法中，r是满足$c_r \geqslant r$的最小下标:

CC1.[初始化。] 对于$1 \leqslant j \leqslant t+1$，置$c_j \leftarrow n-t-1+j$和$z_j \leftarrow 0$，还置$r \leftarrow 1$(我们假设$0<t<n$)。

CC2.[访问。] 访问组合$c_t \cdots c_2 c_1$。然后置$j \leftarrow r$。

95 **CC3.**[转移。] 如果$z_j \neq 0$，则转到CC5。

CC4.[尝试减少c_j。] 置$x \leftarrow c_j+(c_j \bmod 2)-2$。如果$x \geqslant j$，则置$c_j \leftarrow x$，$r \leftarrow 1$；否则如果$c_j=j$，则置$c_j \leftarrow j-1$，$z_j \leftarrow c_{j+1}-((c_{j+1}+1) \bmod 2)$，$r \leftarrow j$；否则如果$c_j<j$，则置$c_j \leftarrow j$，$z_j \leftarrow c_{j+1}-((c_{j+1}+1) \bmod 2)$，$r \leftarrow \max(1, j-1)$。否则置$c_j \leftarrow x$，$r \leftarrow j$。返回到CC2。

CC5.[尝试增加c_j。] 置$x \leftarrow c_j+2$。如果$x<z_j$，则置$c_j \leftarrow x$；否则，如果$x=z_j$和$z_{j+1} \neq 0$，则置$c_j \leftarrow x-(c_{j+1} \bmod 2)$；否则置$z_j \leftarrow 0$，$j \leftarrow j+1$，并转到CC3(但如果$j>t$，则结束)。如果$c_1>0$，则置$r \leftarrow 1$；否则置$r \leftarrow j-1$返回CC2。▌

46. 等式(40)意味着，当j是使$b_j>k$的极小时，$u_k=(b_j+k+1) \bmod 2$。则(37)和(38)产生下列算法，其中为方便起见我们假定$3 \leqslant s<n$。

CB1.[初始化。] 对于$1 \leqslant j \leqslant s$，置$b_j \leftarrow j-1$；并置$z \leftarrow s+1$，$b_z \leftarrow 1$。(当随后的步骤考察$z$的值时，它是使得$b_z \neq z-1$的最小下标。)

CB2.[访问。] 访问对偶组合$b_s \cdots b_2 b_1$。

CB3.[转移。] 如果b_2为奇数：如果$b_2 \neq b_1+1$，则转到CB4，否则如果$b_1>0$则转到CB5，否则如果b_z是奇数则转到CB6。如果b_2为偶数和$b_1>0$则转到CB9。否则如果$b_{z+1}=b_z+1$则转到CB8，否则转到CB7。

CB4.[增加b_1。] 置$b_1 \leftarrow b_1+1$，并返回CB2。

CB5.[滑过b_1和b_2。] 如果b_3为奇数，则置$b_1 \leftarrow b_1+1$和$b_2 \leftarrow b_2+1$，否则置$b_1 \leftarrow b_1-1$，$b_2 \leftarrow b_2-1$，$z \leftarrow 3$。转到CB2。

CB6.[向左滑动。] 如果z为奇数，则置$z \leftarrow z-2$，$b_{z+1} \leftarrow z+1$，$b_z \leftarrow z$，否则置$z \leftarrow z-1$，$b_z \leftarrow z$。转到CB2。

CB7.[滑过b_z。] 如果b_{z+1}是奇数，则置$b_z \leftarrow b_z+1$，而且如果$b_z \geqslant n$，则结束。否则置$b_z \leftarrow b_z-1$，然后如果$b_z<z$，则置$z \leftarrow z+1$。转到CB2。

CB8.[滑过b_z和b_{z+1}。] 如果b_{z+2}是奇数，则置$b_z \leftarrow b_{z+1}$，$b_{z+1} \leftarrow b_z+1$，而且如果$b_{z+1} \geqslant n$，则结束。否则置$b_{z+1} \leftarrow b_z$，$b_z \leftarrow b_z-1$，然后如果$b_z<z$，则置$z \leftarrow z+2$。转到CB2。

CB9.[减小b_1。] 置$b_1 \leftarrow b_1-1$，$z \leftarrow 2$，并返回CB2。 ▮

注意，本算法是无循环的，蔡斯在(*Cong. Num.* 69 (1989)，233－237)中给出了对于序列$\hat{C}_{st}^R$类似的过程。真正令人感到有趣的是，这个算法精确地定义由上道题中的算法产生的下标$c_t \cdots c_1$的补码。

47. 例如，我们可以使用算法C和它的颠倒(习题38)，且以一个d位数(其二进位数字表示在递归的不同级别上的活动)来代替w_j。需要分开的指针$r_0, r_1, \cdots, r_{d-1}$来记住在每个级别上的$r$个值。(许多其他的解是可能的。)

48. 有排列$\pi_1 \cdots \pi_M$使得Λ_j的第k个元素是$\pi_k \alpha_j \uparrow \beta_{k-1}$，而且当$j$从0变到$N-1$时，$\pi_k \alpha_j$跑遍$\{s_1 \cdot 1, \cdots, s_d \cdot d\}$的所有排列。

历史注记：对于$\{s, t\}$组合的一个同态的转动门方案的第一篇著作，是由艾瓦·托罗克(Éva Török)发表的(*Matematikai Lapok* 19 (1968)，143-146)。他是受多重集合的排列生成的诱导而写成的。许多作者随后依赖于类似构造的同态条件，但本习题表明，同态性并非必要。 96

49. 当$0<r<m$时，我们有$\lim_{z\to q}(z^{km+r}-1)/(z^{lm+r}-1)=1$，而且当$r=0$时这个极限是$\lim_{z\to q}(kmz^{km-1})/(lmz^{lm-1})=k/l$。所以当$a \equiv b$(modulo m)时，我们可以把分子$\Pi_{n-k<a\leqslant n}(z^a-1)$的因式同分母$\Pi_{0<b\leqslant k}(z^b-1)$的因式配对。

注记：这个公式是由格·奥里弗(G. Olive)发现的(*AMM* 72 (1965)，619)。在$m=2$，$q=-1$的特殊情况下，第二个因式仅当n为偶数和k为奇数时消失。对于所有$n \geqslant 0$公式$\binom{n}{k}_q=\binom{n}{n-k}_q$成立，但是$\binom{\lfloor n/m \rfloor}{\lfloor k/m \rfloor}$不总等于$\binom{\lfloor n/m \rfloor}{\lfloor (n-k)/m \rfloor}$。然而，我们在$n \bmod m \geqslant k \bmod m$的情况下确有$\lfloor k/m \rfloor + \lfloor (n-k)/m \rfloor = \lfloor n/m \rfloor$，否则第二个因式为零。

50. 当$n_1 \bmod m+\cdots+n_t \bmod m \geqslant m$时，所述系数为零。否则由等式1.2.6－(43)，它等于

$$\binom{\lfloor (n_1+\cdots+n_t)/m \rfloor}{\lfloor n_1/m \rfloor, \cdots, \lfloor n_t/m \rfloor} \binom{(n_1+\cdots+n_t) \bmod m}{n_1 \bmod m, \cdots, n_t \bmod m}_q$$

这里每一个上标是下标之和。

51. 所有通路显然是在000111和111000之间跑动，因为这些顶点有度数1。在所述等价性之下，14条全部的通路归结为4条。(50)中的通路，在反射和颠倒之下，等价于它本身，它可由δ序列A=3452132523414354123来加以描述。其他三个类是B=3452541453414512543，C=3452541453252154123，D=3452134145341432543。德·亨·莱默发现通路C[*AMM* 72 (1965)，Part II，36-46]；D实质上是由埃迪斯(Eades)、希克基(Hickey)和里得(Read)构造的。

(顺便指出，完美的方案并不真正地稀少，尽管看起来它们难以系统地构造。$(s,t)=(3,5)$的情况中有4 050 046种完美的方案。)

52. 我们可以假设，每个s_j非零，而且$d>1$。那么有偶数个和奇数个反演的排列之间的差，由习题50，为$\binom{\lfloor (s_0+\cdots+s_d)/2 \rfloor}{\lfloor s_0/2 \rfloor, \cdots, \lfloor s_d/2 \rfloor} \geqslant 2$，除非至少有2个多重性$s_j$是奇数。

反之，如果至少有两个多重性为奇数，由格·斯塔科维亚克(G. Stachowiak)给出的一般构造[*SIAM J. Discrete Math*. 5 (1992)，199-206]表明，存在一个完美方案。确实，他的构造适用于各种拓扑排序问题。在多重集合的特殊情况下，它对于$d>1$和$s_0 s_1$为奇数的所有情况给出哈密顿循环，除非当$d=2$，$s_0=s_1=1$和s_2为偶数时例外。

53. 参见*AMM*(美国数学月报) 72 (1965)，第Ⅱ部分，36-46。

54. 假设$st \neq 0$，一条哈密顿通路存在当且仅当s和t不全为偶数；一条哈密顿通路存在当且仅当，此外，($s \neq 2$和$t \neq 2$)或$n=5$。[西·克·恩斯(T. C. Enns)，*Discrete Math*. 122 (1993)，153-165。]

55. (a) [由阿伦·威廉斯(Aaron Williams)给出的解。]序列$0^s 1^t$，W_{st}有正确的性质，如果对于$st>0$，

$$W_{st} = 0W_{(s-1)t},\ 1W_{s(t-1)},\ 10^s 1^{t-1};\ W_{0t} = W_{s0} = \phi$$

而且有一个相当有效的无循环实现：假设$t>0$，

W1.[初始化。] 对于$0 \leqslant j<t$，置$n \leftarrow s+t$，$a_j \leftarrow 1$，而且对于$t \leqslant i \leqslant n$，$a_i \leftarrow 0$，并置$j \leftarrow k \leftarrow t-1$。(这是有技巧的，但它有效。)

W2.[访问。] 访问(s,t)组合$a_{n-1} \cdots a_1 a_0$。

W3.[清除a_j。] 置$a_j \leftarrow 0$和$j \leftarrow j+1$。

W4.[是容易情况吗？] 如果$a_j=1$，则置$a_k \leftarrow 1$，$k \leftarrow k+1$并返回W2。

W5.[包起来。] 如果$j=n$，则结束，否则置$a_j \leftarrow 1$。然后如果$k>0$，则置$a_k \leftarrow 1$，
97 $a_0 \leftarrow 0$，$j \leftarrow 1$，以及$k \leftarrow 0$。返回W2。 ▌

在第二次访问之后，j是使得$a_j a_{j-1}=10$的最小下标，而且k是使$a_k=0$的最小下标。

容易情况精确地出现 $\binom{s+t-1}{s}-1$次；k=0的条件在步骤W5中恰好出现 $\binom{s+t-2}{t}+\delta_{t1}$ 次。奇怪，如果N有组合表示(57)，在算法L中阶N的组合有算法W中的阶 $N-t+\binom{n_v}{v-1}+v-1$。[*Lecture Notes in Comp. Sci.* 3595 (2005)，570-576。]

(b) SET bits $(1\ll t)-1$ (此程序假设s>0和t>0)

```
1H PUSHJ $0,Visit                      访问bits=(a_{s+t-1}…a_1 a_0)_2
   ADDU $0,bits,1; AND $0,$0,bits      置$0←bits&(bits+1)
   SUBU $1,$0,1; XOR $1,$0,$1          置$1←$0⊕($0−1)
   ADDU $0,$1,1; AND $1,$1,bits        置$0←$1+1, $1←$1&bits
   AND $0,$0,bits; ODIF $0,$0,1        置$0←($0&bits)−1
   SUBU $1,$1,$0; ADDU bits,bits,$1    置bits←bits+$1=$0
   SRU $0,bits,s+t; PBZ $0,1B          重复除非a_{s+t}=1  ▮
```

56. [*Discrete Math.* 48 (1984)，163-171。]这个问题等价于“中级猜测”。它指出，通过长度为$2t-1$和权为$\{t-1, t\}$的所有二进制串存在一条格雷通路。事实上，通过特殊形式$\alpha_0\,\alpha_1\cdots\alpha_{2t-2}$的一个$\delta$序列，几乎可以肯定，能生成这样的串。其中$\alpha_k$的元素是$\alpha_0$的那些元素移动$k$次modulo $2t-1$得到的。例如，当t=3时，我们可以以$a_5\,a_4\,a_3\,a_2\,a_1\,a_0$=000111开始，并且重复地交换$a_0\leftrightarrow a_\delta$，其中$\delta$跑遍循环(4134 5245 1351 2412 3523)。对于$t\le 15$，已知中等猜测为真。[参见伊·希尔德兹(I.Shields)和卡·戴·萨维吉，*Cong. Num.* 140 (1999)，161-178。]

57. 是的，当$n\ge m>t$时，对于所有m、n和t，存在一个近乎完美的广义词典顺序的解。在二进位串记号下，一个这样的方案，使用(35)的序列A_{st}，是$1\,A_{(m-t)(t-1)}\,0^{n-m}$, $01\,A_{(m-t)(t-1)}\,0^{n-m-1}$, $\cdots$, $0^{n-m}1\,A_{(m-t)(t-1)}$, $0^{n-m+1}\,1\,A_{(m-1-t)(t-1)}$, $\cdots$, $0^{n-t}1\,A_{0\,(t-1)}$。

58. 通过以$t-1$简化m和n，而后对于每个c_j加上$j-1$，解前边的问题。(情况(a)特别简单，它大概是由克泽尼(Czerny)发现的。)

59. 对于从$0^{n-t}1^t$在k步之内可达到的弦的数目g_{mntk}，生成函数$G_{mnt}(z)=\sum g_{mntk}z^k$，满足 $G_{mmt}(z)=\binom{m}{t}_z$ 和 $G_{m(n+1)t}(z)=G_{mnt}(z)+z^{m-(t-1)m}\binom{m-1}{t-1}_z$，因为后一项考虑使$c_t=n$和$c_1>n-m$的情况。一个完美方案仅当$|G_{mnt}(-1)|\le 1$时才是可能的。但如果$n\ge m>t\ge 2$时，由(49)，这个条件仅当$m=t+1$或$(n-t)t$为奇数时才成立。所以当$t$=4和$m$>5时没有完美的解。(许多弦在$n=t+2$时仅有两个邻居，所以人们能容易地排除这个情况。当$n$为偶数时，所有具有$n\ge m>5$和$t$=3的情况显然有完美的通路。)

60. 以下的解使用词典顺序，而且小心地确保每次访问的平均计算数量有界。我们可以假设$stm_s\cdots m_0\ne 0$和$t\le m_s+\cdots+m_1+m_0$。

Q1. [初始化。] 对于$s\ge j\ge 1$，置$q_j\leftarrow 0$和$x=t$。

Q2. [分配。] 置$j\leftarrow 0$。然后当$x>m_j$时，置$q_j\leftarrow m_j$，$x\leftarrow x-m_j$，$j\leftarrow j+1$，并重复直到$x\le m_j$为止。最后置$q_j\leftarrow x$。

Q3. [访问。] 访问有界合成$q_s+\cdots+q_1+q_0$。

Q4. [拣出最右的那些单位。] 如果j=0，则置$x\leftarrow q_0-1$，$j\leftarrow 1$。否则如果q_0=0，则置$x\leftarrow q_j-1$，$q_j\leftarrow 0$以及$j\leftarrow j+1$。否则转到Q7。

Q5. [满了吗？] 如果$j>s$，则结束。否则如果$q_j=m_j$，则置$x\leftarrow x+m_j$，$q_j\leftarrow 0$，$j\leftarrow$
98 $j+1$。并且重复这一步骤。

Q6. [增加q_j。] 置$q_j\leftarrow q_j+1$。然后如果$x=0$，则置$q_0\leftarrow 0$，并返回Q3。(在此情况下，$q_{j-1}=\cdots=q_0=0$)。否则转到Q2。

Q7. [增加和减少。] (现在对于$j>i\geq 0$，$q_i=m_i$。)当$q_j=m_j$时，置$j\leftarrow j+1$，并且重复直到$q_j<m_j$为止(但如果$j>s$时，则结束)。然后置$q_j\leftarrow q_j+1$，$j\leftarrow j-1$，$q_j\leftarrow q_j-1$。如果$q_0=0$，则置$j\leftarrow 1$。返回到Q3。 ▮

例如，如果$m_s=\cdots=m_0=9$，合成3+9+9+7+0+0的后继者为4+0+0+6+9+9，4+0+0+7+8+9, 4+0+0+7+9+8, 4+0+0+8+7+9，…。

61. 如果$t<0$或$t>m_s+\cdots+m_0$，令$F_s(t)=\phi$；否则令$F_0(t)=t$，且当$s>0$时

$$F_s(t)=0+F_{s-1}(t),1+F_{s-1}(t-1)^R,2+F_{s-1}(t-2),\cdots,m_s+F_{s-1}(t-m_s)^{R^{m_s}}$$

这个序列可被证明具有所要求的性质。事实上，在习题4的对应之下，当被限制为由界$m_s,\cdots,m_0$所定义的子序列时，它等价于(31)的同态序列K_{st}所定义的合成。[参见蒂·沃尔什，*J. Combinatorial Math. and Combinatorial Computing* 33 (2000), 323-345；他已经无循环地实现它。]

62. (a)一个有行和r和$c_1+\cdots+c_n-r$的$2\times n$的偶然性表等价于求解$r=a_1+\cdots+a_n$且满足$0\leq a_1\leq c_1,\cdots,0\leq a_n\leq c_n$。

(b) 我们通过对于$j=1,\cdots,n$和$i=1,\cdots,m$，设置$a_{ij}\leftarrow\min(r_i-a_{i1}-\cdots-a_{i(j-1)}, c_j-a_{1j}-\cdots-a_{(i-1)j})$，能顺序地计算它。或者，如果$r_1\leq c_1$置$a_{11}\leftarrow r_1$，$a_{12}\leftarrow\cdots\leftarrow a_{1n}\leftarrow 0$，而且对于剩下的行则以$r_1$减$c_1$；如果$r_1>c_1$，置$a_{11}\leftarrow c_1$，$a_{21}\leftarrow\cdots\leftarrow a_{m1}\leftarrow 0$，并且以$c_1$减$r_1$来做剩下的列。第二个方法表明，至多有$m+n-1$个元素为非零。我们也可以写下显式公式

$$a_{ij}=\max(0,\ \min(r_i, c_j, r_1+\cdots+r_i-c_1-\cdots-c_{j-1}, c_1+\cdots+c_j-r_1-\cdots-r_{i-1}))$$

(c) 如同在(b)中一样，得到相同的矩阵。

(d) 颠倒(b)和(c)中的左和右，在两种情况下答案是

$$a_{ij}=\max(0,\ \min(r_i, c_j, r_{i+1}+\cdots+r_m-c_1-\cdots-c_{j-1}, c_1+\cdots+c_j-r_i-\cdots-r_m))$$

(e) 这里我们选择比如说，行方式的顺序：如同对于r_1的有界合成那样，生成头一行，且有界$(c_1,\cdots,c_n)$；而对于每行$(a_{11},\cdots,a_{1n})$，以同样方式递归地生成剩下的行，但有列之和为$(c_1-a_{11},\cdots,c_n-a_{1n})$。大多数动作发生在底部两行上，但当对于较早的行进行改变时，后边的行必须重新初始化。

63. 如果a_{ij}和a_{kl}为正，通过设置$a_{ij}\leftarrow a_{ij}-1$，$a_{il}\leftarrow a_{il}+1$，$a_{kj}\leftarrow a_{kj}+1$，$a_{kl}\leftarrow a_{kl}-1$，我们得到另一个偶然性表。我们要来证明，其顶点为对于$(r_1,\cdots,r_m;\ c_1,\cdots,c_n)$的偶然性表的图$G$，如果通过这样一个变换，这些顶点彼此可被得到，则它们是相邻的，且这个图有一个哈密顿通路。

当$m=n=2$时，G是一个简单通路。当$m=2$和$n=3$时，G有一个二维结构，而从这个结构中我们看到，每一个顶点是至少两个哈密顿通路的起始点，而且有不同的

结束点。当$m=2$和$n\geqslant4$时，我们可以归纳地证明，从任何顶点到任何其他顶点，G实际上有哈密顿通路。

当$m\geqslant3$和$n\geqslant3$时，如同在62(e)的答案中那样，我们可以把这个问题从m简化为$m-1$，如果我们谨慎从事“不把自己置于窘境”的话。即是，我们必须避免达到这样一种状态，其中对于某个$a, b, c>0$，底部两行的非零元素有$\binom{1a0}{0bc}$的形式，而且对$m-2$行的一个变动迫使它成为$\binom{0a1}{0bc}$。对于行$m-1$和m的上一轮的变化可以避 99
免这样的陷阱，除非$c=1$和它以$\binom{0\,a{+}1\,0}{1\,b{-}1\,1}$或$\binom{1\,a{-}1\,1}{0\,b{+}1\,0}$开始。但这种状态也可加以避免。

(基于习题61的一个广义词典顺序方法将颇为简单，而且它几乎总是对于每步只做4个改变。但它有时一次要更新$2\min(m,n)$个元素。)

64. 当$x_1\cdots x_s$是一个二进制串，而且A是一个子立体表时，令$A\oplus x_1\ldots x_s$表示从左到右，通过$(a_1\oplus x_1, \ldots, a_s\oplus x_s)$来代替$A$的每个子立体中的数字$(a_1, \cdots, a_s)$。例如，$0*1**10\oplus1010=1*1**00$。则以下的相互递归定义一个格雷循环，因为当$st>0$时，$A_{st}$给出从$0^s*^t$到$10^{s-1}*^t$的一个格雷通路，而$B_{st}$给出$0^s*^t$到$*01^{s-1}*^{t-1}$的一条格雷通路：

$$A_{st}=0B_{(s-1)t},*A_{s(t-1)}\oplus001^{s-2},1B^R_{(s-1)t}$$
$$B_{st}=0A_{(s-1)t},1B_{(s-1)t}\oplus010^{s-2},*A_{s(t-1)}\oplus1^s$$

当$s<2$时，串001^{s-2}和010^{s-2}只不过是0^s；A_{s0}是格雷二进制码；$A_{0t}=B_{0t}=*^t$。(顺便指出，稍微更简单的结构

$$G_{st}=*G_{s(t-1)},a_tG_{(s-1)t},a_{t-1}G^R_{(s-1)t},\qquad a_t=t\bmod2$$

定义从$*^t0^s$到$a_{t-1}*^t0^{s-1}$的一个令人愉快的格雷通路。

65. 如果一条通路P被认为等价于P^R和$P\oplus x_1\cdots x_s$，则总数可以如同在习题33中那样，系统地加以计算，而且对于$s+t\leqslant6$，有下列结果：

通路	循环
1	1
1 1	1 1
1 2 1	1 1 1
1 3 3 1	1 1 1 1
1 5 10 4 1	1 2 1 1 1
1 6 36 35 5 1	1 2 3 1 1 1
1 9 310 4630 218 6 1	1 3 46 4 1 1 1

一般地说，当$s=1$时有$t+1$条通路，而当$t=1$时有$\binom{\lceil s/2\rceil+2}{2}-(s\bmod2)$条。对于$s\leqslant2$循环是惟一的。当$s=t=5$时近似地有$6.869\times10^{170}$条通路和$2.495\times10^{70}$个循环。

66. 当$n<t$时，令$G(n,0)=\varepsilon$，$G(n,t)=\phi$，而且对于$1\leqslant t\leqslant n$，令$G(n,t)$为

$$\hat{g}(0)G(n-1,t),\ \hat{g}(1)G(n-1,\ t)^R,\ldots,\ \hat{g}(2^t-1)G(n-1,t)^R,\hat{g}(2^t-1)G(n-1,t-1)$$

其中$\hat{g}(k)$是包含格雷二进制数$g(k)$且在其顶部有它最低有效位的一个t个二进位的

列。在这个一般的公式中，我们含蓄地加一行零于$G(n-1,t-1)$的基底下边。

当$t=1$时，这个值得注意的规则给出通常的格雷二进制码，并省略0…00。一个循环的格雷码是不可能的，因为$\binom{n}{t}_2$是奇数。

67. 对应于算法C的组成的一条格雷通路意味着有一条通路，其中所有转换是$0^k1^l\leftrightarrow 1^l0^k$且$\min(k,l)\le 2$。事实上，也许在每一转换中有一个循环且$\min(k,l)=1$。

68. (a) $\{\phi\}$; (b) ϕ。

69. 使得$\kappa_t N<N$的最小的N是$\binom{2t-1}{t}+\binom{2t-3}{t-1}+\cdots+\binom{1}{1}+1=\frac{1}{2}\left(\binom{2t}{t}+\binom{2t-2}{t-1}+\cdots+\binom{0}{0}+1\right)$，

100 因为$\binom{n}{t-1}<\binom{n}{t}$当且仅当$n\ge 2t-1$。

70. 当$N'<\binom{2t-3}{t}$时，从恒等式

$$\kappa_t\left(\binom{2t-3}{t}+N'\right)-\left(\binom{2t-3}{t}+N'\right)=\kappa_t\left(\binom{2t-2}{t}+N'\right)-\left(\binom{2t-2}{t}+N'\right)=\binom{2t-2}{t}\frac{1}{t-1}+\kappa_{t-1}N'-N'$$

我们得出结论，极大值是$\binom{2t-2}{t}\frac{1}{t}+\binom{2t-4}{t-1}\frac{1}{t-2}+\cdots+\binom{2}{2}\frac{1}{1}$，而且当$t>1$时它在$N$的$2^{t-1}$个值处出现。

71. 令C_t是t团簇，由算法L所访问的头$\binom{1414}{t}+\binom{1009}{t-1}$个$t$组合定义在具有1000000条边的1415个顶点上的图。如果$|C_t|$更大些，则$|\partial^{t-2}C_t|$将超过1000000。因此对于所有$t\ge 2$，由$P_{(1000000)2}$定义的单个图有t团簇的极大个数。

72. 对于$m_s>\cdots>m_u\ge u\ge 1$，$M=\binom{m_s}{s}+\cdots+\binom{m_u}{u}$，其中$\{m_s,\cdots,m_u\}=\{s+t-1,\cdots,n_v\}\setminus\{n_t,\cdots,n_{v+1}\}$。(同习题15作比较，它给出$\binom{s+t}{t}-1-N$ 。)

如果$\alpha=\alpha_{n-1}\cdots\alpha_0$是对应于组合$n_t\cdots n_1$的二进位串，则$v$是1加上$\alpha$中尾部1的个数，而且$u$是诸0的最右运程的长度。例如，当$\alpha=1010001111$时，我们有$s=4$，$t=6$；$M=\binom{8}{4}+\binom{7}{3}$，$u=3$，$N=\binom{9}{6}+\binom{7}{5}$，$v=5$。

73. A和B是交叉相交的⇔对于所有的$\alpha\in A$和$\beta\in B$，$\alpha\not\subseteq U\setminus\beta\Leftrightarrow A\cap\partial^{n-s-t}B^-=\phi$，其中$B^-=\{U\setminus\beta\mid\beta\in B\}$是$(n-t)$组合的一个集合。由于$Q_{Nnt}=P_{N(n-t)}$，我们有$|\partial^{n-s-t}B^-|\ge|\partial^{n-s-t}P_{N(n-t)}|$，而且$\partial^{n-s-t}P_{N(n-t)}=P_{N's}$，其中$N'=\kappa_{s+1}\ldots\kappa_{n-t}N$。因此如果$A$和$B$是交叉相交的，则我们有$M+N'\le|A|+|\partial^{n-s-t}B^-|\le\binom{n}{s}$，以及$Q_{Mns}\cap P_{N's}=\phi$。

反之如果$Q_{Mns}\cap P_{N's}\ne\phi$，我们有$\binom{n}{s}<M+N'\le|A|+|\partial^{n-s-t}B^-|$，所以$A$和$B$不可能是交叉相交的。

74. $|\varrho Q_{Nnt}|=\kappa_{n-t}N$ (见习题94)。还有，如同在(58)和(59)中那样论断，我们发现在这种特定情况下，$\varrho P_{N5}=(n-1)P_{N5}\cup\cdots\cup 10P_{N5}\cup\{543210,\ldots,987654\}$，而且一般说来，$|\varrho P_{Nt}|=(n+1-n_t)N+\binom{n_t+1}{t+1}$。

75. 恒等式$\binom{n+1}{k}=\binom{n}{k}+\binom{n-1}{k-1}+\cdots+\binom{n-k}{0}$，即等式1.2.6−(10)，如果$n_v>v$，则给出另

外一个表示，但(60)不受影响，因为我们有 $\binom{n+1}{k-1}=\binom{n}{k-1}+\binom{n-1}{k-2}+\cdots+\binom{n-k+1}{0}$。

76. 通过把 $\binom{v-1}{v-1}$ 加到(57)表示$N+1$；于是使用上道习题来推演$\kappa_t(N+1)-\kappa_t N=\binom{v-1}{v-2}=v-1$。

77. [戴 · 爱 · 戴金，*Nanta Math*. 8, 2 (1975)，78-83。]如同在习题75中那样，我们对扩充表示 $M=\binom{m_t}{t}+\cdots+\binom{m_u}{u}$ 和 $N=\binom{n_t}{t}+\cdots+\binom{n_v}{v}$ 进行工作，如果最后的下标u或v为零，称它们为*不适当的*。称N为*灵活的*——如果它同时有适当的和不适当的表示，即如果$n_v>v>0$。

(a) 给定一个整数S，求$M+N$使得$M+N=S$和$\kappa_t M+\kappa_t N$是极小的，且M尽可能地大。如果$N=0$，则我们完成了。否则极大-极小操作保持$M+N$和$\kappa_t M+\kappa_t N$两者。所以我们可假设，在M和N的适当表示中$v\geqslant u\geqslant 1$。如果N是不灵活的，则由习题76，$\kappa_t(M+1)+\kappa_t(N-1)=(\kappa_t M+u-1)+(\kappa_t N-v)<\kappa_t M+\kappa_t N$；因此$N$必须是灵活的。但那样我们可以应用极大-极小操作到$M$以及$N$的不适当表示上，并增加$M$；矛盾。

这个证明表明，等式成立当且仅当$MN=0$时，这个事实首先由弗 · 索 · 麦考莱于1927年指出。

(b) 当$M+N=S$时，我们尝试来极小化$\max(\kappa_t M, N)+\kappa_{t-1}N$，这次把$N$表示为 $\binom{n_{t-1}}{t-1}+\cdots+\binom{n_v}{v}$。如果$n_{t-1}<m_t$，极大-极小操作仍可合用；保持$m_t$不变，它保持$M+N$和$\kappa_t M+\kappa_{t-1}N$以及$\kappa_t M>N$的关系。如同在(a)中一样，如果$N\neq 0$，我们导致矛盾，所以可以假设$n_{t-1}\geqslant m_t$。 101

如果$n_{t-1}>m_t$，我们有$N>\kappa_t M$而且还有$\lambda_t N>M$；因此$M+N<\lambda_t N+N=\binom{n_{t-1}+1}{t}+\cdots+\binom{n_v+1}{v}$，而且我们有$\kappa_t(M+N)\leqslant\kappa_t(\lambda_t N+N)=N+\kappa_{t-1}N$。

最后，如果$n_{t-1}=m_t=a$，令 $M=\binom{a}{t}+M'$ 和 $N=\binom{a}{t-1}+N'$。于是 $\kappa_t(M+N)=\binom{a+1}{t-1}+\kappa_{t-1}(M'+N')$，$\kappa_t M=\binom{a}{t-1}+\kappa_{t-1}M'$，而且 $\kappa_{t-1}N=\binom{a}{t-2}+\kappa_{t-2}N'$；对$t$用归纳法可得出这一结果。

78. [尤 · 埃克霍夫(J. Eckhoff)和杰 · 威格纳(G. Wegner)，*Periodica Math. Hung*. 6 (1975)，137－142；安 · 约 · 威 · 希尔顿，*Periodica Math. Hung*. 10 (1979)，25-30。]令$M=|A_1|$和$N=|A_0|$；我们可以假设$t>0$和$N>0$。然后通过对$m+n+t$用归纳法，我们得到$|\partial A|=|\partial A_1\cup A_0|+|\partial A_0|\geqslant\max(|\partial A_1|,|A_0|)+|\partial A_0|\geqslant\max(\kappa_t M, N)+\kappa_{t-1}N\geqslant\kappa_t(M+N)=|P_{|A|t}|$。

反之，令$A_1=P_{Mt}+1$和$A_0=P_{N(t-1)}+1$；这个记号意味着，例如{210, 320}+1={321, 431}。然后$\kappa_t(M+N)\leqslant|\partial A|=|\partial A_1\cup A_0|+|(\partial A_0)0|=\max(\kappa_t M, N)+\kappa_{t-1}N$，因为$\partial A_1=P_{(\kappa_t M)(t-1)}+1$。[舒曾伯格于1959年发现，$\kappa_t(M+N)\leqslant\kappa_t M+\kappa_{t-1}N$当且仅当$\kappa_t M\geqslant N$。]

对于第一个不等式，令A和B是t组合的不相交集合且$|A|=M$，$|\partial A|=\kappa_t M$，$|B|=N$，$|\partial B|=\kappa_t N$，于是$\kappa_t(M+N)=\kappa_t|A\cup B|\leqslant|\partial(A\cup B)|=|\partial A\cup\partial B|=|\partial A|+|\partial B|=\kappa_t M+\kappa_t N$。

79. 事实上，当N由(57)给出时，$\mu_t\,(M+\lambda_{t-1}M)=M$且$\mu_t\,N+\lambda_{t-1}\mu_t\,N=N+(n_2-n_1)$ $[v=1]$。

80. 如果$N>0$且$t>1$，如同在(57)中那样表示N并令$N=N_0+N_1$，其中

$$N_0=\binom{n_t-1}{t}+\cdots+\binom{n_v-1}{v},\qquad N_1=\binom{n_t-1}{t-1}+\cdots+\binom{n_v-1}{v-1}$$

令$N_0=\binom{y}{t}$和$N_1=\binom{z}{t-1}$，然后通过对t和$\lfloor x\rfloor$用归纳法，我们有$\binom{x}{t}=N_0+\kappa_t N_0\geqslant\binom{y}{t}+\binom{y}{t-1}=\binom{y+1}{t}$；$N_1=\binom{x}{t}-\binom{y}{t}\geqslant\binom{x}{t}-\binom{x-1}{t}=\binom{x-1}{t-1}$，而且$\kappa_t\,N=N_1+\kappa_{t-1}N_1\geqslant\binom{z}{t-1}+\binom{z}{t-2}=\binom{z+1}{t-1}\geqslant\binom{x}{t-1}$。

[罗瓦斯泽实际上证明了一个更强的结果；参见习题1.2.6-66。类似地，我们有$\mu_t N\geqslant\binom{x-1}{t-1}$；参见布约纳(Björner)、弗朗克尔和斯坦利，*Combinatorica* 7 (1987)，27-28。]

81. 例如，如果$\hat{P}_{N5}$最大的元素是66433，我们有

$$\hat{P}_{N5}=\{00000,\cdots,55555\}\cup\{60000,\cdots,65555\}\cup\{66000,\cdots,66333\}\cup\{66400,\cdots,66433\}$$

所以$N=\binom{10}{5}+\binom{9}{4}+\binom{6}{3}+\binom{5}{2}$。它的下阴影是大小为$\binom{9}{4}+\binom{8}{3}+\binom{5}{2}+\binom{4}{1}$的

$$\partial\hat{P}_{N5}=\{0000,\cdots,5555\}\cup\{6000,\cdots,6555\}\cup\{6600,\cdots,6633\}\cup\{6640,\cdots,6643\}$$

如果Q_{N95}的最小元素为66433，我们有

$$\hat{Q}_{N95}=\{99999,\cdots,70000\}\cup\{66666,\cdots,66500\}\cup\{66444,\cdots,66440\}\cup\{66433\}$$

所以$N=\left(\binom{13}{9}+\binom{12}{8}+\binom{11}{7}\right)+\left(\binom{8}{6}+\binom{7}{5}\right)+\binom{5}{4}+\binom{3}{3}$。它的上阴影是大小为$\left(\binom{14}{9}+\binom{13}{8}+\binom{12}{7}\right)+\left(\binom{9}{6}+\binom{8}{5}\right)+\binom{6}{4}+\binom{4}{3}=N+\kappa_9 N$的$\varrho\,\hat{Q}_{N95}=\{999999,\cdots,700000\}\cup\{666666,\cdots,665000\}\cup\{664444,\cdots,664400\}\cup\{664333,\cdots,664330\}$。只要$N<\binom{s+t}{t}$，每个组合的大小$t$实质上是没有关系的；例如，在我们已经考虑的情况下$Q_{N98}$的最小
102 元素为99966433。

82. (a)这个推导将要是$\sum_{k\geq0}r_k(x)$，但该级数发散。

[非正式地说，$\tau(x)$的图示出所有2^{-k}的奇数倍时相对量2^{-k}的“坑”。高木贞治发表在(*Proc. Physico-Math. Soc. Japan*(2) 1 (1903)，176-177)的开创性论文已被翻译成英文，收录在他的论文集中(Iwanami Shoten，1973)中。]

(b) 由于当$k>0$时$r_k(1-t)=(-1)^{\lceil 2^k t\rceil}$，我们有$\int_0^{1-x}r_k(t)\mathrm{d}t=\int_x^1 r_k(1-u)\mathrm{d}u=-\int_x^1 r_k(u)\mathrm{d}u=\int_0^x r_k(d)\mathrm{d}u$。第二个等式从$r_k\left(\frac{1}{2}t\right)=r_{k-1}(t)$这一事实得出。部分(d)说明，当$x$为有理数时，这两个等式足以定义$\tau(x)$。

(c) 由于对于$0<x\leq 1$，$\tau(2^{-a}x)=a2^{-a}x+2^{-a}\tau(x)$，当$2^{-a-1}\leq\varepsilon\leq 2^{-a}$时，我们有$\tau(\varepsilon)=a\varepsilon+O(\varepsilon)$。因此对于$0<\varepsilon\leq 1$，$\tau(\varepsilon)=\varepsilon\lg\frac{1}{\varepsilon}+O(\varepsilon)$。

(d) 假设$0\leq p/q\leq 1$。如果$p/q\leq 1/2$，我们有$\tau(p/q)=p/q+\tau(2p/q)/2$；否则$\tau(p/q)=(q-p)/q+\tau(2(q-p)/q)/2$。因此我们可以假设$q$为奇数。当$q$为奇数时，当$p$为偶数时，令$p'=p/2$；而当$p$为奇数时，令$p'=(q-p)/2$。则对于$0<p<q$，$\tau(p/q)=2\tau(p'/q)-2p'/q$；这$q-1$个方程的方程组有惟一解。例如，对于$q=3, 4, 5, 6, 7$的值是2/3，2/3；1/2，1/2，1/2；8/15，2/3，2/3，8/15；1/2，2/3，1/2，2/3，1/2；22/49，30/49，32/49，32/49，30/49，22/49。

(e) $<\frac{1}{2}$的解为$x=\frac{1}{4},\frac{1}{4}-\frac{1}{16},\frac{1}{4}-\frac{1}{16}-\frac{1}{64},\frac{1}{4}-\frac{1}{16}-\frac{1}{64}-\frac{1}{256},\cdots,\frac{1}{6}$

(f) 对于$x=\frac{1}{2}\pm\frac{1}{8}\pm\frac{1}{32}\pm\frac{1}{128}\pm\cdots$，一个不可数集合，值$\frac{2}{3}$可实现。

83. 给定任何整数$q>p>0$，考虑在有向图

$$\begin{array}{ccccccccccccc}
0 & \leftarrow & 1 & \leftarrow & 2 & \leftarrow & 3 & \leftarrow & 4 & \leftarrow & 5 & \leftarrow & \cdots \\
\updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & & \\
1 & \rightarrow & 2 & \rightarrow & 3 & \rightarrow & 4 & \rightarrow & 5 & \rightarrow & 6 & \rightarrow & \cdots
\end{array}$$

中从0开始的通路，以$v\leftarrow -p$开始，计算相关的值v；水平移动改变$v\leftarrow 2v$；从节点a的垂直移动改变$v\leftarrow 2(qa-v)$。如果我们以相同的v值两次到达一个节点，则这条通路停止。如果在该节点$v<-q$或$v\geq qa$，则对上节点a不允许进行转换。对于$v<0$或$v\geq q(a+1)$，则不允许对下节点a进行转换。这些限制强迫这条通路的大多数步骤。(在上一行的节点a意味着“解$\tau(x)=ax-v/q$”；在下一行意味着“解$\tau(x)=v/q-ax$”。)经验测试提议，所有这样的通路是有限的。于是，方程$\tau(x)=p/q$有由序列x_0, x_1, $x_2\cdots$定义的解$x=x_0$，其中在一个水平步骤中$x_k=\frac{1}{2}x_{k+1}$，而在一个垂直步骤中$x_k=1-\frac{1}{2}x_{k+1}$；终于，对于某个$j<k$，$x_k=x_j$。如果$j>0$，而且如果q不是2的一个次幂，则当$x>1/2$时这些全都是$\tau(x)=p/q$的解。

例如，这个过程确定，仅当x是83581/87040时，$\tau(x)=1/5$且$x>1/2$。这惟一的通路产生$x_0=1-\frac{1}{2}x_1, x_1=\frac{1}{2}x_2,\cdots,x_{18}=\frac{1}{2}x_{19}$，而且$x_{19}=x_{11}$。类似地，恰有两个值$x>1/2$而且$\tau(x)=3/5$，并有分母$2^{46}(2^{56}-1)/3$。

而且，看起来在通过节点0的有向图中的所有循环定义p和q的值，使得$\tau(x)=p/q$有不可数多的解。例如，对应于循环(01)，(0121)，(012321)，这样的值是2/3，8/15，8/21。值32/63对应于(012121)而且也对应于(0121012345454321)，以及不返回到0的两个其他通路。

84. [弗朗克尔、松本真、拉泽萨以及德重典英，*J. Combinatorial Theory* A69 (1995)，125-148。]如果$a\leq b$，则有

$$\binom{2t-1-b}{t-a}\Big/T=t^{\underline{a}}(t-1)^{\underline{b-a}}/(2t-1)^{\underline{b}}=2^{-b}(1+f(a,b)t^{-1}+O(b^4/t^2))$$ 103

其中，$f(a,b)=a(1+b)-a^2-b(1+b)/4=f(a+1,b)-b+2a$。因此如果$N$有组合表示(57)，而且如果我们设置$n_j=2t-1-b_j$，则有

$$\frac{t}{T}\left(\kappa_t N-N\right)=\frac{b_t}{2^{b_t}}+\frac{b_{t-1}-2}{2^{b_{t-1}}}+\frac{b_{t-2}-4}{2^{b_{t-2}}}+\cdots+\frac{O(\log t)^3}{t}$$

当b_j超过$2\lg t$时，这些项是可忽略的。因此人们可以证明

$$\tau\left(\sum_{j=0}^{l}2^{-e_j}\right)=\sum_{j=0}^{l}(e_j-2j)2^{-e_j}$$

85. 由(63)，$N-\lambda_{t-1}N$有和$\kappa_t N-N$相同的渐近形式，因为$\tau(x)=\tau(1-x)$，$2\mu_t N-N$也一样，直到$O(T(\log t)^3/t^2)$。因为当$b<2\lg t$时，$\binom{2t-1-b}{t-a}=2\binom{2t-2-b}{t-a}(1+O(\log t)/t)$。

86. $x\in X^{\circ\sim}\Leftrightarrow\bar{x}\notin X^{\circ}\Leftrightarrow\bar{x}\notin X$或$\bar{x}\notin X+e_1$或…或$\bar{x}\notin X+e_n\Leftrightarrow x\in X^{\sim}$或 $x\in X^{\sim}-e_1$或…或$x\in X^{\sim}-e_n\Leftrightarrow x\in X^{\sim+}$。

87. 利用 $X\subseteq Y^{\circ}$的事实，所有三者都为真，当且仅当 $X^{+}\subseteq Y$：(a) $X\subseteq Y^{\circ}\Leftrightarrow X^{\sim}\supseteq Y^{\circ\sim}=Y^{\sim+}\Leftrightarrow Y^{\sim}\subseteq X^{\sim\circ}$；(b) $X^{+}\subseteq X^{+}\Rightarrow X\subseteq X^{+\circ}$；因此$X^{\circ}\subseteq X^{\circ+\circ}$。而且 $X^{\circ}\subseteq X^{\circ}\Rightarrow X^{\circ+}\subseteq X$；因此 $X^{\circ+\circ}\subseteq X^{\circ}$。(c) $\alpha M<N\Leftrightarrow S_M^{+}\subseteq S_N\Leftrightarrow S_M\subseteq S_N^{\cdot}\Leftrightarrow M\leqslant\beta N$。

88. 如果$\nu x<\nu y$，则$\nu(x-e_k)<\nu(y-e_j)$，所以我们可以假定$\nu x=\nu y$而且在词典顺序下$x>y$。我们必定有$y_j>0$，否则$\nu(y-e_j)$将超过$\nu(x-e_k)$。如果对于$1\leqslant i\leqslant j$，$x_i=y_i$，显然$k>j$，而且$x-e_k\prec y-e_j$。否则对某个$i\leqslant j$，$x_i>y_i$。再次，除非$x-e_k=y-e_j$，否则我们有$x-e_k\prec y-e_j$。

89. 从下表

$j=0$	1	2	3	4	5	6	7	8	9	10	11
$e_j+e_1=e_1$	e_0	e_4	e_5	e_2	e_3	e_8	e_9	e_6	e_7	e_{11}	e_{10}
$e_j+e_2=e_2$	e_4	e_0	e_6	e_1	e_8	e_3	e_{10}	e_5	e_{11}	e_7	e_9
$e_j+e_3=e_3$	e_5	e_6	e_7	e_8	e_9	e_{10}	e_0	e_{11}	e_1	e_2	e_4

我们求得$(\alpha 0,\alpha 1,\cdots,\alpha 12)=(0,4,6,7,8,9,10,11,11,12,12,12,12)$，$(\beta 0,\beta 1,\cdots,\beta 12)=(0,0,0,0,1,1,2,3,4,5,6,8,12)$。

90. 令$Y=X^{+}$和$Z=C_kX$，而且对于$0\leqslant a<m_k$，令$N_a=|X_k(a)|$，则

$$|Y|=\sum_{a=0}^{m_k-1}|Y_k(a)|=\sum_{a=0}^{m_k-1}|(X_k(a-1)+e_k)\cup(X_k(a)+E_k(0))|$$
$$\geqslant\sum_{a=0}^{m_k-1}\max(N_{a-1},\alpha N_a)$$

其中$a-1$代表$(a-1)\bmod m_k$而且α函数来自于$(n-1)$维圆环体，因为由归纳法$|X_k(a)+E_k(0)|\geqslant\alpha N_a$。而且

$$|Z^+| = \sum_{a=0}^{m_k-1} |Z_k^+(a)| = \sum_{a=0}^{m_k-1} |(Z_k(a-1)+e_k)\cup(Z_k(a)+E_k(0))|$$
$$= \sum_{a=0}^{m_k-1} \max(N_{a-1}, \alpha N_a)$$

因为$Z_k(a-1)+e_k$和$Z_k(a)+E_k(0)$两者在$n-1$维中都是标准的。 104

91. 令在一个全压缩数组的行a中有N_a点，其中行0在底部；于是$l=N_{-1}\geqslant N_0\geqslant\cdots\geqslant N_{m-1}\geqslant N_m=0$。我们首先证明，存在一个最优的$X$，对于它而言，除非当$N_a=0$或$N_a=l$时，不然"坏"条件$N_a=N_{a+1}$绝不出现。因为如果$a$是最小的坏下标，假设$N_{a-1}>N_a=N_{a+1}=\cdots=N_{a+k}>N_{a+k+1}$。于是我们总能对$N_{a+k}$减1和加1到某个满足$b\leqslant a$的$N_b$，而不增加$|X^+|$，除非在$k=1$和$N_{a+2}=N_{a+1}-1$以及对于$0\leqslant b\leqslant a$，$N_b=N_a+a-b<l$的情况。进一步剖析这样的情况，如果对于某个$c>a+1$，$N_{c+1}<N_c=N_{c-1}$，我们可以置$N_c\leftarrow N_c-1$和$N_a\leftarrow N_a+1$。由此或者减少$a$或者增加$N_0$。否则我们可以求出一个下标$d$使得对于$a<c<d$，$N_c=N_{a+1}+a+1-c>0$，以及或者$N_d=0$或者$N_d<N_{d-1}-1$。那么对于$a<c<d$，对$N_c$减1就行了，而随后对于$0\leqslant b<d-a-1$，对$N_b$加1。(重要的是要注意到，如果$N_d=0$，我们有$N_0\geqslant d-1$，因此$d=m$意味着$l=m$。)

每当$N_a\neq l$和$N_{a+1}\neq 0$时，重复这样的转换直到$N_a>N_{a+1}$为止。我们达到(86)的情况，如同在正文中那样，这个证明即可完成。

92. 令$x+k$表示超过x和有权$\nu x+k$的$T(m_1,\cdots,m_{n-1})$元素按词典顺序之最小者——如果任何这样的元素存在的话。例如，如果$m_1=m_2=m_3=4$和$x=211$，我们有$x+1=212$，$x+2=213$，$x+3=223$，$x+4=233$，$x+5=333$且$x+6$不存在；一般说来，$x+k+1$是由$x+k$通过加上可予增加的最右分量得到的。如果$x+k=(m_1-1,\cdots,m_{n-1}-1)$，我们置$x+k+1=x+k$。然后如果$S(k)$是$\preceq x+k$的$T(m_1,\cdots,m_{n-1})$所有元素之集合，我们有$S(k+1)=S(k)^+$。而且，在$a$中结尾的$S$的元素是其头$n-1$个分量在$S(m-1-a)$中的那些。

本题的结果可以更直观地表述如下：当我们生成n维标准集合$S_1, S_2,\cdots$时，在每层上的$(n-1)$维标准集合在每个点被加到层$m-1$后，就变成彼此分散开来了。类似地，在每个点被加到0层之前，它们变成彼此的核。

93. (a)假设当适当地排序时，参数是$2\leqslant m_1'\leqslant m_2'\leqslant\cdots\leqslant m_n'$，并令$k$是满足$m_k\neq m_k'$的极小值。然后取$N=1+\text{rank}(0,\cdots,0,m_k'-1,0,\cdots,0)$。(我们必须假设$\min(m_1,\cdots,m_n)\geqslant 2$，因为等于1的参数可放置于任何处。)

(b) 仅仅对于$n=2$的情况，掩盖在习题91的答案里边。当n为更大的值时，这个证明通过归纳法而被加入。

94. 求补把词典顺序颠倒过来，而且把ϱ改变成∂。

95. 对于定理K，令$d=n-1$和$s_0=\cdots=s_d=1$。对于定理M，令$d=s$和$s_0=\cdots=s_d=t+1$。

96. 在这样的表示下，N是居于词典顺序下$n_t n_{t-1}\cdots n_1$之前的$\{s_0\cdot 0, s_1\cdot 1, s_2\cdot 2,\cdots\}$的$t$组合的个数。因为广义系数$\binom{S(n)}{t}$计算其最左分量为$<n$的多重组合。

如果我们通过在最右边的非零项$\binom{S(n_v)}{v}$停止而截断这个表示，我们就得到(60)

的一个漂亮的推广。

$$|\partial P_{Nt}|=\binom{S(n_t)}{t-1}+\binom{S(n_{t-1})}{t-2}+\cdots+\binom{S(n_v)}{v-1}$$

[参见乔·弗·克莱门兹(*J. Combinatorial Theory* A37 (1984), 91－97)。对于验证推论C的正确性，需要不等式$s_0 \geqslant s_1 \geqslant \cdots \geqslant s_d$，但对于$|\partial P_{Nt}|$的计算则不必要。对于$t \geqslant k > v$，某些项$\binom{S(n_k)}{k}$可能为零。例如当$N=1$时，$t=4$，$s_0=3$且$s_1=2$，我们

105 有$N=\binom{S(1)}{4}+\binom{S(1)}{3}=0+1$。]

97. (a)这个四面体有四个顶点、六个边、四个面：$(N_0, \cdots, N_4)=(1, 4, 6, 4, 1)$。类似地，八面体有$(N_0, \cdots, N_6)=(1, 6, 8, 8, 0, 0, 0)$，而二十面体有$(N_0, \cdots, N_{12})=(1, 12, 30, 20, 0, \cdots, 0)$。十六面体，也称作3立体，有8个顶点、12个边和6平方面。振动把每个平方面分成两个三角形并且引进新的边。所以我们有$(N_0, \cdots, N_8)=(1, 8, 18, 12, 0, \cdots, 0)$。最后，十二面体的振动的五面体的面导致$(N_0, \cdots, N_{20})=(1, 20, 54, 36, 0, \cdots, 0)$。

(b) $\{210, 310\} \cup \{10, 20, 21, 30, 31\} \cup \{0, 1, 2, 3\} \cup \{\varepsilon\}$

(c) 对于$0 \leqslant t \leqslant n$，$0 \leqslant N_t \leqslant \binom{n}{t}$，和对于$1 \leqslant t \leqslant n$，$N_{t-1} \geqslant \kappa_t N_t$。如果我们定义$\lambda_0 1=\infty$的话，对于$1 \leqslant t \leqslant n$，第二个条件等价于$\lambda_{t-1} N_{t-1} \geqslant N_t$。这些条件对于定理K来说是必要的。而如果$A=\cup P_{Nt'}$，且是充分的。

(d) 不在一个简单的复合体中的元素的补，即集合$\{\{0, \cdots, n-1\}\backslash\alpha \mid \alpha \notin C\}$形成一个简单的复合体。(我们也可验证必要充分条件成立：$N_{t-1} \geqslant \kappa_t N_t \Leftrightarrow \lambda_{t-1} N_{t-1} \geqslant N_t \Leftrightarrow \kappa_{n-t+1} \overline{N}_{n-t+1} \leqslant \overline{N}_{n-t}$，因为由习题94，$\kappa_{n-t} \overline{N}_{n-t+1}=\binom{n}{t}-\lambda_{t-1} N_{t-1}$。

(e) 00000↔14641; 10000↔14640; 11000↔14630; 12000↔14620; 13000↔14610; 14000↔14600; 12100↔14520; 13100↔14510; 14100↔14500; 13200↔14410; 14200↔14400; 13300↔14400; 以及自对偶的情况14300, 13310。

98. 以下由斯·莱纳森(S. Linusson)给出的过程[*Combinatorica* 19 (1999), 255-266]要比更明显的一个方法快得多，他也考虑多重集合更一般的问题。令$L(n, h, l)$统计对于$0 \leqslant t \leqslant l$，满足$N_t=\binom{n}{t}$和对于$t>h$，$N_{t+1}<\binom{n}{t+1}$且$N_t=0$的能行向量个数。那么，除非$-1 \leqslant l \leqslant h \leqslant n$，否则有$L(n, h, l)=0$；以及$L(n, h, h)=L(n, h, -1)=1$，对于$l<n$，$L(n, n, l)=L(n, n-1, l)$。当$n>h \geqslant l \geqslant 0$时，我们可以计算$L(n, h, l)=\sum_{j=l}^{h} L(n-1, h, j) L(n-1, j-1, l-1)$，这是由定理K得出的一个递归。(每一大小向量对应于复合体$\cup P_{Nt'}$，而且$L(n-1, h, j)$表示不含极大元素$n-1$的组合，而$L(n-1, j-1, l-1)$表示那些含极大元素的组合。)最后，总的和是$L(n)=\sum_{l=-1}^{n} L(n, n, l)$。

我们有$L(0), L(1), L(2), \cdots=2, 3, 5, 10, 26, 96, 553, 5461, 100709, 3718354,$

289725509, …; $L(100) \approx 3.2299 \times 10^{1842}$。

99. 一个简单的复合体的极大元素形成一个丛组；反之，包含在一个丛组的元素中的组合形成一个简单的复合体。因此两个概念实质上是等价的。

(a) 如果$(M_0, M_1, \cdots, M_n)$是一个丛组的大小向量，则如果$N_n=M_n$和对于$0 \le t<n$，$N_t = M_t + \kappa_{t+1} N_{t+1}$，$(N_0, N_1, \cdots, N_n)$是一个简单复合体的大小向量。反之，如果我们用词典顺序下的最初的N_t个t组合，则每一个这样的$(N_0, \cdots, N_n)$生成一个$(M_0, \cdots, M_n)$。[乔·弗·克莱门兹在(*Discrete Math.* 4 (1973). 123-128)中把这个结果推广到一般的多重集合上。]

(b) 在答案97(e)的顺序下，它们是00000，00001，10000，00040，01000，00030，02000，00120，03000，00310，04000，00600，00100，00020，01100，00210，02100，00500，00200，00110，01200，00400，00300，01010，01300，00010。注意$(M_0, \cdots, M_n)$是能行的当且仅当$(M_n, \cdots, M_0)$是能行的，所以在这个表示中我们有不同类型的对偶性。

100. 如同在推论C的证明中那样，把A表示为$T(m_1, \cdots, m_n)$的一个子集。那么当A由N个词典顺序下最小的点$x_1 \cdots x_n$组成时，就得到νA的极大值。

这个证明由归结为A被压缩的情况开始。所谓压缩其意义是，它的t多重组合是对于每个t，为$P_{|A \cap T_t|t}$。于是，如果y是$\in A$的最大元素，而且如果x是$\notin A$的最小元素，则我们证明$x<y$意味着$\nu x>\nu y$，因此$\nu(A \setminus \{y\} \cup \{x\})>\nu A$。因为如果$\nu x=\nu y-k$，我们可以求得$\partial^k y$的一个元素，它大于$x$，而这同$A$是被压缩的假设矛盾。 106

101. (a)一般说来，当$f(x_1, \cdots, x_n)$被恰好N_t个权为t的二进制串$x_1 \cdots x_n$所满足时，$F(p)=N_0 p^n+N_1 p^{n-1}(1-p)+\cdots+N_n(1-p)^n$。因此我们求得$G(p)=p^4+3p^3(1-p)+p^2(1-p)^2$；$H(p)=p^4+p^3(1-p)+p^2(1-p)^2$。

(b) 在对应$f(x_1, \cdots, x_n)=1 \Leftrightarrow \{j-1 | x_j=0\} \in C$之下，一个单调公式$f$等价于简单复合体$C$。因此单调布尔函数的函数$f(p)$是满足习题97(c)的条件的那些函数，因而通过选择在词典顺序下最后的N_{n-t}个t组合(它们是头N_s个s组合的补)，我们得到一个适当的函数：{3210}，{321, 320, 310}，{32} 给出$f(w, x, y, z)=wxyz \vee xyz \vee wyz \vee wxz \vee yz=wxz \vee yz$。

马·鲍·舒曾伯格发现，通过注意到$f(1/(1+u))=(N_0+N_1u+\cdots+N_n u^n)/(1+u)^n$，我们能容易地从$f(p)$求出参数$N_t$。人们可以证明，$H(p)$不等价于有任何个数变量的一个单调函数等价，因为$(1+u+u^2)/(1+u)^4=(N_0+N_1u+\cdots+N_n u^n)/(1+u)^n$意味着$N_1=n-3$，$N_2=\binom{n-3}{2}+1$，以及$\kappa_2 N_2=n-2$。

但一般来说，判定这个问题的任务并非那么简单。例如，函数$(1+5u+5u^2+5u^3)/(1+u)^5$同5个变量的任何单调函数都不匹配，因为$\kappa_3 5=7$；但它等于$(1+6u+10u^2+10u^3+5u^4)/(1+u)^6$。对于6个变量，它工作得很好。

102. (a)在I中，选择度数为t的N_t个线性无关多项式；按词典顺序来对它们的项进行排序，并且取它们的线性组合，使得按词典顺序最小的项是不同的单调项。令I'由所有这些单调项的倍数组成。

(b)在I'中，度数为t的每个单调项实质上是一个t多重组合。例如$x_1^3x_2x_5^4$对应于55552111。如果M_t是对于次数t的独立单调项的集合，则理想的性质等价于说$M_{t+1} \supseteq \varrho M_t$。

在给定的例子中，$M_3=\{x_0x_1^2\}$; $M_4=\varrho M_3 \cup \{x_0x_1x_2^2\}$；$M_5=\varrho M_4 \cup \{x_1x_2^4\}$，因为$x_2^2(x_0x_1^2-2x_1x_2^2)-x_1(x_0x_1x_2^2)=-2x_1x_2^4$，而且此后$M_{t+1}=\varrho M_t$。

(c) 由定理M，我们可假设$M_t=\hat{Q}_{Mst}$。令$N_t=\binom{n_{ts}}{s}+\cdots+\binom{n_{t2}}{2}+\binom{n_{t1}}{1}$，其中$s+t \geqslant n_{ts}>\cdots>n_{t2}>n_{t1}\geqslant 0$。于是$n_{ts}=s+t$当且仅当$n_{t(s-1)}=s-2$，…，$n_{t1}=0$，而且我们有

$$N_{t+1}\geqslant N_t+\kappa_s N_t=\binom{n_{ts}+[n_{ts}\geqslant s]}{s}+\cdots+\binom{n_{t2}+[n_{t2}\geqslant 2]}{2}+\binom{n_{t1}+[n_{t1}\geqslant 1]}{1}$$

因此当t增加时序列$(n_{ts}-t-\infty[n_{ts}<s],\cdots,n_{t2}-t-\infty[n_{t2}<2],n_{t1}-t-\infty[n_{t1}<1])$是词典顺序下非递减的，其中我们在有$n_{tj}=j-1$的分量中插入‘$-\infty$’。由习题1.2.1－15(d)，这样一个序列不可能无穷多次地增加而不超过极大值$(s,-\infty,\cdots,-\infty)$。

103. 令P_{Nst}是用如下方式确定的一个序列的头N个元素：对于在词典顺序下的每个二进制串$x=x_{s+t-1}\cdots x_0$，通过在词典顺序之下(认为1<*)，以所有可能的方式来把t个1改变成若干个*，写下$\binom{\nu x}{t}$个子立体。例如，如果x=0101101和t=2，我们生成子立体0101*0*，010*10*，010**01, 0*0110*，0*01*01, 0*0*101。

[参见本·林德斯特洛姆(*Arkiv för Mat.* 8 (1971)，245-257)；类似于推论C的一个推广见康·恩吉尔(K. Engel)，*Sperner Theory* (Cambridge Univ. Press, 1997)，

107 定理8.1.1。]

104. 在交叉顺序下的头N个串有所期望的性质。[特·伍·丹(T.N.Danh)和戴·爱·戴金，*J. London Math. Soc.* (2)55 (1997)，417－426。]

注：通过发现，权为t的N个词典顺序的开头的串的“1阴影”，(即只通过删去一个二进位得到的串)，由权数为t的开头μ_t N个串组成。鲁·阿尔斯威德(R. Ahlswede)和蔡宁把丹－戴金定理推广以允许二进位的插入、删去和/或传输[*Combinatorica* 17 (1997)，11-29; *Applied Math. Letters* 11, 5 (1998)，121－126]。尤维·里克(Uwe Leck)已经证明，不存在具有类似的极小阴影性质的三进制串的全序[预印件，98/6(Univ. Rostock，1998)，6页]。

105. 在循环中每个数出现同样次数。等价地说，$\binom{n-1}{t-1}$必须是t的一个倍数。假定相对于t，n不太大，则这个必要条件看来也是充分的。这样一个结果也可能是真的，但却不能证明。[参见钟金芳蓉、格拉罕(Graham)、葛立恒以及迪亚科尼斯(Diaconis)，*Discrete Math.* 110(1992)，55－57。]

以下一些习题考虑t=2和t=3的情况，对于它们已知一些优美的结果。对于t=4和t=5，已经推导出类似的但更复杂的结果，而对于t=6的情况部分地解决了。对于(n,t)=(12, 6)的情况，是当前未知万有循环存在性的最小者。

106. 令mod (2m+1)的差是1, 2, …, m, 1, 2, …, m, …, 重复2m+1次。例如，对于

$m=3$，循环是(013602561450346235124)。这有效，因为$1+\cdots+m=\binom{m+1}{2}$是同$2m+1$互质的。[*J. École Polytechnique* 4，Cahier 10 (1810)，16-48。]

107. 7个双倍的 , , ...,  可以以3^7种办法插入{0, 1, 2, 3, 4, 5, 6}的3组合的万有循环中。这样的万有循环的个数是完全图K_7的欧几里得尾部的个数。如果我们认为$(a_0\,a_1\cdots a_{20})$等价于$(a_1\cdots a_{20}\,a_0)$但不等价于颠倒顺序循环$(a_{20}\cdots a_1\,a_0)$，则只可以证明该数为129, 976, 320。所以答案为284, 258, 211, 840。

[这个问题首先是由迈 · 赖斯(M. Reiss)于1859年解决的，但他的方法是如此复杂，以致人们对其结果表示怀疑。参见(*Nouvelles Annales de Mathématiques* 8 (1849)，74; 11 (1852)，115; *Annali di Matematica Pura ed Applicata* (2) 5 (1871-1873)，63-120)。一个简单得多的解，它确认了赖斯的断言，是由菲 · 约里瓦尔德(P. Jolivald)和加 · 塔里(G. Tarry)找到的，他们也枚举了K_9的欧几里德的尾部。参见(*Comptes Rendus Association Française pour l'Avancement des Sciences* 15，part 2 (1886)，49-53; É. Lucas, *Récréations Mathématiques* 4 (1894)，123-151)。布仁丹 · 达 · 麦凯(Brendan D. McKay)和罗伯特 · 威 · 罗宾逊(Robert W. Robinson)找到一个更好的方法，使得他们通过使用尾部的数目是

$$(m-1)!^{2m+1}[z_0^{2m}z_1^{2m-2}\cdots z_{2m}^{2m-2}]\det(a_{jk})\prod_{1\leqslant j<k\leqslant 2m}(z_j^2+z_k^2)$$

这一事实，经由K_{21}，继续进行枚举，其中当$j\neq k$时，$a_{jk}=-1/(z_j^2+z_k^2)$；$a_{jj}=-1/(2z_j^2)+\sum_{0<k<2m}1/(z_j^2+z_k^2)$。参见*Combinatorics, Probability, and Computing* 7 (1998)，437-449。]

卡 · 弗莱 · 圣特－马里(C. Flye Sainte-Marie)在(*L'Intermédiaire des Mathématiciens* 1 (1894)，164-165)中说明，K_7的欧几里得尾部包括在{0, 1, ⋯, 6}的排列之下有7重对称性的2 × 720个(即庞索特(Poinsot)循环及其颠倒)。加上具有3重对称性的32 × 1680个，再加上非对称的25778 × 5040个循环。

108. 对于$n<7$，除在$n=4$的平凡情况外，不可能有解。当$n=7$时，有12 255 208 × 7!个万有循环，但不考虑$(a_0\,a_1\cdots a_{34})$和$(a_1\cdots a_{34}\,a_0)$是相同的，包括像在习题105中示例循环那样的有5重对称性的那些在内。 108

当$n\geqslant 8$时，我们可以像布 · 杰克逊(B. Jackson)在(*Discrete Math.* 117 (1993)，141-150)中所建议的那样系统地进行，也请见格 · 哈尔伯特(G. Hurlbert)(*SIAM J. Disc. Math.* 7 (1994)，598-604)：把每个3组合放进"标准循环顺序"$c_1c_2c_3$，其中$c_2=(c_1+\delta)\bmod n$，$c_3=(c_2+\delta')\bmod n$，$0<\delta$, $\delta'<n/2$，而且或者$\delta=\delta'$或者$\max(\delta,\delta')<n-\delta-\delta'\neq(n-1)/2$或($1<\delta<n/4$和$\delta'=(n-1)/2$)或($\delta=(n-1)/2$和$1<\delta'<n/4$)。例如，当$n=8$时，$(\delta,\delta')$可允许的值是(1, 1)，(1, 2)，(1, 3)，(2, 1)，(2, 2)，(3, 1)，(3, 3)。当$n=11$时它们是(1, 1)，(1, 2)，(1, 3)，(1, 4)，(2, 1)，(2, 2)，(2, 3)，(2, 5)，(3, 1)，(3, 2)，(3, 3)，(4, 1)，(4, 4)，(5, 2)，(5, 5)。然后构造对于$0\leqslant c<n$和$1\leqslant\delta<n/2$的具

有顶点(c, δ)，和在标准循环顺序下对于每一组合有边$(c_1, \delta)\to(c_2, \delta')$的有向图。这个有向图是连通和平衡的，所以由定理2.3.4.2D它有一个欧几里得尾部。(当n为奇数时，关于$(n-1)/2$的奇异规则使有向图连通。当n=8时，欧几里得尾部可加以选择使之有n重对称性，但不是当n=12时。)

109. 当n=1时，循环(000)是平凡的；当n=2时，没有循环；当n=4时，实际上仅有2个，即(0001112223330202131 3)和(00011120203332221313)。当$n\geqslant 5$时，令多重组合$d_1d_2d_3$处于标准循环顺序，如果$d_2=(d_1+\delta-1) \bmod n$，$d_3=(d_2+\delta'-1) \bmod n$以及$(\delta, \delta')$对于上一答案中的$n+3$是可允许的话。构造对于$0\leqslant d<n$和$1\leqslant\delta<(n+3)/2$具有顶点$(d, \delta)$，和对于在标准循环顺序之下的每一多重组合$d_1d_2d_3$具有边$(d_1, \delta)\to(d_2, \delta')$的有向图；然后求一个欧几里得尾部。

也许对于$\{0, 1, \cdots, n-1\}$，t多重组合的一个万有循环存在，当且仅当对于$\{0, 1, \cdots, n+t-1\}$，$t-$多重组合的一个万有循环存在。

110. 检查运程的一个好方法是来计算数$b(S)=\sum\{2^{p(c)}\mid c\in S\}$，其中$(p(\mathtt{A}), \cdots, p(\mathtt{K}))=(1, \cdots, 13)$，然后置$l\leftarrow b(S)\&-b(S)$并且检查$b(S)+l=l<<s$，还检查$((l<<s)|(l>>1))\&a=0$，其中$a=2^{p(c_1)}|\cdots|2^{p(c_5)}$。当$S$以格雷码的顺序跑遍所有31个非空的子集时，容易维持$b(S)$和$\sum\{v(c)|c\in S\}$的值。对于$x=(0, \cdots, 29)$，答案为(1009008，99792，2813796，505008，2855676，697508，1800268，751324，1137236，361224，388740，51680，317340，19656，90100，9168，58248，11196，2708，0，8068，2496，444，356，3680，0，0，0，76，4)；于是均值为≈4.769和方差为≈9.768。

没有指头的双手有时滑稽地叫做19，因为该数不能由卡片做成。
——乔·亨·戴维森，*Dee's Hand-Book of Cribbage* (1839)

注记：在纸牌游戏“crib”中不允许四牌同花，那么分配较为容易计算，结果是(1022208，99792，2839800，508908，2868960，703496，1787176，755320，1118336，358368，378240，43880，310956，16548，88132，9072，57288，11196，2264，0，7828，2472，444，356，3680，0，0，0，76，4)；均值和方差
109 减少到近似为4.735和9.667。

7.2.1.4节

1.

m^n	$m^{\underline{n}}$	$m!\left\{{n \atop m}\right\}$
$\binom{m+n-1}{n}$	$\binom{m}{n}$	$\binom{n-1}{n-m}$
$\left\{{n \atop 0}\right\}+\cdots+\left\{{n \atop m}\right\}$	$[m\geq n]$	$\left\{{n \atop m}\right\}$
$\left|{m+n \atop m}\right|$	$[m\geq n]$	$\left|{n \atop m}\right|$

2. 一般说来，给定任何整数$x_1\geqslant\cdots\geqslant x_m$，我们得到所有整数$m$元组$a_1\cdots a_m$，使

得$a_1 \geqslant \cdots \geqslant a_m$，$a_1+\cdots+a_m=x_1+\cdots+x_m$，而且通过初始化$a_1\cdots a_m \leftarrow x_1\cdots x_m$和$a_{m+1} \leftarrow x_m-2$，使$a_m, \cdots, a_1 \geqslant x_m, \cdots, x_1$。特别是，如果$c$是任何整数常数，通过对于$1<j\leqslant m$，初始化$a_1 \leftarrow n-mc+c$，$a_j \leftarrow c$以及$a_{m+1} \leftarrow c-2$，而且假定$n \geqslant cm$，我们得到整数$m$元组使得$a_1 \geqslant \cdots \geqslant a_m \geqslant c$和$a_1+\cdots+a_m=n$。

3. 对于$1 \leqslant j \leqslant m$，置 $a_j = \lfloor (n+m-j)/m \rfloor = \lceil (n+1-j)/m \rceil$。参见*CMath* §3.4。

4. 我们必须有$a_m \geqslant a_1-1$；因此对于$1 \leqslant j \leqslant m$，$a_j = \lfloor (n+m-j)/m \rfloor$，其中$m$是满足$\lfloor n/m \rfloor \geqslant r$的最大整数，即$m=\lfloor n/r \rfloor$。

5. [参见尤金 · 马 · 克林科(Eugene M. Klimko)，*BIT* 13 (1973)，38–49。]

C1. [初始化。] 置$c_0 \leftarrow 1$，$c_1 \leftarrow n$，$c_2\cdots c_n \leftarrow 0\cdots 0$，$l_0 \leftarrow 1$，$l_1 \leftarrow 0$。(我们假定$n>0$。)

C2. [访问。] 访问由部分计数$c_1\cdots c_n$和链接$l_0 l_1 \cdots l_n$所表示的分划。

C3. [转移。] 置$j \leftarrow l_0$和$k \leftarrow l_j$。如果$c_j=1$，则转到C6；否则，如果$j>1$，则转到C5。

C4. [把1+1改为2。] 置$c_1 \leftarrow c_1-2$，$c_2 \leftarrow c_2+1$。然后如果$c_1=0$，置$l_0 \leftarrow 2$，而且如果$k \neq 2$，则置$l_2 \leftarrow l_1$。如果$c_1>0$且$k \neq 2$，则置$l_2 \leftarrow l_1$以及$l_1 \leftarrow 2$。返回C2。

C5. [把$j \cdot c_j$改变成$(j+1)+1+\cdots+1$。] 置$c_1 \leftarrow j(c_j-1)-1$并转到C7。

C6. [把$k \cdot c_k+j$改变成$(k+1)+1+\cdots+1$。] 如果$k=0$则结束。否则置$c_j \leftarrow 0$；然后置$c_1 \leftarrow k(c_k-1)+j-1$，$j \leftarrow k$以及$k \leftarrow l_k$。

C7. [调整链接。] 如果$c_1>0$，则置$l_0 \leftarrow 1$，$l_1 \leftarrow j+1$，否则置$l_0 \leftarrow j+1$。然后置$c_j \leftarrow 0$和$c_{j+1} \leftarrow c_{j+1}+1$。如果$k \neq j+1$，则置$l_{j+1} \leftarrow k$。返回C2。 ▌

注意此算法是无循环的，但它实际上不比算法P更快。步骤C4，C5和C6分别被执行$p(n-2)$，$2p(n)-p(n+1)-p(n-2)$以及$p(n+1)-p(n)$次。因此当n很大时步骤C4是最重要的。(参见习题45及由芬纳(Fenner)和洛伊舟(Loizou)在[*Acta. Inf.* 16 (1981)，237-252]中给出的详尽分析。)

6. 置$k \leftarrow a_1$和$j \leftarrow 1$。然后，当$k>a_{j+1}$时，置$b_k \leftarrow j$和$k \leftarrow k-1$，直到$k=a_{j+1}$为止。如果$k>0$，则置$j \leftarrow j+1$并重复直到$k=0$为止。(我们以对偶形式$a_j-a_{j+1}=d_j$使用(11)，其中$d_1\cdots d_n$是$b_1 b_2\cdots$的部分计数表示。注意，本算法的运行时间实际上同a_1+b_1，即输出长度加上输入长度成比例。)

7. 由(11)的对偶，我们有 $b_1\cdots b_n = n^{a_n}(n-1)^{a_{n-1}-a_n}\cdots 1^{a_1-a_2}0^{n-a_1}$。

8. 转置费尔利斯框图对应于反射和求补二进位串(15)。所以我们可以简单地交换和颠倒诸p和诸q，并得到分划$b_1 b_2\cdots=(q_t+\cdots+q_1)^{p_1}(q_t+\cdots+q_2)^{p_2}\cdots(q_t)^{p_t}$。

9. 由归纳法，如果$a_k=l-1$且$b_l=k-1$，则增加a_k和b_l保持相等性。 110

10. (a)通过加“1”，得到每个节点的左儿子，右儿子则通过增加最右数字得到。这个孩子存在当且仅当父节点以不相等的数字结束。n的所有分划以词典顺序出现在n级上。

(b) 通过把“11”改成2而得到左儿子；它存在当且仅当父节点至少包含两个1。右儿子通过删去一个1和增加超过1的最小部分得到，它存在当且仅当至少有一个1，

而且最小的更大部分恰出现一次。所有n分成为m个部分的分划以词典顺序出现在$n-m$级上；整棵树的前根次序给出整个的词典顺序。[特·伊·芬纳和乔·洛伊舟(G. Loizou)，*Comp. J.* **23** (1980)，332-337。]

11. $[z^{100}]1/((1-z)(1-z^2)(1-z^5)(1-z^{10})(1-z^{20})(1-z^{50})(1-z^{100}))=4563$，而且$[z^{100}](1+z+z^2)(1+z^2+z^4)\cdots(1+z^{100}+z^{200})=7$。[参见乔·玻里亚(G. Pólya)，*AMM* **63** (1956)，689-697。]在无穷序列$\Pi_{k\geq 1}(1+z^k+z^{2k})(1+z^{2k}+z^{4k})(1+z^{5k}+z^{10k})$中，$z^{10^n}$的系数是$2^{n+1}-1$，而且$z^{10^n-1}$的系数是$2^n$。

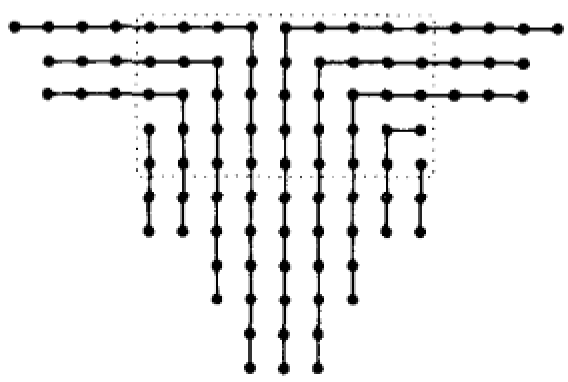
12. 为证明$(1+z)(1+z^2)(1+z^3)\cdots=1/((1-z)(1-z^3)(1-z^5)\cdots)$，把左边写成为

$$\frac{(1-z^2)(1-z^4)(1-z^6)}{(1-z)(1-z^2)(1-z^3)}\cdots$$

从分子和分母中消除公因式。或者，在恒等式$(1+z)(1+z^2)(1+z^4)(1+z^8)\cdots=1/(1-z)$中以$z^1, z^3, z^5, \cdots$代替$z$，并且把结果乘在一起。[*Novi Comment.Acad. Sci. Pet.* **3** (1750)，125-169，§47。]

13. 把分划$c_1\cdot 1+c_2\cdot 2+\cdots$分成$\lfloor c_1/2\rfloor\cdot 2+\lfloor c_2/2\rfloor\cdot 4+\cdots+r_1\cdot 1+r_3\cdot 3+\cdots$，其中$r_m=(c_m \bmod 2)+2(c_{2m} \bmod 2)+4(c_{4m} \bmod 2)+\cdots$。[*Johns Hopkins Univ. Circular* **2** (1882)，72。]

14. 希尔威斯特对应作为一个框图最好理解，其中奇排列的点被中心化并被分成不相交的钩。例如，在有五个不同的奇部分，分划17+15+15+9+9+9+9+5+5+3+3通过框图对应于具有4个间隙的全不同分划19+18+16+13+12+9+5+4+3。



反之，分成$2t$个不同部分的一个分划可以惟一地写成形式$(a_1+b_1-1)+(a_1+b_2-2)+(a_2+b_2-3)+(a_2+b_3-4)+\cdots+(a_{t-1}+b_t-2t+2)+(a_t+b_t-2t+1)+(a_t+b_{t+1}-2t)$，其中$a_1>a_2>\cdots>a_t\geq t$且$b_1>b_2>\cdots>b_t\geq b_{t+1}=t$。它对应于$(2a_1-1)+\cdots+(2a_t-1)+(2A_1-1)+\cdots+(2A_t-1)$，其中$A_1+\cdots+A_t$是$(b_1-t)+\cdots+(b_t-t)$的共轭。$t$的值实际上是一个“德尔菲矩形”的大小。

当$n=10$时，有关的奇分划是9+1，7+3，7+1+1+1，5+5，5+3+1+1，5+1+1+1+1+1，3+3+3+1，3+3+1+1+1+1，3+1+⋯+1，1+⋯+1，分别对应于不同
111 部分的分划6+4，5+4+1，7+3，4+3+2+1，6+3+1，8+2，5+3+2，7+2+1，9+1，10。[参见希尔威斯特在*Amer. J. Math.* **5** (1882)，251-330和**6** (1883)，334-336的著名论文。]

15. 迹k的每个自共轭分划对应于n分成k个不同奇部分(钩)的一个分划。因此我们可以以乘积$(1+z)(1+z^3)(1+z^5)\cdots$或以和$1+z^1/(1-z^2)+z^4/((1-z^2)(1-z^4))+z^9((1-z^2)(1-z^4)(1-z^6))+\cdots$来写生成函数。[*Johns Hopkins Univ. Circular* **3** (1883), 42-43。]

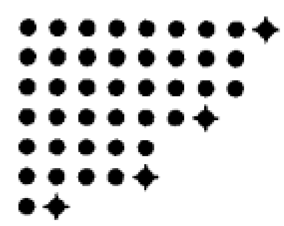
16. 德尔菲方块含k^2个点，剩下的点对应于两个独立的分划且最大部分$\leqslant k$。因此，如果我们使用w来统计部分且使用z来统计点，我们发现

$$\prod_{m=1}^{\infty}\frac{1}{1-wz^m}=\sum_{k=0}^{\infty}\frac{w^k z^{k^2}}{(1-z)(1-z^2)\cdots(1-z^k)(1-wz)(1-wz^2)\cdots(1-wz^k)}$$

[这个看起来令人印象深刻的公式是习题19的更令人印象深刻的恒等式的特殊情况$x=y=0$。]

17. (a)$((1+uvz)(1+uvz^2)(1+uvz^3)\cdots)/((1-uz)(1-uz^2)(1-uz^3)\cdots)$。

(b) 一个合并分划可以通过一个广义菲尔利斯框图表示，其中我们把所有部分合并在一起，如果$a_i\geqslant b_j$，则把a_i放在b_j之上，然后标记每个b_j最右边的点。例如，合并分划(8, 8, 5; 9, 7, 5, 2)有如右框图，且把标记点划作‘✦’；标记仅在角落处出现。因此，转置的框图对应于另一个合并分划。在此情况下它是(7, 6, 6, 4, 3; 7, 6, 4, 1)。(参见城市东明和丹·斯坦顿(D. Stanton)，*Pacific J. Math*. **127** (1987)，103-120；希·科提尔(S. Corteel)和杰·洛弗佐伊(J. Lovejoy)，*Trans. Amer. Math. Soc.* **356** (2004)，1623-1635；以及雨果·帕克准备发表在*The Ramanujan Journal*中的论文“Partition bijections，a survey”。)



有$t>0$部分的每个合并分划这样一来对应于一个“共轭”，其中最大部分是t。而且对于这样的合并分划，生成函数是$((1+vz)\cdots(1+vz^{t-1}))/((1-z)\cdots(1-z^t))$乘以$(vz^t+z^t)$，其中$vz^t$对应于$b_1=t$的情况，而$z^t$对应于$r=0$或$b_1<t$的情况。

(c) 因此我们得到在答案1.2.6-58中的一般z多项式定理的一个形式

$$\frac{(1+uvz)(1+uvz^2)(1+uvz^3)}{(1-uz)(1-uz^2)(1-uz^3)}\cdots=\sum_{t=0}^{\infty}\frac{(1+v)(1+vz)}{(1-z)(1-z^2)}\cdots\frac{(1+vz^{t-1})}{(1-z^t)}u^t z^t$$

18. 当给定诸c和诸d时，这些方程显然确定诸a和诸b，所以我们要来证明从诸a和诸b，诸c和诸d是惟一确定的。以下算法从右到左确定诸c和诸d:

A1. [初始化。] 置$i\leftarrow r$，$j\leftarrow s$，$k\leftarrow 0$，而且$a_0\leftarrow b_0\leftarrow\infty$。

A2. [转移。] 如果$i+j=0$，则停止。否则如果$a_i\geqslant b_j-k$，则转到A4。

A3. [吸收a_i。] 置$c_{i+j}\leftarrow a_i$，$d_{i+j}\leftarrow 0$，$i\leftarrow i-1$，$k\leftarrow k+1$，并返回A2。

A4. [吸收b_j。] 置$c_{i+j}\leftarrow b_j-k$，$d_{i+j}\leftarrow 1$，$j\leftarrow j-1$，$k\leftarrow k+1$，并返回A2。 ▌

也有从左到右的一个方法:

B1. [初始化。] 置$i\leftarrow 1$，$j\leftarrow 1$，$k\leftarrow r+s$，且$a_{r+1}\leftarrow b_{s+1}\leftarrow -\infty$。

B2. [转移。] 如果$k=0$，则停止。否则置$k\leftarrow k-1$，然后如果$a_i\leqslant b_j-k$，则转到B4。

B3. [吸收a_i。] 置$c_{i+j-1} \leftarrow a_i$，$d_{i+j-1} \leftarrow 0$，$i \leftarrow i+1$，并返回B2。

112 B4. [吸收b_j。] 置$c_{i+j-1} \leftarrow b_j - k$，$d_{i+j-1} \leftarrow 1$，$j \leftarrow j+1$，并返回B2。 ▌

在两种情况下，转移都是被强迫的，而且得到的序列满足$c_1 \geqslant \cdots \geqslant c_{r+s}$。注意，$c_{r+s}=\min(a_r, b_s)$和$c_1=\max(a_1, b_1-r-s+1)$。

由此我们已经以不同方式证明了习题17(c)的恒等式。这个思想的推广导致拉曼奴燕的“带有许多参数的著名公式”的一个组合证明。

$$\sum_{n=-\infty}^{\infty} w^n \prod_{k=0}^{\infty} \frac{1-bz^{k+n}}{1-az^{k+n}} = \prod_{k=0}^{\infty} \frac{(1-a^{-1}bz^k)(1-a^{-1}w^{-1}z^{k+1})(1-awz^k)(1-z^{k+1})}{(1-a^{-1}bw^{-1}z^k)(1-a^{-1}z^{k+1})(1-az^k)(1-wz^k)}$$

[*参考文献*：哥·哈·哈迪，拉曼奴燕(1940)，等式(12.12.2)；多·泽尔伯格，*Europ.J. Combinatorics* **88** (1987)，461-463；阿·加·伊(A. J. Yee)，*J. Comb. Theory* **A105** (2004)，63-77。]

19. [*Crelle* **34** (1847)，285-328。]由习题17(c)，对于k提示的求和是

$$\left(\sum_{l\geq 0} v^l \frac{(z-bz)\cdots(z-bz^l)}{(1-z)\cdots(1-z^l)} \frac{(1-uz)\cdots(1-uz^l)}{(1-auz)\cdots(1-auz^l)}\right) \cdot \prod_{m=1}^{\infty} \frac{1-auz^m}{1-uz^m}$$

而且对于l求和是类似的，但通过$u \leftrightarrow v$，$a \leftrightarrow b$，$k \leftrightarrow l$进行。而且当$b=auz$时，对于k和l两者，求和归结为

$$\prod_{m=1}^{\infty} \frac{(1-uvz^{m+1})(1-auz^m)}{(1-uz^m)(1-vz^m)}$$

现在令$u=wxy$，$v=1/(yz)$，$a=1/x$和$b=wyz$；把这个无穷乘积同对l的求和等置。

20. 为得到$p(n)$，我们需要对以前条目的$\sqrt{8n/3}$作近似加或减，而且大多这些条目是$\Theta(\sqrt{n})$个二进位长的。因此在$\Theta(n)$个步骤内计算$p(n)$，且总的时间是$\Theta(n^2)$。(直截了当地使用(17)将花费$\Theta(n^{5/2})$个步骤。)

21. 由于$\sum_{n=0}^{\infty} q(n)z^n = (1+z)(1+z^2)\cdots$ 等于$(1-z^2)(1-z^4)\cdots P(z)=(1-z^2-z^4+z^{10}+z^{14}-z^{24}-\cdots)P(z)$，我们有

$$q(n)=p(n)-p(n-2)-p(n-4)+p(n-10)+p(n-14)-p(n-24)-\cdots$$

[也有单独关于q的一个“纯递归”，它类似于在下道习题中对于$\sigma(n)$的递归。]

22. 从(21)我们有$\sum_{n=1}^{\infty} \sigma(n)z^n = \sum_{m,n\geq 1} mz^{mn} = z\frac{d}{dz}\ln P(z) = (z+2z^2-5z^5-7z^7+\cdots)/(1-z-z^2+z^5+z^7+\cdots)$。[*Bibliothèque Impartiale* **3** (1751)，10-31。]

23. 置$u=w$和$v=z/w$来得出

$$\prod_{k=1}^{\infty}(1-z^k w)(1-z^k/w)(1-z^k)=\sum_{n=-\infty}^{\infty}(-1)^n w^n z^{n(n+1)/2}/(1-w)$$
$$=\sum_{n=0}^{\infty}(-1)^n(w^{-n}-w^{n+1})z^{n(n+1)/2}/(1-w)$$
$$=\sum_{n=0}^{\infty}(-1)^n(w^{-n}+\cdots+w^n)z^{n(n+1)/2}$$

当$|z|<1$且w接近1时，这些乘法是合法的。现在置$w=1$。

[参见在答案14中所引用的希尔威斯特的论文的§57。雅可比的证明见他的专著*Fundamenta Nova Theoriæ Functionum Ellipticarum* (1829)的§66。] 113

24. (a) 由(18)和习题23，对所有整数j和k求和，$[z^n]A(z)=\sum(-1)^{j+k}(2k+1)[3j^2+j+k^2+k=2n]$。当$n \bmod 5=4$时，所有的贡献有$j \bmod 5=4$和$k \bmod 5=2$，但那么一来，$(2k+1) \bmod 5=0$。

(b) 由等式4.6.2－(5)，当p为素数时，$B(z)^p\equiv B(z^p)$ (modulo p)。

(c) 取$B(z)=P(z)$。因为$A(z)=P(z)^{-4}$。[*Proc. Cambridge Philos. Soc.* **19** (1919), 207-210。一个类似的证明指出，当$n \bmod 7=5$时，$p(n)$是7的倍数。拉曼奴燕继续进行得到漂亮的公式$p(5n+4)/5=[z^n]P(z)^6/P(z^5)$；$p(7n+5)/7=[z^n](P(z)^4/P(z^7)^3+7z P(z)^8/p(z^7)^7)$。阿特金(Atkin)和斯温纳顿－戴尔(Swinnerton-Dyer)在[*Proc. London Math. Soc.* (3) **4**(1953)，84-106]中证明，如同由弗·迪森(F. Dyson)所猜测的那样，$5n+4$和$7n+5$的分划可以根据mod 5或mod 7的(最大部分——部分的个数)的分别的值，被分成相同大小的类。一个稍微更复杂的组合统计也证明，当$n \bmod 11=6$时，$p(n) \bmod 11=0$；参见弗·杰·加尔文(F.G.Garvan)，*Trans. Amer. Math. Soc.* **305** (1988)，47-77。]

25. [通过对于所述的恒等式两边进行微商，可证明这个提示。它是1826年由尼·亨·阿贝尔(N.H.Abel)发现的一个漂亮公式的特殊情况$y=1-x$:

$$\mathrm{Li}_2(x)+\mathrm{Li}_2(y)=\mathrm{Li}_2\left(\frac{x}{1-y}\right)+\mathrm{Li}_2\left(\frac{y}{1-x}\right)-\mathrm{Li}_2\left(\frac{xy}{(1-x)(1-y)}\right)-\ln(1-x)\ln(1-y)$$

参见阿贝尔的[*Euvres Complètes* **2**(Christiania: Grφndahl，1881)，189-193。]

(a) 令$f(x)=\ln(1/(1-e^{-xt}))$，然后$\int_1^x f(x)dx=-\mathrm{Li}_2(e^{-tx})/t$ 和 $f^{(n)}(x)=(-t)^n e^{tx}\sum_k\left\langle {n-1 \atop k}\right\rangle e^{ktx}/(e^{tx}-1)^n$，所以欧拉的求和公式给出，当$t\to 0$时，$\mathrm{Li}_2(e^{-t})/t+\frac{1}{2}\ln(1/(1-e^{-t}))+O(1)=(\zeta(2)+t\ln(1-e^{-t})-\mathrm{Li}_2(1-e^{-t}))/t-\frac{1}{2}\ln t+O(1)=\zeta(2)/t+\frac{1}{2}\ln t+O(1)$。

(b) 我们有 $\sum_{m,n>1} e^{-mnt}/n=\frac{1}{2\pi i}\sum_{m,n>1}\int_{1-i\infty}^{1+i\infty}(mnt)^{-z}\Gamma(z)d2/n$，它求和成为 $\frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty}\zeta(z+1)\zeta(z)t^{-z}\Gamma(z)dz$。在$z=1$处的极给出$\zeta(2)/t$；在$z=0$处的双极给

出 $-\zeta(0)\ln t+\zeta'(0)=\frac{1}{2}\ln t-\frac{1}{2}\ln 2\pi$ ；在$z=-1$处的极给出 $-\zeta(-1)\zeta(0)t=B_2B_1t=-t/24$，$\zeta(z+1)\zeta(z)$的零值删除了$\Gamma(z)$的其他极，所以对于任意大的$M$，结果是 $\ln P(e^{-t})=\zeta(2)/t+\frac{1}{2}\ln(t/2\pi)-t/24+O(t^M)$ 。

26. 令 $F(n)=\sum_{k=1}^{\infty}e^{-k^2/n}$ 。对于所有的x，我们可以通过 $f(x)=e^{-x^2/n}[x>0]+\frac{1}{2}\delta_{x0}$ 或$f(x)=e^{-x^2/n}$来使用(25)，因为 $2F(n)+1=\sum_{k=-\infty}^{\infty}e^{-k^2/n}$ 。让我们来选择后一个，则对于$\theta=0$，如果我们替换$u=y+\pi mni$，(25)的右边是快速收敛的

$$\lim_{M\to\infty}\sum_{m=-M}^{M}\int_{-\infty}^{\infty}e^{-2\pi miy-y^2/n}dy=\sum_{m=-\infty}^{\infty}e^{-\pi^2m^2n^2}\int_{-\infty}^{\infty}e^{-u^2/n}du$$

而且这个积分是$\sqrt{\pi n}$。[此结果是泊松的开创性论文420页上的公式(15)。]

27. 令 $g_n=\sqrt{\pi/6t}\,e^{-n^2\pi^2/6t}\cos\frac{n\pi}{6}$ 。然后 $\int_{-\infty}^{\infty}f(y)\cos 2\pi mydy=g_{2m+1}+g_{2m-1}$，所以我们有

$$\frac{e^{-t/24}}{P(e^{-t})}=g_1+g_{-1}+2\sum_{m=1}^{\infty}(g_{2m+1}+g_{2m-1})=2\sum_{m=-\infty}^{\infty}g_{2m+1}$$

项g_{6n+1}和g_{-6n-1}组合在一起给出(30)的第n项。[参见马·伊·克诺普(M.I.Knopp)，
114 *Modular Functions in Analytic Number Theory* (1970)，第3章。]

28. (a, b, c, d)参见*Trans*，*Amer. Math. Soc.* 43 (1938)，271-295。事实上，莱默借助于习题4.5.4-23的雅可比符号求出 $A_{p^e}(n)$ 的显式公式:

$$A_{2^e}(n)=(-1)^e\left(\frac{-1}{m}\right)2^{e/2}\sin\frac{4\pi m}{2^{e+3}},\quad \text{如果}(3m)^2\equiv 1-24n\ (\text{modulo } 2^{e+3})$$

$$A_{3^e}(N)=(-1)^{e+1}\left(\frac{m}{3}\right)\frac{2}{\sqrt{3}}3^{e/2}\sin\frac{4\pi m}{3^{e+1}},\quad \text{如果}(8m)^2\equiv 1-24n\ (\text{modulo } 3^{e+1})$$

$$A_{p^e}(n)=\begin{cases}2\left(\frac{3}{p^e}\right)p^{e/2}\cos\frac{4\pi m}{p^e}, & \text{如果}(24m)^2\equiv 1-24n\ (\text{modulo } p^e)\text{，}p\geqslant 5\text{且}24n \bmod p\neq 1\\ \left(\frac{3}{p^e}\right)p^{e/2}[e=1], & \text{如果}24n \bmod p=1\text{且}p\geqslant 5\end{cases}$$

(e) 对于$3<p_1<\cdots<p_t$和$e_1\cdots e_t\neq 0$，如果 $n=2^a3^bp_1^{e_1}\cdots p_t^{e_t}$ ，则$A_k(n)\neq 0$的概率是 $2^{-t}(1+(-1)^{[e_1=1]}/p_1)\ldots(1+(-1)^{[e_t=1]}/p_t)$ 。

29. $z_1z_2\cdots z_m/((1-z_1)(1-z_1z_2)\cdots(1-z_1z_2\cdots z_m))$。

30. 由(39)，(a) $\left|{n+1 \atop m}\right|$ 和(b) $\left|{m+n \atop m}\right|$。

31. 头一个解[小马歇尔·霍尔(Marshall Hall Jr.)，*Combinatorial Theory* (1967)，§4.1]：由递归式(39)，我们可以直接证明，对于$0 \leqslant r<k!$，存在一个多项式 $f_{k,r}(n)=n^{k-1}/(k!(k-1)!)+O(n^{k-2})$，它使得 $\left|{n \atop k}\right|=f_{n,n \bmod k!}(n)$。

第二个解：由于 $(1-z)\ldots(1-z^m)=\prod_{p\perp q}(1-e^{2\pi ip/q}z)^{\lfloor m/q\rfloor}$，其中的乘积是对于满足$0 \leqslant p<q$的所有既约分数$p/q$进行的，而(41)中$z^n$的系数可表达为单位根乘以$n$的多项式之和，即像 $\sum_{p\perp q} e^{2\pi ipn/q} f_{pq}(n)$ 这样，其中$f_{pq}(n)$是次数小于m/q的一个多项式。因此存在常数使得 $\left|{n \atop 2}\right|=a_1n+a_2+(-1)^n a_3$；$\left|{n \atop 3}\right|=b_1n^2+b_2n+b_3=(-1)^n b_4+\omega^n b_5+\omega^{-n}b_6$，其中 $\omega=e^{2\pi i/3}$，等等。这些常数通过对于小的n的值来确定，因而头两种情况是

$$\left|{n \atop 2}\right|=\frac{1}{2}n-\frac{1}{4}+\frac{1}{4}(-1)^n;\quad \left|{n \atop 3}\right|=\frac{1}{12}n^2-\frac{7}{72}-\frac{1}{8}(-1)^n+\frac{1}{9}\omega^n+\frac{1}{9}\omega^{-n}$$

由此得出，$\left|{n \atop 3}\right|$是最接近于$n^2/12$的整数。类似地，$\left|{n \atop 4}\right|$是最接近于$(n^3+3n^2-9n[n$奇数$])/144$的整数。

对于$\left|{n \atop 2}\right|$、$\left|{n \atop 3}\right|$和$\left|{n \atop 4}\right|$的精确公式，不含对于底限函数的简化，最先由吉·弗·马尔法梯(G.F.Malfatti)求得，见*Memorie di Mat. e Fis. Società Italiana* 3 (1786)，571-663。沃·约·阿·科尔曼(W.J.A.Colman)在[*Fibonacci Quarterly* 21 (1983)，272-284]中证明，$\left|{n \atop 5}\right|$是最接近于$(n^4+10n^3+10n^2-75n-45n(-1)^n)/2880$的整数，他还对$\left|{n \atop 6}\right|$和$\left|{n \atop 7}\right|$给出类似的公式。

32. 由于$\left|{m+n \atop m}\right| \leqslant p(n)$，且相等当且仅当$m \geqslant n$时，我们有$\left|{n \atop m}\right| \leqslant p(n-m)$且相等当且仅当$2m \geqslant n$时。

33. 分成m个部分的一个分划对应于至多$m!$个合成；因此$\binom{n-1}{m-1} \leqslant m!\left|{n \atop m}\right|$。结果，$p(n) \geqslant (n-1)!/((n-m)!\ m!\ (m-1)!)$，而且当$m=\sqrt{n}$时，斯特林的近似公式证明$\ln p(n) \geqslant 2\sqrt{n}-\ln n-\frac{1}{2}-\ln 2\pi$。

34. $a_1>a_2>\cdots>a_m>0$当且仅当$a_1-m+1 \geqslant a_2-m+2 \geqslant \cdots \geqslant a_m \geqslant 1$，而且分成$m$个不同部分的分划对应于$m!$个合成。因此，由上一个答案，我们有

$$\frac{1}{m!}\binom{n-1}{m-1} \leqslant \left|{n \atop m}\right| \leqslant \frac{1}{m!}\binom{n+m(m-1)/2}{m-1}$$ 115

[参见汉·古普塔(H.Gupta)，*Proc. Indian Acad. Sci.* A16 (1942)，101-102。当$n=\Theta(m^3)$时对于$\left|{n \atop m}\right|$的一个详尽的渐近公式出现在习题3.3.2-30中。]

35. (a) $x=\frac{1}{C}\ln\frac{1}{C}\approx-0.194$ 。

(b) $x=\frac{1}{C}\ln\frac{1}{C}-\frac{1}{C}\ln\ln 2\approx 0.092$；一般地，我们有$x=\frac{1}{C}\left(\ln\frac{1}{C}-\ln\ln\frac{1}{F(x)}\right)$ 。

(c) $\int_{-\infty}^{\infty} x dF(x)=\int_0^{\infty}(Cu)^{-2}(\ln u)e^{-1/(Cu)}du=-\frac{1}{C}\int_0^{\infty}(\ln C+\ln v)e^{-v}dv=(\gamma-\ln C)$ $/C\approx 0.256$ 。

(d) 类似地，$\int_{-\infty}^{\infty} x^2e^{-Cx}\exp(-e^{-Cx}/C)dx=(\gamma^2+\zeta(2)-2\gamma\ln C+(\ln C)^2)/C^2$ ≈ 1.0656 。因此方差精确地为$\zeta(2)/C^2=1$。

[概率分布 $e^{-e^{(a-x)/b}}$ 通常叫做费希尔(Fisher)－梯皮特(Tippett)分布；参见*Proc. Cambridge Phil. Soc.* **24** (1928)，180-190。]

36. 对于$j_r-(m+r-1)\geqslant\cdots\geqslant j_2-(m+1)\geqslant j_1-m\geqslant 1$求和，得出

$$\begin{aligned}\Sigma_r&=\sum_t\left|\begin{matrix}t-rm-r(r-1)/2\\ r\end{matrix}\right|\frac{p(n-t)}{p(n)}\\&=\frac{\alpha}{1-\alpha}\frac{\alpha^2}{1-\alpha^2}\cdots\frac{\alpha^r}{1-\alpha^r}\alpha^{rm}(1+O(n^{-1/2+2\varepsilon}))+E\\&=\frac{n^{-1/2}}{\alpha^{-1}-1}\frac{n^{-1/2}}{\alpha^{-2}-1}\cdots\frac{n^{-1/2}}{\alpha^{-r}-1}\exp(-Crx+O(rn^{-1/2+2\varepsilon}))+E\end{aligned}$$

其中E是考虑$t>n^{1/2+\varepsilon}$ 的情况的误差项。前导因式$n^{-1/2}/(\alpha^{-j}-1)$是$\frac{1}{jC}(1+O(jn^{-1/2}))$ 。

而且容易验证$E=O(n^{\log n}e^{-Cn^{\varepsilon}})$，即使我们使用粗糙的上限$\left|\begin{smallmatrix}t-rm-r(r-1)/2\\ r\end{smallmatrix}\right|<t^r$，因为

$$\sum_{t>xN}t^re^{-t/N}=O\left(\int_{xN}^{\infty}t^re^{-t/N}dt\right)=O(N^{r+1}x^re^{-x}/(1-r/x))$$

其中 $N=\Theta(\sqrt{n}),x=\Theta(n^{\varepsilon}),r=O(\log n)$ 。

37. 这样一个分划在Σ_0中被统计一次，在Σ_1中被统计q次，在Σ_2中被统计$\binom{q}{2}$次……，因此在以$(-1)^r\Sigma_r$结尾的部分和中被精确地统计 $\sum_{j=0}^{r}(-1)^j\binom{q}{j}=(-1)^r\binom{q-1}{r}$次。当$r$为奇数时，这个计数至多为$\delta_{q0}$，而当$r$为偶数时它至少是$\delta_{q0}$。[一个类似的论断表明，习题1.3.3－26的推广的原理也有这个分类的性质，参考文献：卡·博恩费尔罗尼(C.Bonferroni)，*Pubblicazioni del Reale Istituto Superiore de Scienze Economiche e Commerciale di Firenze* **8** (1936)，3-62。]

38. $z^{l+m-1}\binom{l+m-2}{m-1}_z=z^{l+m-1}(1-z^l)\ldots(1-z^{l+m-2})/((1-z)\ldots(1-z^{m-1}))$。

39. 如果$\alpha=a_1\cdots a_m$是至多有m个部分的分划，如果$a_1<1$，令$f(\alpha)=\infty$，否则令

$f(\alpha)=\min\{j|a_1>l+a_{j+1}\}$。令$g_k$是对于满足$f(\alpha)>k$的分划的生成函数。满足$f(\alpha)=k<\infty$的分划以不等式

$$a_1\geqslant a_2\geqslant\cdots\geqslant a_k\geqslant a_1-l>\alpha_{k+1}\geqslant\cdots\geqslant a_{m+1}=0$$

来表征。因此$a_1a_2\cdots a_m=(b_k+l+1)(b_1+1)\cdots(b_{k-1}+1)b_{k+1}\cdots b_m$，其中$f(b_1\cdots b_m)\geqslant k$，而且其逆也为真。由此得出$g_k=g_{k-1}-z^{l+k}g_{k-1}$。

[参见*American J. Math.* 5 (1882)，254-257。]

40. $z^{m(m+1)/2}\binom{l}{m}_z=(z-z^l)(z^2-z^l)\dots(z^m-z^l)/((1-z)(1-z^2)\dots(1-z^m))$。这个公式实质上是习题1.2.6-58的$z$项式定理。

41. 参见杰·阿尔默克维斯特(G.Almkvist)和乔·埃·安德鲁斯(G. E. Andrews)。*J. Number Theory* 38 (1991)，135－144。 [116]

42. 阿·弗尔希克(A.Vershik)[*Functional Anal. Applic.* 30 (1996)，90-105，定理4.7]已经指出公式

$$\frac{1-e^{-c\varphi}}{1-e^{-c(\theta+\varphi)}}e^{-ck/\sqrt{n}}+\frac{1-e^{-c\theta}}{1-e^{-c(\theta+\varphi)}}e^{-ca k/\sqrt{n}}\approx 1$$

其中常数c必须选择为θ和φ的一个函数，使得这形状的面积是n。如果$\theta\varphi<2$，则常数c为负，如果$\theta\varphi>2$则c为正。当$\theta\varphi=2$时，这个形状就归结为一条直线

$$\frac{k}{\theta\sqrt{n}}+\frac{a_k}{\varphi\sqrt{n}}\approx 1$$

如果$\varphi=\infty$我们有$c=\sqrt{\mathrm{Li}_2(t)}$，其中$t$满足$\theta=\left(\ln-\frac{1}{1-t}\right)\Big/\sqrt{\mathrm{Li}_2(t)}$。

43. 我们有$a_1>a_2>\cdots>a_k$，当且仅当共轭分划包括部分1, 2, …, $k-1$的每一个至少一次。这样分划的个数为$p(n-k(k-1)/2)$；这个总数包括对于$a_k=0$的$\left|{n-(k-1)(k-2)/2 \atop k-1}\right|$种情况。

44. 假设$n>0$。带有不相等的最小部分(或仅有一个部分)的个数为$p(n+1)-p(n)$，即不以1结尾的$n+1$的分划的个数，因为我们通过改变最小部分而从后者得到前者。因此答案为$2p(n)-p(n+1)$。[参见鲁·约·博斯科维奇(R. J. Boscovich)，*Giornale de'Letterati* (Rome，1748)，15。其最小三个部分相等的分划数为$3p(n)-p(n+1)-2p(n+2)+p(n+3)$。关于最小部分的其他限制的类似公式也可推导出来。]

45. 由等式(37)，我们有$p(n-j)/p(n)=1-Cjn^{-1/2}+(C^2j^2+2j)/(2n)-(8C^3j^3+60Cj^2+Cj+12C^{-1}j)/(48n^{3/2})+O(j^4n^{-2})$。

46. 如果$n>1$，$T_2'(n)=p(n-1)-p(n-2)<p(n)-p(n-1)=T_2''(n)$。因为$p(n)-p(n-1)$是不以1结尾的$n$的分划数；如果我们增加最大的部分，则每一个这样的$n-1$的分划生成对于$n$的分划。但差别是相当小的：$(T_2''(n)-T_2'(n))/p(n)=C^2n+O(n^{-3/2})$。

47. 在这个提示中的恒等式由对(21)求微商得到；参见习题22。当$c_1+2c_2+\cdots$

$+nc_n=n$时，由对n的归纳法，得到部分计数$c_1\cdots c_n$的概率是

$$\Pr(c_1\cdots c_n)=\sum_{k=1}^{n}\sum_{j=1}^{c_k}\frac{kp(n-jk)}{np(n)}\Pr(c_1\cdots c_{k-1}(c_k-j)c_{k+1}\cdots c_n)$$
$$=\sum_{k=1}^{n}\sum_{j=1}^{c_k}\frac{k}{np(n)}=\frac{1}{p(n)}$$

[*Combinatorial Algorithms* (Academic Press, 1975)，第10章。]

48. 在步骤N5中j有一个固定的特定值的概率为$6/(\pi^2 j^2)+O(n^{-1/2})$，而且$jk$的平均值的阶为$\sqrt{n}$。在步骤N4中所花费平均时间为$\Theta(n)$，因此平均运行时间的阶为$n^{3/2}$。(更精确的分析将是合意的。)

49. (a) 我们有$F(z)=\sum_{k=1}^{\infty}F_k(z)$，其中$F_k(z)$是对于其最小部分为$\geqslant k$的所有分划的生成函数，即$1/((1-z^k)(1-z^{k+1})\cdots)-1$。

(b) 令$f_k(n)=[z^n]F_k(z)/p(n)$，则$f_1(n)=1$；$f_2(n)=1-p(n-1)/p(n)=Cn^{-1/2}+O(n^{-1})$；$f_3(n)=(p(n)-p(n-1)-p(n-2)+p(n-3))/p(n)=2C^2n^{-1}+O(n^{-3/2})$；而且$f_4(n)=6C^3n^{-3/2}+O(n^{-2})$。(参见习题45。)结果是$f_{k+1}(n)=k!C^k n^{-k/2}+O(n^{-(k+1)/2})$；特别是$f_5(n)=O(n^{-2})$，因此$f_5(n)+\cdots+f_n(n)=O(n^{-1})$，因为$f_{k+1}(n)<f_k(n)$。

117 把每样东西都加在一起得到$[z^n]F(z)=p(n)(1+C/\sqrt{n}+O(n^{-1}))$。

50. (a) 当$0\leqslant k<m$时，由归纳法$c_m(m+k)=c_{m-1}(m-1+k)+c_m(k)=m-1-k+c(k)+1$。

(b) 因为对于$0\leqslant k\leqslant m$，$\left|{m+k \atop m}\right|=p(k)$，

(c) 当$n=2m$时，算法H实质上生成m的分划，而且我们知道$j-1$是在刚生成的分划的共轭中的最小部分——除非当$j-1=m$时，在其共轭仅有一个部分的分划$1\cdots1$之后。

(d) 如果α的所有部分超过k，则令$\alpha k^{q+1}j$对应于$\alpha(k+1)$。

(e) 对于其第二个最小的部分$\geqslant k$的所有分划，生成函数$G_k(z)$为$(z+\cdots+z^{k-1})F_k(z)+F_k(z)-z^k/(1-z)=F_{k+1}(z)/(1-z)$，其中$F_k(z)$在上题中定义。结果$C(z)=(F(z)-F_1(z))/(1-z)+z/(1-z)^2$。

(f) 我们可以如同在上题中那样证明，对于$k\leqslant5$，$[z^n]G_k(n)/p(n)=O(n^{-k/2})$；因此$c(m)/p(m)=1+O(m^{-1/2})$。对于小的$m$，比值$(c(m)+1)/p(m)$很容易计算，在$m=7$时它达到2.6的一个极大值，而后，它就稳定地减少。所以严格地注意渐近误差的界将完成这个证明。

注：伯·弗里斯梯特(B.Fristedt)[*Trans.Amer. Math. Soc.* 337 (1993)，703-735]在给出其他结果的当中还证明了，在n的一个随机分划中k的数目以渐近概率e^{-x}大于$Cx\sqrt{n}$。

52. 在词典顺序下，$\left|{64+13 \atop 13}\right|$个64的分划有$a_1\leqslant13$；而当中的$\left|{50+10 \atop 10}\right|$个有$a_1=14$和$a_2\leqslant10$；等等。因此由这个提示，在词典顺序下分划14 11 9 6 4 3 2 1^{15}前边恰有

$p(64) - 1000000$个分划，使它成为在颠倒词典顺序下的第100万个。

53. 如同在上题中那样，100的 $\left|{80 \atop 12}\right|$ 个分划有$a_1=32$和$a_2 \leqslant 12$，等等。因此，按词典顺序其中$a_1=32$的第100万个分划是32 13 12 8 7 6 5 5 1^{12}。算法H产生它的共轭，即20 8 8 8 6 5 4 3 3 3 3 2 1^{19}。

54. (a) 明显地为真。这个问题恰好是一个准备动作。

(b)真的，但不那么明显。通过考虑费尔利斯框图，以及当$k=a_l'$ 时的等式，如果 $\alpha^T = a_1'a_2'\ldots$，当$k \leqslant \alpha_l'$ 时，我们有

$$a_1+\cdots+a_k+a_1'+\cdots+a_k' \leqslant n+kl$$

因此如果$\alpha \succeq \beta$，而且对于某个l，$a_1'+\cdots+a_l' > b_1'+\cdots+b_l'$，当$k=b_l'$ 时，我们有$n+kl=b_1+\cdots+b_k+b_1'+\cdots+b_l' < a_1+\cdots+a_k+a_1'+\cdots+a_l' \leqslant n+kl$，矛盾。

(c) 如果$c_1c_2\cdots$是一个分划，递归式$c_k=\min(a_1+\cdots+a_k,\ b_1+\cdots+b_k)-(c_1+\cdots+c_{k-1})$显然定义一个最大的下界。而且它是的，因为如果$c_1+\cdots+c_k=a_1+\cdots+a_k$，我们有$0 \leqslant \min(a_{k+1},\ b_{k+1}) \leqslant \min(a_{k+1},\ b_{k+1}+b_1+\cdots+b_k-a_1-\cdots a_k)=c_{k+1} \leqslant a_{k+1} \leqslant a_k=c_k+(c_1+\cdots+c_{k-1})-(a_1+\cdots+a_{k-1}) \leqslant c_k$。

(d) $\alpha \vee \beta=(\alpha^T \wedge \beta^T)^T$。(双重共轭是必要的，因为类似于部分(c)中的一个面向极大的递归可能会不成立。)

(e) $\alpha \wedge \beta$有max (l, m)部分而$\alpha \vee \beta$有min (l, m)部分(考虑它们的共轭的头一个组成)。

(f) 通过部分(c)的推导，对于$\alpha \wedge \beta$为真。对于$\alpha \vee \beta$为假(尽管在图32中为真)。例如，(17 16 5 4 3 2)$\vee$(17 9 8 7 6)=(17 16 5 5 4)。

参考文献：托 · 布里劳斯基(T.Brylawski)，*Discrete Mathematics* 6 (1973)，201-219。

55. (a) 如果$\alpha \vdash \beta$ 而且$\alpha \succeq \gamma \succeq \beta$ 其中$\gamma=c_1c_2\cdots$，则对于除$k=l$和$k=l+1$之外的所有k，$a_1+\cdots+a_k=c_1+\cdots+c_k=b_1+\cdots+b_k$；因此$\alpha$覆盖$\beta$，由此$\beta^T$ 覆盖α^T。

反之，如果$\alpha \succeq \beta$且$\alpha \neq \beta$，我们可以求得$\gamma \succeq \beta$，使得$\alpha \vdash \gamma$或$\gamma^T \vdash \alpha^T$，如下：求满足$a_k > b_k$的最小的k，以及满足$a_k > a_{l+1}$的最小的l。如果$a_l > a_{l+1}+1$，通过$c_k=a_k-[k=l]+[k=l+1]$定义$\gamma=c_1c_2\cdots$。如果$a_l=a_{l+1}+1$，求满足$a_{l+1} > a_{l'+1}$的最小的l'。而且如果$a_{l'} > a_{l'+1}+1$，令$c_k=a_k-[k=l']+[k=l'+1]$，否则$c_k=a_k-[k=l]+[k=l'+1]$。 118

(b) 如同在(15)中那样，考虑α和β为n个0和n个1的串。则$\alpha \vdash \beta$当且仅当$\alpha \to \beta$，而且$\beta^T \vdash \alpha^T$当且仅当$\alpha \Rightarrow \beta$，其中‘$\to$’表示用101^q01代替形如011^q10的子串，而“$\Rightarrow$”表示对于某个$q \geqslant 0$，用100^q01来代替形如010^q10的子串。

(c) 一个分划至多覆盖$[a_1>a_2]+\cdots+[a_{m-1}>a_m]+[a_m \geqslant 2]$个其他的。在$a_m=1$的情况下，分划$\alpha=(n_2+n_1-1)(n_2-2)(n_2-3)\cdots 21$使这个量成为极大。对于$a_m \geqslant 2$的情况给不出改进。(共轭分划，即$(n_2-1)(n_2-2)\cdots 21^{n_1+1}$，就是这个样子。因此，$\alpha$和$\alpha^T$ 也被其他的极大个数所覆盖。)

(d) 等价地，μ的连续部分至多相差1，而且最小部分为1，边界表示没有连续的1。

(e) 使用边界表示和用关系→来代替⊢。如果$\alpha\to\alpha_1$和$\alpha\to\alpha_1'$，则我们可以很容易地证明一个串β的存在，使得$\alpha_1\to\beta$和$\alpha_1'\to\beta$。例如：

$$\begin{array}{ccccc} & \nearrow & 101^q0111^r10 & \searrow & \\ 011^q1011^r10 & & & & 101^q1011^r01 \\ & \searrow & 011^q1101^r01 & \nearrow & \end{array}$$

令$\beta=\beta_2\vdash\cdots\vdash\beta_m$，其中$\beta_m$为极小。那么，通过对于max$(k, k')$的归纳法，我们有$k=m$和$\alpha_k=\beta_m$；而且$k'=m$和$\alpha'_{k'}=\beta_m$。

(f) 置$\beta\leftarrow\alpha^T$；然后重复地置$\beta\leftarrow\beta'$直到β为极小为止，并使用任何方便的分划β'使得$\beta\vdash\beta'$。所需的分划为β^T。

证明：令$\mu(\alpha)$是在部分(e)中的公共值$\alpha_k=\alpha'_{k'}$，我们必须证明$\alpha\succeq\beta$意味着$\mu(\alpha)\succeq\mu(\beta)$。对于$0\leqslant j<k$，有一个序列$\alpha=\alpha_0,\cdots,\alpha_k=\beta$，其中$\alpha_j\to\alpha_{j+1}$或$\alpha_j\Rightarrow\alpha_j+1$。如果$\alpha_0\to\alpha_1$，我们有$\mu(\alpha)=\mu(\alpha_1)$；因此只要证明$\alpha\Rightarrow\beta$和$\alpha\to\alpha'$意味着$\alpha'\succeq\mu(\beta)$即可。但我们有，例如：

$$\begin{array}{ccccc} & \nearrow\!\!\!\nearrow & 100^q0111^r10 & \searrow & \\ 010^q1011^r10 & & & & 100^q1011^r01 \\ & \searrow & 010^q1101^r01\to 010^{q-1}10011^r01 & \nearrow\!\!\!\nearrow & \end{array}$$

因为我们可以假定$q>0$；而且其余情况类似。

(g) λ_n 的诸部分是，对于$1\leqslant k<n_2$，$a_k=n_2+[k\leqslant n_1]-k$；$\lambda_n^T$ 的诸部分是，对于$1\leqslant k\leqslant n_2$，$b_k=n_2-k+[n_2-k<n_1]$。(f) 的算法在$\binom{n_2+1}{3}-\binom{n_2-n_1}{2}$步之后从$n^1$达到$\lambda_n^T$，因为每步使$\sum kb_k=\sum\binom{a_k+1}{2}$加1。

(h) 当$n\geqslant 3$时，长度为$2n-4$的通路n，$(n-1)1$，$(n-2)2$，$(n-2)11$，$(n-3)21$，$\cdots$，321^{n-5}，31^{n-3}，221^{n-4}，21^{n-2}，1^n是最短的。

可以证明，最长的通路有$m=2\binom{n_2}{3}+n_1(n_2-1)$步。对于$0\leqslant j<l$，一个这样的通路有$\alpha_0,\cdots,\alpha_k,\cdots,\alpha_l,\cdots,\alpha_m$的形式，其中$\alpha_0=n^1$；$\alpha_k=\lambda_n$；$\alpha_l=\lambda_n^T$；$\alpha_j\vdash\alpha_{j+1}$。而且对于$k\leqslant j<m$，$\alpha_{j+1}^T\vdash\alpha_j^T$。

参考文献：卡·格林(C.Greene)和丹·J·克莱特曼(D.J.Kleitman)，*Europ.J. Combinatorics* 7 (1986)，1-10。

56. 假设$\lambda=u_1\cdots u_m$和$\mu=v_1\cdots v_m$。以下(未优化的)算法应用习题54的理论来生成在协词典顺序下的分划，并且维持$\alpha=a_1a_2\cdots a_m\preceq\mu$以及$\alpha^T=b_1b_2\cdots b_l\preceq\lambda^T$。为求出
119 的后继，我们首先求出能增加b_j的最大的j。然后，我们有$\beta=b_1\cdots b_{j-1}(b_j+1)1\cdots1\preceq\lambda^T$，因此所希后继是$\beta^T\wedge\mu$。此算法维持辅助表$r_j=b_j+\cdots+b_l$，$s_j=v_1+\cdots+v_j$以及$t_j=w_j+w_{j+1}+\cdots$，其中$\lambda^T=w_1w_2\cdots$。

M1. [初始化。] 置$q\leftarrow0$，$k\leftarrow u_1$。对于$j=1,\cdots,m$，当$u_{j+1}<k$时，置$t_k\leftarrow q\leftarrow q+j$和$k\leftarrow k-1$。然后再次置$q\leftarrow0$，而且对于$j=1,\cdots,m$，置$a_j\leftarrow v_j$，$s_j\leftarrow q\leftarrow q+a_j$。

然后再次置$q\leftarrow 0$，以及$k\leftarrow l\leftarrow a_1$。对于$j=1, \cdots, m$，当$a_{j+1}<k$时置$b_k\leftarrow j$，$r_k\leftarrow q\leftarrow q+j$，以及$k\leftarrow k-1$。最后置$t_1\leftarrow 0$，$b_0\leftarrow 0$，$b_{-1}\leftarrow -1$。

M2. [访问。] 访问分划$a_1\cdots a_m$和/或它的共轭$b_1\cdots b_l$。

M3. [求j。] 令j是使得$r_{j+1}>t_{j+1}$和$b_j\neq b_{j-1}$的最大的$<l$的整数。如果$j=0$则此算法结束。

M4. [增加b_j。] 置$x\leftarrow r_{j+1}-1$，$k\leftarrow b_j$，$b_j\leftarrow k+1$且$a_{k+1}\leftarrow j$。(a_{k+1}以前的值是$j-1$。现在使用实质上是习题54(c)的方法，我们打算更新$a_1\cdots a_k$来在列$j+1$，$j+2$，…分布x个点。)

M5. [多数化。] 置$z\leftarrow 0$，而后对于$i=1, \cdots, k$，做以下工作：置$x\leftarrow x+j$，$y\leftarrow \min(x, s_i)$，$a_i\leftarrow y-z$，$z\leftarrow y$；如果$i=1$，则置$l\leftarrow p\leftarrow a_1$和$q\leftarrow 0$；如果$i>1$，当$p>a_i$时置$b_p\leftarrow i-1$，$r_p\leftarrow q\leftarrow q+i-1$，$p\leftarrow p-1$。最后当$p>j$时，置$b_p\leftarrow k$，$r_p\leftarrow q\leftarrow q+k$，$p\leftarrow p-1$，返回M2。 ▮

57. 如果$\lambda=\mu^T$，显然仅有一个这样的矩阵，它实质上是λ的费尔利斯框图。而且条件$\lambda\preceq\mu^T$是必要的，因为如果$\mu^T=b_1 b_2\cdots$，我们有$b_1+\cdots+b_k=\min(c_1, k)+\min(c_2, k)+\cdots$，而这个量必定不小于在头$k$行中1的个数。最后，如果有对于$\lambda$和$\mu$的一个矩阵，以及如果$\lambda$覆盖$\alpha$，通过从任何特定的行把1移动到有较少的1的另外的行，我们可以很容易地构造α和μ的一个矩阵。

注：这个结果通常叫做盖尔(Gale)－赖泽(Ryser)定理，这是因为由戴·盖尔[*Pacific J. Math.* 7 (1957), 1073–1082]和赫·约·赖泽[*Canadian J. Math.* 9 (1957), 371-377]的著名文章所致。但是具有行的和为λ和列的和为μ的0－1矩阵的个数是在初等对称函数$\sum x_{i_1}^{c_1}x_{i_2}^{c_2}\cdots$的乘积中，单项对称函数$e_{r_1}e_{r_2}\cdots$的系数，其中

$$e_r=[z^r](1+x_1 z)(1+x_2 z)(1+x_3 z)\cdots$$

在这个范畴中，这个结果至少自20世纪30年代以来就为人所知了。参见在[*Proc. London Math. Soc.* (2)40 (1936), 40-70]中关于$\prod_{m,n>0}(1+x_m y_n)$的约·埃·里特伍德的公式。[凯利在*Philosophical Trans.*147 (1857), 489-499中更早得多地证明了，词典顺序$\lambda\leqslant\mu^T$是必要的。]

58. [罗·富·缪尔黑德(R.F.Muirhead), *Proc. Edinburgh Math. Soc.* 21 (1903), 144-157。]条件$\alpha\succeq\beta$是必要的，因为我们可以置$x_1=\cdots=x_k=x$和$x_{k+1}=\cdots=x_n=1$，且令$x\to\infty$。它是充分的，因为我们仅须当α覆盖β时证明它。然后，比如说，如果部分$(a_1 a_2)$变成(a_1-1, a_2+1)，则左边是右边加上非负的量

$$\frac{1}{2m!}\sum x_{p_1}^{a_2}x_{p_2}^{a_2}\cdots x_{p_m}^{a_m}\left(x_{p_1}^{a_1-a_2-1}-x_{p_2}^{a_1-a_2-1}\right)(x_{p_1}-x_{p_2})$$

[历史注记：缪尔黑德的论文是现在叫做多数化概念的最早记载。不久之后，马·奥·罗仁兹在(*Quarterly Publ. Amer. Stat. Assoc.* 9 (1905), 209-219)上给出一个等价的定义。罗仁兹对于测定财富的非一致分布感兴趣。还有另外一个等价概 120
念，由伊·舒尔在(*Sitzungsberichte Berliner Math. Gesellschaft* 22 (1923), 9-20)上

系统阐述。"多数化"是由哈迪、里特伍德和波利亚起的名字，他们在(*Messenger of Math*. 58 (1929)，145-152)中确定了它的最基本性质，参见习题2.3.4.5-17。由阿·瓦·马歇尔(A.W.Marshall)和英·奥尔金(I.Olkin)(Academic Press，1979)所写的一本杰出的书*Inequalities*，完全专注于这个课题。]

59. 对于n=0, 1, 2, 3, 4和6的惟一通路必定有所指出的对称性。对于n=5有一个这样的通路，即11111，2111，221，311，32，41，5。对于n=7，有4条这样的通路:

1111111, 211111, 22111, 2221, 322, 3211, 31111, 4111, 511, 421, 331, 43, 52, 61, 7;
1111111, 211111, 22111, 2221, 322, 421, 511, 4111, 31111, 3211, 331, 43, 52, 61, 7;
1111111, 211111, 31111, 22111, 2221, 322, 3211, 4111, 421, 331, 43, 52, 511, 61, 7;
1111111, 211111, 31111, 22111, 2221, 322, 421, 4111, 3211, 331, 43, 52, 511, 61, 7

没有其他的了，因为对于所有$n \geq 8$，至少有两个共轭分划存在(参见习题16)。

60. 对于$L(6, 6)$，使用(59)；否则在每一处使用$L'(4, 6)$和$L'(3, 5)$。

在$M(4, 18)$中，在443322和4432221之间插入444222，4442211。

在$M(5, 11)$中，在62111和6221之间插入52211，5222。

在$M(5, 20)$中，在5552111和555221之间插入5542211，554222。

在$M(6, 13)$中，在62221和6322之间插入72211，7222。

在$L(4, 14)$中，在43322和432221之间插入44222，442211。

在$L(5, 15)$中，在552111和55221之间插入542211，54222。

在$L(7, 12)$中，在72111和7221之间插入62211，6222。

62. 对于n=7, 8和9，除了两种情况，即n=8，m=3，α=3221和n=9，m=4，α=432外，命题成立。

64. 如果$n=2^k q$，其中q为奇数，令ω_n表示分划$(2^k)^q$，即q个部分等于2^k。对于$n>0$，递归规则

$$B(n)=B(n-1)^R 1, 2\times B(n/2)$$

其中$2\times B(n/2)$表示加倍$B(n/2)$的所有部分(或者如果n为奇数则使序列为空)，定义一条令人愉快的格雷通路，如果我们令$B(0)$是0的惟一分划，则此通路以$\omega_{n-1}1$开始并以ω_n结束。因此

$$B(1)=1,\ B(2)=11, 2,\ B(3)=21, 111,\ B(4)=1111, 211, 22, 4$$

在这个序列所满足的重要性质当中，以下事实为其中之一，即当n为偶数时

$$B(n)=(2\times B(0))1^n, (2\times B(1))1^{n-2}, (2\times B(2))1^{n-4}, \ldots, (2\times B(n/2))1^0$$

例如

$$B(8)=11111111, 2111111, 221111, 41111, 4211, 22211, 2222, 422, 44, 8$$

当$n \geq 2$时，以下算法无循环地生成$B(n)$:

K1. [初始化。] 置$c_0 \leftarrow p_0 \leftarrow 0$，$p_1 \leftarrow 1$。如果$n$为偶数，置$c_1 \leftarrow n$，$t \leftarrow 1$；否则令$n-1=2^k q$，其中$q$为奇数，并置$c_1 \leftarrow 1$，$c_2 \leftarrow q$，$p_2 \leftarrow 2^k$，$t \leftarrow 2$。

K2.[偶访问。] 访问分划 $p_t^{c_t}\cdots p_1^{c_1}$。(现在$c_t+\cdots+c_1$为偶数。)

K3.[改变最大部分。] 如果$c_t=1$，把最大部分分开；如果$p_t\neq 2p_{t-1}$，置$c_t\leftarrow 2$，$p_t\leftarrow p_t/2$，否则置$c_{t-1}\leftarrow c_{t-1}+2$，$t\leftarrow t-1$。但如果$c_t>1$，则合并两个最大部分；如果$c_t=2$，置$c_t\leftarrow 1$，$p_t\leftarrow 2p_t$，否则置$c_t\leftarrow c_t-2$，$c_{t+1}\leftarrow 1$，$p_{t+1}\leftarrow 2p_t$，$t\leftarrow t+1$。 121

K4.[奇访问。] 访问分划 $p_t^{c_t}\cdots p_1^{c_1}$。(现在$c_t+\cdots+c_1$为奇数。)

K5.[改变次最大部分。] 现在我们希望来应用以下变换："暂时删去$c_t-[t$为偶数$]$个最大部分，然后应用步骤K3，然后恢复被删去的部分。"更精确地说，有9种情况：(1a)如果c_t为奇数和$t=1$，则结束。(1b1) 如果c_t为奇数，$c_{t-1}=1$，而且$p_{t-1}=2p_{t-2}$，则置$c_{t-2}\leftarrow c_{t-2}+2$，$c_{t-1}\leftarrow c_t$，$p_{t-1}\leftarrow p_t$，$t\leftarrow t-1$；(1b2) 如果$c_t$为奇数，$c_{t-1}=1$，而且$p_{t-1}\neq 2p_{t-2}$则置$c_{t-1}\leftarrow 2$，$p_{t-1}\leftarrow p_{t-1}/2$。(1c1) 如果$c_t$为奇数，$c_{t-1}=2$，而且$p_t=2p_{t-1}$，则置$c_{t-1}\leftarrow c_t+1$，$p_{t-1}\leftarrow p_t$，$t\leftarrow t-1$；(1c2) 如果$c_t$为奇数，$c_{t-1}=2$，而且$p_t\neq 2p_{t-1}$，则置$c_{t-1}\leftarrow 1$，$p_{t-1}\leftarrow 2p_{t-1}$。(1d1) 如果$c_t$为奇数，$c_{t-1}>2$，而且$p_t=2p_{t-1}$，则置$c_{t-1}\leftarrow c_{t-1}-2$，$c_t\leftarrow c_t+1$；(1d2) 如果$c_t$为奇数，$c_{t-1}>2$，而且$p_t\neq 2p_{t-1}$，则置$c_{t+1}\leftarrow c_t$，$p_{t+1}\leftarrow p_t$，$c_t\leftarrow 1$，$p_t\leftarrow 2p_{t-1}$，$c_{t-1}\leftarrow c_{t-1}-2$，$t\leftarrow t+1$。(2a)如果$c_t$为偶数，且$p_t=2p_{t-1}$，则置$c_t\leftarrow c_t-1$，$c_{t-1}\leftarrow c_{t-1}+2$。(2b)如果$c_t$为偶数，且$p_t\neq 2p_{t-1}$，则置$c_{t+1}\leftarrow c_t-1$，$p_{t+1}\leftarrow p_t$，$c_t\leftarrow 2$，$p_t\leftarrow p_t/2$，$t\leftarrow t+1$。返回K2。 ▌

[当在一行中实施两次时，在K3和K5中的变换对它们自己实行返工。这个构造是由托·科特哈斯特(T.Colthurst)和迈·克里比尔(M.Kleber)在有待发表的"A Gray path on binary partitions"中给出的。欧拉在1750年曾在他的论文的§50中考虑过这种分划的个数。]

65. 如果 $p_1^{e_1}\ldots p_r^{e_r}$ 是m的素因子分解，则这样的因子分解的个数是$p(e_1)\cdots p(e_r)$，而且我们可以令$n=\max(e_1,\cdots,e_r)$。其实，对于满足$0\leqslant x_k<p(e_k)$的每一个r元组$(x_1,\cdots,x_r)$，我们可以令 $m_j=p_1^{a_{1j}}\ldots p_r^{a_{rj}}$，其中$a_{k1}\cdots a_{kn}$是$e_k$的第$x_k+1$个分划。于是我们可以同分划的一个格雷码一起使用一个反射的格雷码。

66. 令$a_1\cdots a_m$是满足特定不等式的一个m元组。我们可以把它排序成非递增的次序$a_{x_1}\geqslant\cdots\geqslant a_{x_m}$。其中如果我们要求排序是稳定的，则排列$x_1\cdots x_m$是惟一确定的。参见等式5－(2)。

如果$j\prec k$，我们有$a_j\geqslant a_k$，因此j出现在排列$x_1\cdots x_m$中k的左边。因此$x_1\cdots x_m$是由算法7.2.1.2V输出的排列之一。而且，当$a_j=a_k$且$j<k$时，由稳定性，j也将在k的左边。因此当$x_i>x_{i+1}$是一个"下降"时，a_{x_i}严格地大于$a_{x_{i+1}}$。

为生成n的所有有关的分划，取每一个拓扑排列$x_1\cdots x_m$，并生成$n-t$的分划$y_1\cdots y_m$，其中t是$x_1\cdots x_m$的下标(参见5.1.1节)。对于$1\leqslant j\leqslant m$，置$a_{x_j}\leftarrow y_j+t_j$，其中$t_j$是在$x_i\cdots x_m$中$x_j$右边的下降个数。

例如，如果$x_1\cdots x_m$=314592687，我们要来生成满足$a_3>a_1\geqslant a_4\geqslant a_5\geqslant a_9>a_2\geqslant$

$a_6 \geq a_8 > a_7$的所有情况。在此情况下，t=1+5+8=14；所以我们置$a_1 \leftarrow y_2+2$，$a_2 \leftarrow y_6+1$，$a_3 \leftarrow y_1+3$，$a_4 \leftarrow y_3+2$，$a_5 \leftarrow y_4+2$，$a_6 \leftarrow y_7+1$，$a_7 \leftarrow y_9$，$a_8 \leftarrow y_8+1$以及$a_9 \leftarrow y_5+2$。在习题29的意义下，广义的生成函数$\sum z_1^{a_1}\ldots z_9^{a_9}$为：

$$\frac{z_1^2 z_2 z_3^3 z_4^2 z_5^2 z_6 z_8 z_9^2}{(1-z_3)(1-z_3z_1)(1-z_3z_1z_4)(1-z_3z_1z_4z_5)\cdots(1-z_3z_1z_4z_5z_9z_2z_6z_8z_7)}$$

当$\prec$是任何给定的偏序时，对于n的所有这样的分划，通常生成函数因此是$\sum z^{\operatorname{ind}\alpha}/((1-z)(1-z^2)\cdots(1-z^m))$，其中的求和是对于算法7.2.1.2V的所有输出$\alpha$进行的。

[关于这些思想的重要扩展和应用，参见理·彼·斯坦利，*Memoirs Amer. Math. Soc.* **119** (1972)。关于由顶向下分划的信息，也可参见伦·卡利兹(L.Carlitz)，*Studies in Foundations and Combinatorics* (New York: Academic Press,

122 1978)，101-129。]

67. 如果$n+1=q_1\cdots q_r$，其中所有因子$q_1,\cdots,q_r$都≥ 2，我们得到一个完美分划$\{(q_1-1)\cdot 1, (q_2-1)\cdot q_1, (q_3-1)\cdot q_1q_2, \cdots, (q_r-1)\cdot q_1\cdots q_{r-1}\}$，该分划以一个明显的方式对应于混合进制记号。(这些因子q_j的顺序是重要的。)

反之，所有完美分划都以这种方式产生出来。假设多重集合$M=\{k_1\cdot p_1, \cdots, k_m\cdot p_m\}$是一个完美分划，其中$p_1<\cdots<p_m$；则对于$1\leq j\leq m$，我们必须有$p_j=(k_1+1)\cdots(k_{j-1}+1)$，因为$p_j$是$M$的一个子多重子集的最小和，$M$不是$\{k_1\cdot p_1, \cdots, k_{j-1}\cdot p_{j-1}\}$的子多重子集。

具有最少元素的n的完美分划出现当且仅当所有的q_j是素数，因为每当$p>1$和$q>1$时，$pq-1>(p-1)+(q-1)$。因此，例如，11的极小完美分划对应于通常的因式分解2·2·3，2·3·2和3·2·2。参考文献：*Quarterly Journal of Mathematics* **21** (1886)，367-373。

68. (a) 对于某个i和j，如果$a_i+1\leq a_j-1$，则我们可以把$\{a_i, a_j\}$改变成$\{a_i+1, a_j-1\}$。由此使乘积增加$a_j-a_i-1>0$。因此仅在习题3的最优地平衡分划下出现最优。[路·乌廷格(L.Oettinger)和约·德比斯(J.Derbès)，*Nouv. Ann. Math.* **18** (1859)，442；**19** (1860)，117-118。]

(b) 没有为1的部分；如果$a_j\geq 4$，我们可以把它改成$2+(a_j-2)$而不减少乘积。因此我们可以假定所有部分为2或3。通过把2+2+2改成3+3，我们得到一个改进，因此至多有两个2。因此当n mod 3为0时，最优是$3^{n/3}$；当n mod 3为1时，$4\cdot 3^{(n-4)/3}=3^{(n-4)/3}\cdot 2\cdot 2=(4/3^{4/3})\,3^{n/3}$。当$n$ mod 3为2时，$3^{(n-2)/3}\cdot 2=(2/3^{2/3})3^{n/3}$。[奥·梅布纳(O.MeiBner)，*Mathematisch－naturwissenschaftliche Blätter* **4** (1907)，85。]

69. 所有$n>2$都有解$(n, 2, 1, \cdots, 1)$。通过以$s_2\cdots s_N\leftarrow 1\cdots 1$开始，以及每当$ak-b<N$时置$s_{ak-b}\leftarrow 0$(其中$a=x_1\cdots x_t-1$，$b=x_1+\cdots+x_t-t-1$，$k\geq x_1\geq\cdots\geq x_t$，以及$a>1$)，我们可以“筛出”$<N$的其他情况，因为$k+x_1+\cdots+x_t+(ak-b-t-1)=kx_1\cdots x_t$。仅当$(x_1\cdots x_t-1)x_1-(x_1+\cdots+x_t)<N-t$时，才需要考虑序列$(x_1, \cdots, x_t)$。我们也可继续减小

N使得$s_N=1$。这样一来，当N初始为2^{30}时，仅仅(32766，1486539，254887，1511，937，478，4)序列$(x_1, \cdots, x_t)$需要加以试验，而仅有的幸存者结果是2，3，4，6，24，114，174和444。[参见恩·特洛斯特(E.Trost)，*Elemente der Math.* **11** (1956)，135；迈·米希尤里维奇(M.Misiurewicz)，*Elemente der Math.* **21** (1966)，90。]

注：当$N\to\infty$时，不大可能有新的幸存者了，但需要有新的思想来把它们删除掉。最简单的序列$(x_1, \cdots, x_t)$=(3)和(2, 2)，已经排除满足$n \bmod 6 \neq 0$的所有>5的n；这一事实可以被用来把计算加速6的一个因子。序列(6)和(3, 2)排除剩下的40%(即排除形如$5k-4$和$5k-2$的所有n)；序列(8)、(4, 2)和(2, 2, 2)排除剩下的3/7；具有$t=1$的序列意味着$n-1$必须是素数；其中$x_1\cdots x_t=2^r$的序列排除$n \bmod (2^r-1)$的大约$p(r)$个剩余；其中$x_1\cdots x_t$是r个不同素数的乘积的序列将排除$n \bmod (x_1\cdots x_{t-1})$的大约$\varpi_r$个剩余。

70. 每一步把n的一个分划变成另一个，所以我们最终必须达到一个重复的循环。许多分划只不过是对于费尔利斯框图每个从东北到西南的对角线执行一个循环移位，把它

$$\text{从}\ \begin{matrix} x_1 & x_2 & x_4 & x_7 & x_{11} & x_{16} & \cdots \\ x_3 & x_5 & x_8 & x_{12} & x_{17} & x_{23} & \cdots \\ x_6 & x_9 & x_{13} & x_{18} & x_{24} & x_{31} & \cdots \\ x_{10} & x_{14} & x_{19} & x_{25} & x_{32} & x_{40} & \cdots \\ x_{15} & x_{20} & x_{26} & x_{33} & x_{41} & x_{50} & \cdots \\ x_{21} & x_{27} & x_{34} & x_{42} & x_{51} & x_{61} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{matrix}\ \text{改变成}\ \begin{matrix} x_1 & x_3 & x_6 & x_{10} & x_{15} & x_{21} & \cdots \\ x_2 & x_4 & x_7 & x_{11} & x_{16} & x_{22} & \cdots \\ x_5 & x_8 & x_{12} & x_{17} & x_{23} & x_{30} & \cdots \\ x_9 & x_{13} & x_{18} & x_{24} & x_{31} & x_{39} & \cdots \\ x_{14} & x_{19} & x_{25} & x_{32} & x_{40} & x_{49} & \cdots \\ x_{20} & x_{26} & x_{33} & x_{41} & x_{50} & x_{60} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{matrix}\ ;$$

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换句话说，它们应用排列ρ=(1)(23)(456)(78910)…到诸单元。仅当ρ在上面的一个点处引入空单元时出现例外，例如，当x_{11}非空时，x_{10}可能为空。但通过把顶部的行下移，我们可以得到正确的新框图。在对这样一些情况应用ρ之后，把它排序到它适当的位置。这样一种移动总是减少被占用的对角的个数，所以它不能是一个循环的一部分。因此每个循环完全由ρ所组成。

如果在一个循环的分划中，一个对角的任何元素为空，剩下一个对角的所有元素必为空。因为，比如说x_5为空，则重复应用ρ将使x_5相邻于下一个对角的每个单元x_7, x_8, x_9, x_{10}。因此，如果对于$n_2 > n_1 \geqslant 0$的$n=\binom{n_2}{2}+\binom{n_1}{1}$，循环状态精确地是有$n_2-1$个完全填满的对角和在下一个对角的$n_1$个点。[这一结果是由约·布兰特(J.Brandt)给出的，*Proc. Amer. Math. Soc.* **85** (1982)，483-486。问题的来源不清楚；参见马丁·加德纳，*The Last Recreations* (1997)，第2章。]

71. 当$n=1+\cdots+m>1$时，起始的分划$(m-1)(m-1)(m-2)\cdots 211$从循环状态算起有$m(m-1)$的距离，而这是极大值。[井草洁，*Math. Magazine* **58** (1985)，259-271；格·埃添尼(G.Etienne)，*J. Combin. Theory* **A58** (1991)，181-197。]在一般情况下，格里格斯(Griggs)和何志昌[*Advances in Appl. Math.* **21** (1998)，205-227]猜测，对所有的$n>1$，到一个循环的极大距离是$\max(2n+2-n_1(n_2+1), n+n_2+1, n_1(n_2+1))-2n_2$；对于$n<100$，他们的猜测已得到验证。而且，当$n_2=2n_1+\{-1, 0, 2\}$

时，最坏情况的开始分划看来是惟一的。

72. (a) 对于$n-jk$的每一个分划α，把在分划$n=j\cdot k+\alpha$中k的第j个出现同在$k\cdot j+\alpha$中j的第k个出现加以交换。例如，当n=6时，这些交换是

6　51，42，　411，33，321，　3111，222，　2211，　21111，　111111

a　b1，fg，c1g，hi，jkl，d1kh，n2i，m21n，elmjf，ledcba

(b) $p(n-k)+p(n-2k)+p(n-3k)+\cdots$。[阿 · 霍 · 马 · 霍尔，*AMM* 93 (1986)，475-476]。

7.2.1.5节

1. 每当在步骤H6中m被置为等于r时，把它改回成$r-1$。

2. L1. [初始化。] 对于$1\leqslant j\leqslant n$置$l_j\leftarrow j-1$和$a_j\leftarrow 0$。并置$h_1\leftarrow n$，$t\leftarrow 1$，并把l_0置为任何方便的非零值。

L2. [访问。] 访问通过$l_1\cdots l_n$和$h_1\cdots h_t$表示的t块分划。(对应于这个分划的限制增长串为$a_1\cdots a_n$。)

L3. [求j。] 置$j\leftarrow n$；然后当l_j=0时置$j\leftarrow j-1$，以及$t\leftarrow t-1$。

L4. [把j移到下一块。] 如果j=0则结束。否则置$k\leftarrow a_j+1$，$h_k\leftarrow l_j$，$a_j\leftarrow k$。如果$k=t$，则置$t\leftarrow t+1$且$l_j\leftarrow 0$；否则置$l_j\leftarrow h_{k+1}$，最后置$h_{k+1}\leftarrow j$。

L5. [把$j+1$，…，n移到块1。] 当$j<n$时，置$j\leftarrow j+1$，$l_j\leftarrow h_1$，$a_j\leftarrow 0$以及$h_1\leftarrow j$，返回L2。 ▮

3. 令$\tau(k,n)$是满足如下条件的串$a_1\cdots a_n$的个数：即对于$1\leqslant j\leqslant n$，$0\leqslant a_j\leqslant 1+\max(k-1,a_1,\cdots,a_{j-1})$；因此，$\tau(k,0)=1$，$\tau(0,n)=\varpi_n$，以及$\tau(k,n)=k\tau(k,n-1)+\tau(k+1,n-1)$。[斯 · 吉 · 威廉森(S.G.Williamson)把$\tau(k,n)$称作"尾部系数"；参见*SICOMP* 5 (1976)，602-617。]在一个给定的限制增长串$a_1\cdots a_n$之前，由算法H生成的串的个数是$\sum_{j=1}^{n}a_j\tau(b_j,n-j)$，其中$b_j=1+\max(a_1,\cdots,a_{j-1})$。在尾部系数的一个预先计算的表的帮助下，向后工作，我们发现，当$a_1\cdots a_{12}$=010220345041时
124 这个公式产生999999。

4. 以在图库中对应的出现个数为下标，每种类型最普通的代表是$\texttt{zzzzz}_0$，$\texttt{ooooh}_0$，$\texttt{xxxix}_0$，$\texttt{xxxii}_0$，$\texttt{ooops}_0$，$\texttt{llull}_0$，$\texttt{llala}_0$，$\texttt{eeler}_0$，$\texttt{iitti}_0$，$\texttt{xxiii}_0$，$\texttt{ccxxv}_0$，$\texttt{eerie}_1$，$\texttt{llama}_1$，$\texttt{xxvii}_0$，$\texttt{oozed}_5$，$\texttt{uhuuu}_0$，$\texttt{mamma}_1$，$\texttt{puppy}_{28}$，$\texttt{anana}_0$，$\texttt{hehee}_0$，$\texttt{vivid}_{15}$，$\texttt{rarer}_3$，$\texttt{etext}_1$，$\texttt{amass}_2$，$\texttt{again}_{137}$，$\texttt{ahhaa}_0$，$\texttt{esses}_1$，$\texttt{teeth}_{25}$，$\texttt{yaaay}_0$，$\texttt{ahhhh}_2$，$\texttt{pssst}_2$，$\texttt{seems}_7$，$\texttt{added}_6$，$\texttt{lxxii}_0$，$\texttt{books}_{184}$，$\texttt{swiss}_3$，$\texttt{sense}_{10}$，$\texttt{eneded}_3$，$\texttt{check}_{160}$，$\texttt{level}_{18}$，$\texttt{tepee}_4$，$\texttt{slyly}_5$，$\texttt{never}_{154}$，$\texttt{sells}_6$，$\texttt{motto}_{21}$，$\texttt{whooo}_2$，$\texttt{trees}_{384}$，$\texttt{going}_{307}$，$\texttt{which}_{151}$，$\texttt{there}_{174}$，$\texttt{three}_{100}$，$\texttt{their}_{3834}$，

(参见索 · 戈罗姆布，*Math. Mag.* 53 (1980)，219-221。只有两个不同字母的词当然很稀少，带有下标0的这里所列出的18个代表，在更大的词典中或在因特网的英语语言页中可找到。)

5. (a) $112=\rho(0225)$。这个序列是$r(0), r(1), r(4), r(9), r(16), \cdots$，其中$r(n)$是通过以十进记号(有1个或多个前导0)来表达$n$而得到的，并且应用习题4中的$\rho$函数，然后删除前导零。注意$n/9 \leqslant r(n) \leqslant n$。

(b) $1012=r(45^2)$。这个序列和(a)相同，但排成顺序并删去重复元素。(谁知道$88^2=7744, 212^2=44944$以及$264^2=69696$？)

6. 使用算法7.2.1.2V的拓扑排序算法，连同一个适当的偏序：包括长度为j的c_j个链，并且对于它们最小元素排了序。例如，如果$n=20$，$c_2=3$和$c_3=c_4=2$，我们使用该算法来求出$\{1, \cdots, 20\}$的所有排列$a_1 \cdots a_{20}$，使得$1 \prec 2, 3 \prec 4, 5 \prec 6, 1 \prec 3 \prec 5, 7 \prec 8 \prec 9, 10 \prec 11 \prec 12, 7 \prec 10, 13 \prec 14 \prec 15 \prec 16, 17 \prec 18 \prec 19 \prec 20, 13 \prec 17$，并形成限制增长串$\rho(f(a_1), \cdots, f(a_{20}))$，其中$\rho$在习题4中定义而且$(f(1), \cdots, f(20))=(1, 1, 2, 2, 3, 3, 4, 4, 4, 5, 5, 5, 6, 6, 6, 6, 7, 7, 7, 7)$。当然输出的总数由(48)给出。

7. 精确地为ϖ_n。它们是我们通过颠倒在(2)中块的自左到右的顺序并且去掉“|”符号得到的排列：1234, 4123, 3124, 3412, …, 4321。[参见安·克莱森(A.Claesson), *European J. Combinatorics* 22 (2001)，961-971；塞·基塔耶夫(S.Kitaev)在待发表在*Discrete Math.*中的一篇论文“Partially ordered generalized patterns”里已经发现一个意义深远的推广。令π是$\{0, \cdots, r\}$的一个排列，令g_n是$\{1, \cdots, n\}$这样的排列$a_1 \cdots a_n$的个数，即$a_{k-0\pi} > a_{k-1\pi} > \cdots > a_{k-r\pi} > a_j$意味着$j>k$，而且令$f_n$是这样的排列$a_1 \cdots a_n$的个数，即对于它而言，对于$r<k \leqslant n$，模式$a_{k-0\pi} > a_{k-1\pi} > \cdots > a_{k-r\pi}$全加以避免。于是$\sum_{n\geq 0} g_n z^n/n! = \exp\left(\sum_{n\geq 1} f_{n-1} z^n/n!\right)$]。

8. 对于$\{1, \cdots, n\}$分成m个块的每一个分划，按照它们最小元素的递减顺序来安排这些块，并以所有可能的方式来排列非最小的块元素。例如，如果$n=9$和$m=3$，分划126|38|4579将产生457938126，以及通过在它们当中排列$\{5, 7, 9\}$和$\{2, 6\}$所得到的其他11种情况。(实质上，相同的方法产生出恰有k个循环的所有排列；参见1.3.3节的“不寻常的对应”。)

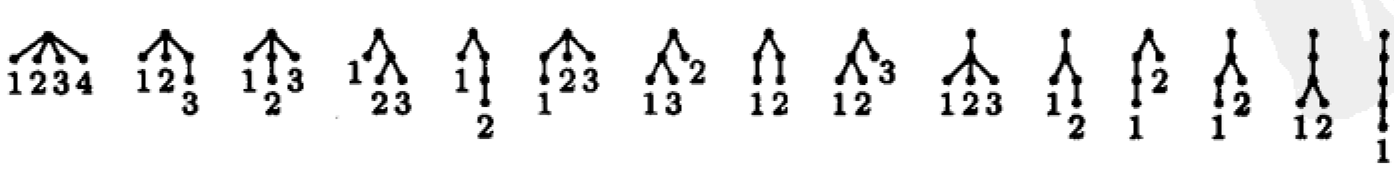
9. 在多重集合$\{k_0 \cdot 0, k_1 \cdot 1, \cdots, k_{n-1} \cdot (n-1)\}$的排列当中，精确地

$$\binom{k_0+k_1+\cdots+k_{n-1}}{k_0, k_1, \cdots, k_{n-1}} \frac{k_0}{(k_0+k_1+\cdots+k_{n-1})} \frac{k_1}{(k_1+\cdots+k_{n-1})} \cdots \frac{k_{n-1}}{k_{n-1}}$$

有限制增长，因为$k_j/(k_j+\cdots+k_{n-1})$是j居于$\{j+1, \cdots, n-1\}$之前的概率。

如果$n>0$，0的平均个数为$1+(n-1)\varpi_{n-1}/\varpi_n=\Theta(\log n)$，因为在所有$\varpi_n$种情况中0的总数为$\sum_{k=1}^{n} k\binom{n-1}{k-1}\varpi_{n-k} = \varpi_n + (n-1)\varpi_{n-1}$。

10. 给定$\{1, \cdots, n\}$的一个分划，通过令$j-1$是其最小元素为j的一个块的所有成员的双亲，来构造$\{0, 1, \cdots, n\}$上的一个有向树。然后重新标号这些树，保持顺序， 125
并删去其他标号。例如，(2)中的15个分划分别对应于



为颠倒这一过程，取一个半标号了的树，并且通过考虑在从根到最小叶的通路上首先遇到的节点，然后从根到次最小叶的通路，等等，对它的节点重新赋予标号。叶的数目是$n+1$减去块的数目。[这个构造同习题2.3.4.4－18和该小节中许多枚举紧密相关。参见彼 · L. 埃尔多斯(P.L.Erdös)和拉 · 阿 · 斯泽基里(L.A.Székely)，*Advances in Applied Math.* 10 (1989)，488-496。]

11. 我们从64855集合分成至多10个块的分划的900个获得纯字母算术，对于这些块来说$\rho(a_1\cdots a_{13})=\rho(a_5\cdots a_8a_1\cdots a_4a_9\cdots a_{13})$，而且也从13 788 536集合分划中的563 527得到，对于它们$\rho(a_1\cdots a_{13})<\rho(a_5\cdots a_8a_1\cdots a_4a_9\cdots a_{13})$。最初的一些例子是aaaa+aaaa=baaac, aaaa+aaaa=bbbbc以及aaaa+aaab=baaac；最后是abcd+efgd=dceab(goat+newt=tango)以及abcd+efgd=dceaf (clad+nerd=dance)。[把一个分划生成程序挂钩到一个字母算术解的思想是由阿兰 · 舒特克里夫(Alan Sutcliffe)给出的。]

12. (a)形成$\rho((a_1a_1')\cdots(a_na_n'))$，其中$\rho$在习题4中定义，因为我们有$x\equiv y$(modulo $\Pi\vee\Pi'$)当且仅当$x\equiv y$(modulo Π)和$x\equiv y$(modulo Π')。

(b) 如同在习题2中那样通过链接表示Π，像在算法2.3.3E中那样表示Π'，而且使用该算法使得当$l_j\neq 0$时，$j\equiv l_j$。(为有效起见，我们可以假设Π至少有和Π'一样多的块。)

(c) 当Π的一个块已被分成两个部分时；即当Π'的两个块已被合并在一起时。

(d) $\binom{t}{2}$。

(e) $(2^{s_1-1}-1)+\cdots+(2^{s_t-1}-1)$。

(f) 真的。令$\Pi\vee\Pi'$有块$B_1|B_2|\cdots|B_t$，其中$\Pi=B_1B_2|B_3|\cdots|B_t$。于是，$\Pi'$实质上是满足$B_1\neq B_2$的$\{B_1,\cdots,B_t\}$的一个分划，而且$\Pi\wedge\Pi'$是通过合并$\Pi'$中包含$B_1$的块和包含$B_2$的块而得到的。[满足这个条件的一个有限格叫做下半模块；参见加 · 毕克霍夫(G.Birkhoff)，*Lattice Theory* (1940)，§I.8。习题7.2.1.4-54的多数化格，当例如α=4111和α'=331时，没有这个性质。]

(g) 假的。例如，令Π=0011，Π'=0101。

(h) Π和Π'的块是$\Pi\vee\Pi'$的块的并，所以我们可以假设$\Pi\vee\Pi'=\{1,\cdots,t\}$。像在部分(b)中一样，当$\Pi$有$t-r$个块时，把$j$同$l_j$合并，在$r$步内得到$\Pi$。应用于$\Pi'$的这些合并每一个都将使块的数目减去0或1。因此$b(\Pi')-b(\Pi\wedge\Pi')\leqslant r=b(\Pi\vee\Pi')-b(\Pi)$。

[在(*Algebra Universalis* 10 (1980)，74-95)中，帕 · 普德拉克(P.Pudlák)和吉 · 图马(J.Tůma)证明，对于适当大的n，每一个有限格是$\{1,\cdots,n\}$的分划格的子格。]

13. [参见*Advances in Math.* 26 (1977)，290-305。]如果一个t块分划的j个最大元素出现在单元块中，但其次的$n-j$个元素则不然，我们就说这个分划有$t-j$的阶。定义“斯特林串”Σ_{nt}为t块分划$\Pi_1,\Pi_2,\cdots$的阶的序列。例如Σ_{43}=122333，于是$\Sigma_{tt}=0$。而且通过在后者中以长度为$\binom{t+1}{2}-\binom{d}{2}$的串$d^d\,(d+1)^{d+1}\cdots t^t$代替后者中的每个数字$d$，我们从$\Sigma_{nt}$得到$\Sigma_{(n+1)t}$；例如

$$\Sigma_{53} = 122333_{\wedge}22333_{\wedge}22333_{\wedge}333_{\wedge}333_{\wedge}333$$ 126

其基本思想是考虑算法H的词典顺序生成过程。假设$\Pi=a_1 \cdots a_n$是阶j的t块分划，则它是其限制增长串以$a_1 \cdots a_{n-t+j}$开始的词典顺序下最小的t块分划。在词典顺序下，由Π覆盖的分划是$\Pi_{12}, \Pi_{13}, \Pi_{23}, \Pi_{14}, \Pi_{24}, \Pi_{34}, \cdots, \Pi_{(t-1)t}$，其中$\Pi_{rs}$意味着“$\Pi$的合并块$r$和$s$”（即，将$s-1$的所有出现改变为$r-1$，然后应用$\rho$来得到一个限制增长串）。如果$\Pi'$是从$\Pi_{1(j+1)}$向上，这些中最后的$\binom{t}{2}-\binom{j}{2}$中的任意一个，则$\Pi$是跟随$\Pi'$的最小$t$块分划。例如，如果$\Pi$=001012034，则$n=9$，$t=5$，$j=3$，而且相关的分划$\Pi'$是$\rho(001012004)$，$\rho(001012014)$，$\rho(001012024)$，$\rho(001012030)$，$\rho(001012031)$，$\rho(001012032)$，$\rho(001012033)$。

因此 $f_{nt}(N)=f_{nt}(N-1)+\binom{t}{2}-\binom{j}{2}$，其中$j$是$\Sigma_{nt}$的第$N$个数字。

14. E1. [初始化。] 对于$1\leqslant j\leqslant n$，置$a_j\leftarrow 0$和$b_j\leftarrow d_j\leftarrow 1$。

E2. [访问。] 访问限制增长串$a_1\cdots a_n$。

E3. [求j。] 置$j\leftarrow n$；然后，当$a_j=d_j$时，置$d_j\leftarrow 1-d_j$和$j\leftarrow j-1$。

E4. [完成了吗？] 如果j=1则结束。否则如果d_j=0，则转到E6。

E5. [下移。] 如果$a_j=0$，则置$a_j\leftarrow b_j$，$m\leftarrow a_j+1$，并转到E7。否则，如果$a_j=b_j$，置$a_j\leftarrow b_j-1$，$m\leftarrow b_j$，并转到E7。否则置$a_j\leftarrow a_j-1$，并返回E2。

E6. [上移。] 如果$a_j=b_j-1$，置$a_j\leftarrow b_j$，$m\leftarrow a_j+1$，并转到E7。否则如果$a_j=b_j$，置$a_j\leftarrow 0$，$m\leftarrow b_j$，并转到E7。否则置$a_j\leftarrow a_j+1$并返回E2。

E7. [固定$b_{j+1}\cdots b_n$。] 对于$k=j+1,\cdots,n$置$b_k\leftarrow m$。返回E2。 ▌

[如同在算法H中一样，这个算法可加以广泛优化，j几乎总是等于n。]

15. 它对应于无穷二进制串01011011011…的头n个数字，因为ϖ_{n-1}为偶数当且仅当$n \bmod 3=0$(参见习题23)。

16. 00012，01012，01112，00112，00102，01102，01002，01202，01212，01222，01022，01122，00122，00121，01121，01021，01221，01211，01201，01200，01210，01220，01020，01120，00120。

17. 当σ=0时对于$A_{\mu\nu}$的“向前”和“向后”生成，以及当σ=1时对于$A'_{\mu\nu}$的“向前”和“向后”生成，以下的解使用两个相互递归的过程$f(\mu,\nu,\sigma)$和$b(\mu,\nu,\sigma)$。为开始这一过程，假设$1<m<n$，首先对于$1\leqslant j\leqslant n-m$置$a_j\leftarrow 0$，并对于$1\leqslant j\leqslant m$置$a_{n-m+j}\leftarrow j-1$，然后调用$f(m,n,0)$。

过程$f(\mu,\nu,\sigma)$：如果μ=2，则访问$a_1\cdots a_n$；否则调用$f(\mu-1,\nu-1,(\mu+\sigma) \bmod 2)$。然后，如果$\nu=\mu+1$，做下列步骤：把$a_\mu$从0改为$\mu-1$并访问$a_1\cdots a_n$；重复地置$a_\nu\leftarrow a_\nu-1$并且访问$a_1\cdots a_n$直到$a_\nu$=0为止。但如果$\nu>\mu+1$，则把$a_{\nu-1}$(如果$\mu+\sigma$为奇数)或$a_\mu$(如果$\mu+\sigma$为偶数)从0改成$\mu-1$；然后如果$a_\nu+\sigma$为奇数则调用$b(\mu,\nu-1,0)$，如果$a_\nu+\sigma$是偶数则调用$f(\mu,\nu-1,0)$；而且当$a_\nu>0$时，置$a_\nu\leftarrow a_\nu-1$并且以同样方式调用$b(\mu,\nu-1,0)$或$f(\mu,\nu-1,0)$，直到$a_\nu$=0为止。

过程$b(\mu,\nu,\sigma)$：如果$\nu=\mu+1$，首先做下列动作：重复地访问$a_1\cdots a_n$并置$a_\nu\leftarrow a_\nu+1$，直到$a_\nu=\mu-1$为止；然后访问$a_1\cdots a_n$并把a_μ从$\mu-1$改为0。但如果$\nu>\mu+1$，则如果$a_\nu+\sigma$为奇，调用$f(\mu,\nu-1,0)$，如果$a_\nu+\sigma$为偶，调用$b(\mu,\nu-1,0)$；然后当$a_\nu<\mu-1$时，置$a_\nu\leftarrow a_\nu+1$，并且再次以同样方式调用$f(\mu,\nu-1,0)$或$b(\mu,\nu-1,0)$，直到$a_\nu=\mu-1$为止。最后把$\alpha_{\nu-1}$(如果$\mu+\sigma$为奇数)或a_μ(如果$\mu+\sigma$为偶数)从$\mu-1$改成0。最后，在
127 两种情况下，如果$\mu=2$则访问$a_1\cdots a_n$，否则调用$b(\mu-1,\nu-1,(\mu+\sigma)\bmod 2)$。

大多数运行时间都花费在$\mu=2$的情况下；基于格雷二进制码(以及从拉斯基的实际序列偏离开)的更快的例程可代替这个情况。当$\mu=\nu-1$时也可用一个流水线过程。

18. 这个序列必须以$01\cdots(n-1)$开始(或结束)。由习题32，当$0\neq\delta_n\neq(1)^{0+1+\cdots+(n-1)}$时，即当$n \bmod 12$为4，6，7或9时，不能有这样的格雷码存在。

$n=1,2,3$的情况容易求解。当$n=5$时有1 927 683 326种解存在，因此对于所有$n\geqslant 8$，除了已被排除的那些情况，大概有无限多个解存在。确实，我们大概可通过在后面的答案28(e)中所考虑的串的所有ϖ_{nk}来求这样一个格雷通路，但当$n\equiv 2k+(2,4,5,7)\pmod{12}$时例外。

注：对于$2<m<n$，在习题30中的广义斯特林数$\left\{{n \atop m}\right\}_{-1}$超过1，所以对于$\{1,\cdots,n\}$分成$m$个块的分划，没有这样的格雷码。

19. (a) 把(6)改成为模式$0,2,\cdots,m,\cdots,3,1$或它的颠倒，如同在结尾顺序(7.2.1.3-(45))中那样。

(b) 我们可以推广(8)和(9)，来得到以$0^{n-m}01\cdots(m-1)$开始及分别以$01\cdots(m-1)\alpha$和$0^{n-m-1}01\cdots(m-1)a$结束的序列$A_{mn\alpha}$和$A'_{mn\alpha}$。其中$0\leqslant a\leqslant m-2$且$\alpha$是有$0\leqslant a_j\leqslant m-2$的任何串$a_1\cdots a_{n-m}$。当$2<m<n$时新的规则是

$$A_{m(n+1)(\alpha a)}=\begin{cases}A_{(m-1)n(b\beta)}x_1,A^R_{mn\beta}x_1,A_{mn\alpha}x_2,\cdots,A_{mn\alpha}x_m & ,\text{如果}m\text{是偶数}\\ A'_{(m-1)nb}x_1,A_{mn\alpha}x_1,A^R_{mn\alpha}x_2,\cdots,A_{mn\alpha}x_m, & \text{如果}m\text{是奇数}\end{cases}$$

$$A'_{m(n+1)a}=\begin{cases}A'_{(m-1)nb}x_1,A_{mn\beta}x_1,A^R_{mn\beta}x_2,\cdots,A^R_{mn\beta}x_m, & \text{如果}m\text{是偶数}\\ A_{(m-1)n(b\beta)}x_1,A^R_{mn\beta}x_1,A_{mn\beta}x_2,\cdots,A^R_{mn\beta}x_m, & \text{如果}m\text{是奇数}\end{cases}$$

这里$b=m-3$，$\beta=b^{n-m}$，而$(x_1,\cdots,x_m)$是从$x_1=m-1$到$x_m=a$的一条通路。

20. 012323212122；一般说来，在习题4的记号下，$(a_1\cdots a_n)^T=\rho(a_n\cdots a_1)$。

21. 数$\langle s_0,s_1,s_2,\ldots\rangle=\langle 1,1,2,3,7,12,31,59,164,339,999,\ldots\rangle$满足递归$s_{2n+1}=\sum_k\binom{n}{k}s_{2n-2k}$，$s_{2n+2}=\sum_k\binom{n}{k}(2^k+1)s_{2n-2k}$。这是由于中间元素同其他元素相关联的方式所致。因此$s_{2n}=n![z^n]\exp((e^{2z}-1)/2+e^z-1)$以及$s_{2n+1}=n![z^n]\exp((e^{2z}-1)/2+e^z+z-1)$。通过考虑头一半上的集合分划，我们也有$s_{2n}=\sum_k\left\{{n \atop k}\right\}x_k$和$s_{2n+1}=\sum_k\left\{{n+1 \atop k}\right\}x_{k-1}$，其中$x_n=2x_{n-1}+(n-1)x_{n-2}=n![z^n]\exp(2z+z^2/2)$。[西·萨·莫兹金在(*Proc. Symp. Pure Math.* 19 (1971), 173)中考虑了序列$\langle s_{2n}\rangle$]。

22. (a) 由(16)， $\sum_{k=0}^{\infty} k^n \Pr(X=k) = e^{-1}\sum_{k=0}^{\infty} k^n / k! = \varpi_n$ 。

(b) $\sum_{k=0}^{\infty} k^n \Pr(X=k) = \sum_{k=0}^{\infty} k^n \sum_{j=0}^{m}\binom{j}{k}(-1)^{j-k}/j!$ ，而且我们可以把内求和扩展到 $j=\infty$，因为当$j>n$时， $\sum_k\binom{j}{k}(-1)^k k^n = 0$。因此我们得到 $\sum_{k=0}^{\infty}(k^n/k!)\sum_{l=0}^{\infty}(-1)^l/l!$ $=\varpi_n$。[参见约·奥·伊尔文(J.O.Irwin)，*J. Royal Stat. Soc.* **A118** (1955)，389-404；詹·皮特曼，*AMM* **104** (1997)，201-209。]

23. (a)由(14)每当$f(x)=x^n$时，这个公式成立，所以一般说来它成立。(因此由(16)，我们也有 $\sum_{k=0}^{\infty} f(k)/k! = ef(\varpi)$ 。)

(b) 假设我们已经证明对于k的这个关系，并令 $h(x)=(x-1)^{\underline{k}}f(x), g(x)=f(x+1)$，于是 $f(\varpi+k+1)=g(\varpi+k)=\varpi^{\underline{k}}g(\varpi)=h(\varpi+1)=\varpi h(\varpi)=\varpi^{\underline{k+1}}f(\varpi)$。[参见雅·陶查德(J.Touchard)，*Ann. Soc. Sci. Bruxelles* **53** (1933)，21-31；由约翰·布里萨德(John Blissard)在(*Quart. J. Pure and Applied Math.* **4** (1861)， 279-305)中发明的这个符号的“黑影演算”(以下简称阴影记号——译者)十分有用。但必须小心处理它，因为$f(\varpi)=g(\varpi)$并不意味着$f(\varpi)\,h(\varpi)=g(\varpi)+h(\varpi)$。] 128

(c) 这个提示是习题4.6.2-16(c)的特殊情况。在(b)中置$f(x)=x^n$和$k=p$，然后产生$\varpi_n\equiv\varpi_{p+n}-\varpi_{1+n}$。

(d) 对p取模(modulo)，多项式x^N-1可由$g(x)=x^p-x-1$整除，因为$x^{p^k}\equiv x+k$和 $x^N\equiv x^p\equiv x^{\underline{p}}\equiv x^p-x\equiv 1$(modulo $g(x)$和p)。因此，如果$h(x)=(x^N-1)x^n/g(x)$。我们有 $h(\varpi)\equiv h(\varpi+p)=\varpi^{\underline{p}}h(\varpi)\equiv(\varpi^p-\varpi)h(\varpi)$ ，而且 $0\equiv g(\varpi)h(\varpi)=\varpi^{N+n}-\varpi^n$ (modulo p)。

24. 这个提示由对e用归纳法得出。因为 $x^{\underline{p^e}}=\prod_{k=0}^{p-1}(x-kp^{e-1})^{\underline{p^{e-1}}}$ 。通过对n用归纳法，我们也可证明$x^n\equiv r_n(x)$(modulo $g_1(x)$ and p)意味着

$$x^{p^{e-1}n}\equiv r_n(x)^{p^{e-1}}(\text{modulo } g_e(x), pg_{e-1}(x),\cdots,p^{e-1}g_1(x),\text{和}p^e)$$

因此对于某个带有整系数的多项式$h_k(x)$， $x^{p^{e-1}N}=1+h_0(x)g_e(x)+ph_1(x)g_{e-1}(x)+\cdots+p^{e-1}h_{e-1}(x)g_1(x)+p^e h_e(x)$。取modulo p^e我们有$h_0(\varpi)\varpi^n\equiv h_0(\varpi+p^e)(\varpi+p^e)^n$ $=\varpi^{\underline{p^e}}h_0(\varpi)\varpi^n\equiv(g_e(\varpi)+1)h_0(\varpi)\varpi^n$ ；因此

$$\varpi^{p^{e-1}N+n}=\varpi^n+h_0(\varpi)g_e(\varpi)\varpi^n+ph_1(\varpi)g_{e-1}(\varpi)\varpi^n+\cdots\equiv\varpi^n$$

[当$p=2$时，可应用一个类似的推导，但我们令$g_{j+1}(x)=g_j(x)^2+2[j=2]$，因而我们得到$\varpi_n\equiv\varpi_{n+3\cdot2^e}$ (modulo 2^e)。这些结果是由马歇尔·哈尔给出的；参见*Bull. Amer. Math. Soc.* **40** (1934)，387；*Amer. J. Math.* **70** (1948)，387-388。关于进一步的信息，参见威·弗·伦能(W. F. Lunnon)，彼·阿·巴·普里桑特斯(P. A. B.

Pleasants)，以及尼·马·史蒂芬(N. M. Stephens)，*Acta Arith.* 35 (1979)，1-16。]

25. 第一个不等式通过把一个更一般得多的原理应用于限制增长串的树而得到：在对于所有非根节点p，$\deg(p) \geqslant \deg(\text{parent }(p))$的任何树中，当在级$k$上，$w_k$是节点的总数时，我们有$w_k/w_{k-1} \leqslant w_{k+1}/w_k$。因为如果在$k-1$级上的$m=w_{k-1}$个节点分别有$a_1$，…，$a_m$个子女，则它们至少有$a_1^2+\cdots+a_m^2$个孙子女；因此$w_{k-1}w_{k+1} \geqslant m(a_1^2+\cdots+a_m^2) \geqslant (a_1+\cdots+a_m)^2 = w_k^2$。

对于第二个不等式，注意$\varpi_{n+1}-\varpi_n=\sum_{k=0}^{n}\left(\binom{n}{k}-\binom{n-1}{k-1}\right)\varpi_{n-k}$；因此

$$\frac{\varpi_{n+1}}{\varpi_n}-1=\sum_{k=0}^{n-1}\binom{n-1}{k}\frac{\varpi_{n-k}}{\varpi_n} \leqslant \sum_{k=0}^{n-1}\binom{n-1}{k}\frac{\varpi_{n-k-1}}{\varpi_{n-1}}=\frac{\varpi_n}{\varpi_{n-1}}$$

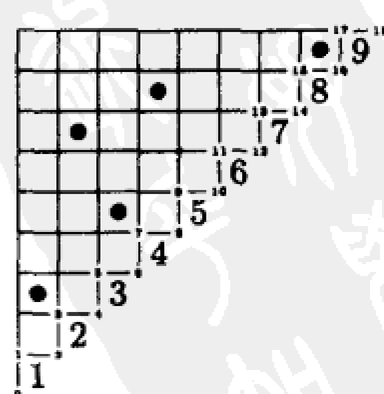
因为，例如$\varpi_{n-3}/\varpi_n=(\varpi_{n-3}/\varpi_{n-2})(\varpi_{n-2}/\varpi_{n-1})(\varpi_{n-1}/\varpi_n)$小于或等于$(\varpi_{n-4}/\varpi_{n-3})(\varpi_{n-3}/\varpi_{n-2})(\varpi_{n-2}/\varpi_{n-1})=\varpi_{n-4}/\varpi_{n-1}$。

26. 从ⓝ①到ⓣⓣ有$\binom{n-1}{n-t}$条向右的通路，我们可以通过0和1来表示它们，其中0意味着“向右”，1意味着“向上”，诸1的位置告诉我们元素中的哪$n-t$个是在具有1的块中。下一步，如果$t>1$，是在远的左边处的另一个顶点，所以我们以定义在剩下的$t-1$个元素上的一个分划的通路继续。例如，分划14|2|3对应于在这些约定之下的通路0010，其中分别的二进位意味着$1 \not\equiv 2$，$1 \not\equiv 3$，$1 \equiv 4$，$2 \not\equiv 3$。[许多其他的解释是可能的，这里提议的约定表明ϖ_{nk}具有$1 \not\equiv 2$，…，$1 \not\equiv k$的分划。这是由哈·W·贝克尔(H. W. Becker)所发现的一个组合性质，参见*AMM* 51 (1944)，47，
129 以及*Mathematics Magazine* 22 (1948)，23-26。]

27. (a) 一般地，$\lambda_0=\lambda_1=\lambda_{2n-1}=\lambda_{2n}=0$，下表也求出经由部分(b)的算法，对应于每个循环的限制增长串：

0,0,0,0,0,0,0,0,0,0	0123	0,0,1,0,0,0,0,0,0,0	0012	0,0,1,1,1,0,0,0,0,0	0102
0,0,0,0,0,0,1,0,0,0	0122	0,0,1,0,0,0,1,0,0,0	0011	0,0,1,1,1,0,1,0,0,0	0100
0,0,0,0,1,0,0,0,0,0	0112	0,0,1,0,1,0,0,0,0,0	0001	0,0,1,1,1,1,1,0,0,0	0120
0,0,0,0,1,0,1,0,0,0	0111	0,0,1,0,1,0,1,0,0,0	0000	0,0,1,1,11,1,1,0,0	0101
0,0,0,0,1,1,1,0,0,0	0121	0,0,1,0,1,1,1,0,0,0	0010	0,0,1,1,2,1,1,0,0,0	0110

(b) “表景”的名字暗示同5.1.4节的一个联系；而且确实在那里建立的理论导致有趣的一对一对应。通过在习题2的链接表表示中每当$l_j \neq 0$时，就在$n+1-j$行的l_j列处放置一个车，我们可以在一个三角形棋盘上表示集合分划(参见习题5.1.3－19的答案)。例如，这里示出135|27|489|6的车表示。同样地，在一个两行数组中可确定非零链接。例如$\binom{12348}{37589}$。参见5.1.4－(11)。



考虑在这个三角形框图左下角开始并沿着右边界的边，最后在右上角处结束的长度为$2n$的一条通路。这个通路的点是：对于$0 \leqslant k \leqslant 2n$，$z_k=(\lfloor k/2\rfloor,\lceil k/2\rceil)$。

而且，z_k 上的矩形和左边的矩形精确地包含一些车，当$i \leqslant \lfloor k/2 \rfloor$和$j> \lceil k/2 \rceil$时，这些车对两行数组贡献坐标对 $\genfrac{}{}{0pt}{}{i}{j}$；在我们的例子中，当$9 \leqslant k \leqslant 12$时，恰好有两个这样的车，即$\binom{2\,4}{7\,8}$。定理5.1.4A告诉我们，这样的两行数组等价于表景(P_k, Q_k)，其中P_k 的元素来自于下行，而Q_k的元素来自上行，而且P_k和Q_k 都有相同的形状。在P表景中使用递减顺序和在Q表景中使用递增顺序是有利的，使得在我们的例子中，对于k=0, 1, 17和18，当P_k 和Q_k 是空的时，它们分别是:

k	P_k	Q_k	k	P_k	Q_k	k	P_k	Q_k
2	3	1	7	7 5	2 3	12	8 7	2 4
3	3	1	8	8 5 7	2 3 4	13	8	4
4	7 3	1 2	9	8 7	2 4	14	8	4
5	7	2	10	8 7	2 4	15	·	·
6	7 5	2 3	11	8 7	2 4	16	9	8

这样一来，如果我们令λ_k 是确定P_k 和Q_k 的共同形状的整数分划时，集合分划导致一个摇摆不定的表景循环$\lambda_0, \lambda_1, \cdots, \lambda_{2n}$。(在我们的例子中，这个循环是0, 0, 1, 1, 11, 1, 2, 2, 21, 11, 11, 11, 11, 1, 1, 0, 1, 0, 0。)而且，t_{2k-1}=0当且仅当行$n+1-k$不包含车，当且仅当k在它的块中是最小的。

反之，从形状序列λ_k 可以重新构造P_k 和Q_k的元素。即，如果t_k =0，则Q_k=Q_{k-1}。否则如果k为偶数，则Q_k 是Q_{k-1}而且数$k/2$被放置在行t_k右边的新单元处；如果k为奇 130
数，则通过使用算法5.1.4D删去行t_k 最右边的条目，而从Q_{k-1} 得到Q_k。一个类似的过程从P_{k+1} 和t_{k+1} 的值定义P_k，所以我们可从P_{2n} 往回工作到P_0，因此形状序列λ_k 足以告诉我们在何处放置诸车。

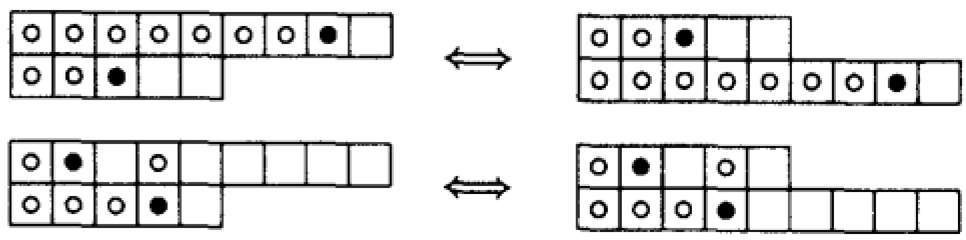
摇摆不定的表景循环是在陈永川、邓玉平、杜若霞、理·彼·斯坦利和颜华菲的论文“匹配和分划的交叉和嵌套”(Crossings and nestings of matchings and partitions)中引入的(预印件，2005)。他们证明，这个构造有重要的(和令人惊讶的)结果。例如，如果集合分划Π对应于摇摆不定的表景循环$\lambda_0, \lambda_1, \cdots, \lambda_{2n}$，我们说它的对偶$\Pi^D$是对应于转置形状序列 $\lambda_0^T, \lambda_1^T, \cdots, \lambda_{2n}^T$ 的集合分划。于是，由习题5.1.4－7，Π包含“在l处的k交叉”，即满足$i_1<\cdots<i_k \leqslant l<j_1<\cdots<j_k$ 和$i_1 \equiv j_1, \cdots, i_k \equiv j_k$ (modulo Π)的下标序列，当且仅当Π^D 包含一个“在l处的k嵌套”，它是满足 $i_1' < \cdots < i_k' \leqslant l < j_k' < \cdots < j_1'$ 和 $i_1' \equiv j_1', \cdots, i_k' \equiv j_k'$ (modulo Π^D)的一个下标序列。也要注意，一个卷积实质上是其中所有块都有大小为1或2的一个集合分划。卷积的对偶是有相同的单元集的一个卷积。特别是，一个完美匹配的对偶(当没有单集时)是一个完美匹配。

而且，一个类似的构造适用于在任何费尔利斯框图中的车的配置，而不仅适

用于对应集合分划的楼梯形状。给定至多有m个部分的费尔利斯框图，这些部分的大小都$\leqslant n$，我们简单地考虑坚持框图的右边的通路$z_0=(0,0),z_1,\cdots,z_{m+n}=(n,m)$。而且假定，当$z_k=z_{k-1}+(1,0)$时 $\lambda_k=\lambda_{k-1}+e_{t_k}$，当$z_k=z_{k-1}+(0,1)$时 $\lambda_k=\lambda_{k-1}-e_{t_k}$，对于楼梯形状我们给出的证明也表明，在费尔利斯框图中车的每一个配置，而且在每行中至多有一个车，在每列中至多也有一个车，它对应于这个种类的一个惟一的表景循环。

[除此之外，还有更多为真的事实。参见S · 福明(S. Fomin)，*J. Combin. Theory* A72 (1995)，277－292；M · 范 · 李欧文(M. van Leeuwen)，*Electronic J. Combinatorics* 3，2 (1996)，论文#R15。]

28. (a) 通过交换在行j和$j+1$处车的位置，当且仅当在较长行的“锅柄”处有一个车，定义车的位置之间的一一对应：



(b) 通过转置所有车，这个关系从定义看是明显的。

(c) 假设$a_1\geqslant a_2\geqslant\cdots$和$a_k>a_{k+1}$，于是我们有

$$R(a_1,a_2,\cdots)=xR(a_1-1,\cdots,a_{k-1}-1,a_{k+1},\cdots)+yR(a_1,\cdots,a_{k-1},a_{k-1},a_{k+1},\cdots)$$

因为头一项统计在其中一个车在行k和列a_k中的情况。而且$R(0)=1$，因为空的配置所致。从这些递归，我们求出

$$R(1)=x+y;\ R(2)=R(1,1)=x+xy+y^2;\ R(3)=R(1,1,1)=x+xy+xy^2+y^3$$
$$R(2,1)=x^2+2xy+xy^2+y^3$$
$$R(3,1)=R(2,2)=R(2,1,1)=x^2+x^2y+xy+2xy^2+xy^3+y^4$$
$$R(3,1,1)=R(3,2)=R(2,2,1)=x^2+2x^2y+x^2y^2+2xy^2+2xy^3+xy^4+y^5$$

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$$R(3,2,1)=x^3+3x^2y+3x^2y^2+x^2y^3+3xy^3+2xy^4+xy^5+y^6$$

(d) 例如，公式$\varpi_{73}(x,y)=x\,\varpi_{63}(x,y)+y\,\varpi_{74}(x,y)$等价于$R(5,4,4,3,2,1)=xR(4,3,3,2,1)+yR(5,4,3,3,2,1)$，此为(c)的一个特殊情况；而且$\varpi_{nn}(x,y)=R(n-2,\cdots,0)$明显地等于$\varpi_{(n-1)1}(x,y)=R((n-2,\cdots,1)$。

(e) 事实上$y^{k-1}\varpi_{nk}(x,y)$是对于$a_2>0,\cdots,a_k>0$的所有限制增长串$a_1\cdots a_n$所述的求和。

29. (a) 如果诸车分别在列$(c_1,\cdots,c_n)$，自由单元的数目是排列$(n+1-c_1)\cdots(n+1-c_n)$的反演的数目。[把图35右边的例子转动180°，并且把这个结果同等式5.1.1-(5)后边的图示作比较。]

(b) 每个$r\times r$配置可以放置在比如说行$i_1<\cdots<i_r$和列$j_1<\cdots<j_r$上，并且在未被选定的行和列中产生$(m-r)(n-r)$个自由单元；在未选定的行和未选定的列处有$(i_2-i_1+1)+2(i_3-i_2-1)+\cdots+(r-1)(i_r-i_{r-1}-1)+r(m-i_r)$，而且在选定的行和选定的列有

类似的数。而且

$$\sum_{1\leqslant i_1<\cdots<i_r\leqslant m} y^{(i_2-i_1+1)+2(i_3-i_2-1)+\cdots+(r-1)(i_r-i_{r-1}-1)+r(m-i_r)}$$

可以看作是对于$r\geqslant a_1\geqslant a_2\geqslant\cdots\geqslant a_{m-r}\geqslant 0$的所有分划上对 $y^{a_1+a_2+\cdots+a_{m-r}}$ 的求和， 所以由定理C，它是 $\binom{m}{r}_y$ 。由(a)，对于选定的行和列，多项式$r!_y$生成自由单元。因此，答案为 $y^{(m-r)(n-r)}\binom{m}{r}_y\binom{n}{r}_y r!_y=y^{(m-r)(n-r)}m!_y\,n!_y/((m-r)!_y\,(n-r)!_y\,r!_y)$。

(c) 对于有t个附加的高度为m的列的费尔利斯框图，左边是生成函数$R_m(t+a_1,\cdots,t+a_m)$。因为有$t+a_m$个方法来把一个车放在行m中，相对于这些选择有 $1+y+\cdots+y^{t+a_m-1}=(1-y^{t+a_m})/(1-y)$ 个自由单元，然后在行$m-1$中有$t+a_{m-1}-1$个可利用单元，等等。

同样，右边等于$R_m(t+a_1,\cdots,t+a_m)$。因为如果把$m-k$个车放在$>t$的列中，则我们必定把k个车放在k个未用的$\leqslant t$的列中。而且我们已经看到，当把k个车放在$k\times t$的一个盘上时，$t!_y/(t-k)!_y$是自由单元的生成函数。

[在这里证明的这个公式可以看作是变量y和y^t的一个多项式恒等式；因此它对于任意t都是正确的，尽管我们假设t是一个非负整数。在y=1的情况下，这个结果是由约·戈德曼(J. Goldman)、城市东明和丹·怀特(D.White)发现的(*Proc. Amer. Math*. Soc. **52** (1975)，485-492)，一般情况是由阿·马·加西亚(A. M. Garsia)和杰·布·蔺梅尔(J. B. Remmel)确立的(*J. Combinatorial Theory* **A41** (1986)，246-275)，他们使用一个类似的论断来证明另外的公式

$$\sum_{t=0}^{\infty} z^t\prod_{j=1}^{m}\frac{1-y^{a_j+m-j+t}}{1-y}=\sum_{k=0}^{n}k!_y\left(\frac{z}{1-yz}\right)\cdots\left(\frac{z}{1-y^k z}\right)R_{m-k}(a_1,\cdots,a_m)$$

(d) 从(c)直接得出的这个断言，也意味着我们有$R(a_1,\cdots,a_m)=R(a_1',\cdots,a_m')$，当且仅当等式对于所有$x$和$y$的任何非零值成立。习题28(d)的珀西多项式$\varpi_{nk}(x,y)$是对于$\binom{n-1}{k-1}$的不同费尔利斯框图的车多项式。例如，$\varpi_{63}(x,y)$枚举形状43321，44221，44311，4432，53221，53311，5332，54211，5422和5431的车的配置。

30. (a) 我们有 $\varpi_n(x,y)=\sum_m x^{n-m}A_{mn}$ ，其中$A_{mn}=R_{n-m}(n-1,\cdots,1)$满足一个简单定律：如果我们不在形状$(n-1,\cdots,1)$的行1中放置一个车，则该行有$m-1$个自由单元、因为在其他行中有$n-m$个车。但是如果确实把一个车放在那里，我们就保 132
留它的0或1或……或$m-1$个单元自由。因此$A_{mn}=y^{m-1}A_{(m-1)(n-1)}+(1+y+\cdots+y^{m-1})A_{m(n-1)}$，而且由归纳法得出，$A_{mn}=y^{m(m-1)/2}\left\{{n \atop m}\right\}_y$ 。

(b) 由公式 $\varpi_{n+1}(x,y)=\sum_k\binom{n}{k}x^{n-k}y^k\varpi_k(x,y)$ 产生

$$A_{m(n+1)} = \sum_k \binom{n}{k} y^k A_{(m-1)k}$$

(c) 从(a)和(b)我们有

$$\frac{z^n}{(1-z)(1-(1+q)z)\cdots(1-(1+q+\cdots+q^{n-1})z)} = \sum_k \left\{ {k \atop n} \right\}_q z^k$$

$$\sum_k \binom{n}{k}_q (-1)^k q^{\binom{k}{2}} e^{(1+q+\cdots+q^{n-k-1})z} = q^{\binom{n}{2}} n!_q \sum_k \left\{ {k \atop n} \right\}_q \frac{z^k}{k!}$$

[第二个公式通过对n的归纳法来证明，因为两边满足微分方程$G'_{n+1}(z)=(1+q+\cdots+q^n)e^z G_n(qz)$，习题1.2.6－58证明当$z=0$时的相等性。]

历史注记：伦纳德·卡利兹在(*Transactions of the Amer. Math. Soc.* **33** (1933)，127-129)中引进了q斯特林数。然后在(*Duke Math. J.* **15** (1948)，987-1000)中，他在(解决其他事情当中)推导等式1.2.6-(45)的一个适当的推广：

$$(1+q+\cdots+q^{m-1})^n = \sum_k \left\{ {n \atop k} \right\}_q q^{\binom{k}{2}} \frac{m!_q}{(m-k)!_q}$$

31. $\exp(e^{w+z}+w-1)$；因此在习题23的阴影记号下$\varpi_{nk}=(\varpi+1)^{n-k}\varpi^{k-1}=\varpi^{n+1-k}(\varpi-1)^{k-1}$。[利·莫泽和马·维曼，*Trans. Royal Soc. Canada* (3) **43** (1954)，第3节，31-37。]事实上，习题28(d)的数$\varpi_{nk}(x,1)$由$\exp((e^{xw+xz}-1)/x+xw)$生成。

32. 我们有$\delta_n=\varpi_n(1,-1)$。而且当$x=1$和$y=-1$时，在习题28(d)的广义珀西三角形中容易想像一个简单的模式：对于$1<k<n$，我们有$|\varpi_{nk}(1,-1)|<1$和$\varpi_{n(k+1)}(1,-1)\equiv\varpi_{nk}(1,-1)+(-1)^n$ (modulo 3)。[在(*JACM* **20** (1973)，512-513)中，吉迪安·厄尔里兹给出了一个等价结果的组合证明。]

33. 通过在习题28(d)中置$x=y=1$，通过像在答案27中对车的放置来表示集合分划导致答案ϖ_{nk}。[$k=n$的情况是由赫·普罗丁格(H. Prodinger)发现的，*Fibonacci Quarterly* **19** (1981)，463-465。]

34. (a) 圭特顿的十四行诗包括方案01010101232323的149种，方案01010101234234的64种，方案01010101234342中的两种，只使用一次的方案有7种(像01100110234432)，还有我们不认为它们是十四行诗的29首诗，因为它们没有14行。

(b) 彼得拉奇的短抒情歌包括方案01100110234234的115种十四行诗，方案01100110232323的109种，方案01100110234324的66种，方案01100110232232的7种，以及像01010101232323这样每个至多使用三次的20个其他方案。

(c) 在斯宾瑟的情歌中，89首十四行诗中的88个使用方案01011212232344；唯一的例外(第8号)是“莎士比亚风格”的。

(d) 莎士比亚的154首十四行诗全部都用稍微容易的方案01012323454566，除了其中的两个(99和126)没有14行外。

133 (e) 布朗宁的来自葡萄牙的44首独行诗，遵照彼特拉查的方案01100110232323。

有时这些诗行将(碰巧)押偶韵，尽管它们并不需要这样；例如，布朗宁最后的十四行诗有方案01100110121212。

顺便指出，在但丁(Dante)的《神曲》(*Divine Comedy*)的冗长篇章中，使用了音韵互锁的方案，其中对于$n=1, 2, \cdots$，$1\equiv 3$和$3n-1\equiv 3n+1\equiv 3n+3$。

35. 每个不完备的n行音韵方案Π对应于$\{1, \cdots, n+1\}$的无单元集分划，其中$(n+1)$通过Π的所有单元集而被分组。[哈·W·贝克尔在(*AMM* 48 (1941)，702)中给出一个代数证明。注意，通过容斥原理，$\varpi'_n=\sum_k\binom{n}{k}(-1)^{n-k}\varpi_k$，而且$\varpi_n=\sum_k\binom{n}{k}\varpi'_k$；在习题23的阴影记号下，我们事实上可以写$\varpi'=\varpi-1$。杰·奥·萨立特(J. O. Shallit)通过置$\varpi_{n(n+1)}=\varpi'_n$，提出了推广的珀西三角形的建议；参见习题38(e)和习题33。事实上，ϖ_{nk}是$\{1, \cdots, n\}$的分划的个数，并且有这样的性质，即$1, \cdots, k-1$不是单元集；参见哈·W·贝克尔，*Bull. Amer. Math. Soc.* 58 (1954)，63。]

36. $\exp(e^z-1-z)$。(一般地说，如果ϑ_n是$\{1, \cdots, n\}$分成为可允许大小$s_1<s_2<\cdots$的子集的分划个数的话，则指数生成函数$\sum_n\vartheta_n z^n/n!$是$\exp(z^{s_1}/s_1!+z^{s_2}/s_2!+\cdots)$，因为$(z^{s_1}/s_1!+z^{s_2}/s_2!+\cdots)^k$是对于分成恰好$k$部分的分划的指数生成函数。)

37. 有长度n的$\sum_k\binom{n}{k}\varpi'_k\,\varpi'_{n-k}$个可能性，因此当$n=14$时为784 071 966。(但普希金的方案不能打破。)

38. (a) 想像以$x_1x_2\cdots x_n=01\cdots(n-1)$开始，然后对于$j=1, 2, \cdots, n$，逐次删去某个元素$b_j$并且把它放在左边。然后对于$1\leqslant k\leqslant|\{b_1, \cdots, b_n\}|$，$x_k$将是第$k$个最近移动的元素；参见习题5.2.3-36。结果数组$x_1\cdots x_n$将返回到它原来的状态，当且仅当$b_n\cdots b_1$是一个限制增长的串时。[罗宾斯(Robbins)和博尔克(Bolker)，*Æquat. Math.* 22 (1981)，281-282。]

换言之，令$a_1\cdots a_n$为一个限制增长串。对于$0\leqslant j<n$置$b_{-j}\leftarrow j$和$b_{j+1}\leftarrow a_{n-j}$，然后对于$1\leqslant j\leqslant n$，通过以下规则，即$b_j$是序列$b_{j-1}, b_{j-2}, \cdots$的第$k_j$个不同元素，由此定义$k_j$。例如，这样一来，串$a_1\cdots a_{16}$=0123032303456745对应于$\sigma$循环6688448628232384。

(b) 这样的通路对应于限制增长串且有$\max(a_1, \cdots, a_n)\leqslant m$，所以答案为$\left\{{n \atop 0}\right\}+\left\{{n \atop 1}\right\}+\cdots+\left\{{n \atop m}\right\}$。

(c) 我们可以假设$i=1$，因为每当$k_1k_2\cdots k_n$是一个σ循环时，序列$k_2\cdots k_nk_1$是一个σ循环。因此答案是满足$a_n=j-1$的限制增长串的数目，即$\left\{{n-1 \atop j-1}\right\}+\left\{{n-1 \atop j}\right\}+\left\{{n-1 \atop j+1}\right\}+\cdots$。

(d) 如果答案是f_n我们必须有$\sum_k\binom{n}{k}f_k=\varpi_n$，因为$\sigma_1$是恒等排列，因此$f_n=\varpi'_n$，即没有单元集的集合分划的个数(习题35)。

(e) 由(a)和(d)，再一次为ϖ'_n。[结果当p为质数时，$\varpi'_p \bmod p=1$]。

39. 置$u=t^{p+1}$来得到 $\frac{1}{p+1}\int_0^\infty e^{-u}u^{(q-p)/(p+1)}du=\frac{1}{p+1}\Gamma\left(\frac{q+1}{p+1}\right)$ 。

40. 我们有$g(z)=cz-n\ \ln\ z$，所以在n/c处出现马鞍点。现在在$\pm n/c\pm mi/c$处矩形通路有角落，而且$\exp\ g\ (n/c+it)=(e^n\ c^n/n^n)\exp(-t^2c^2/(2n)+it^3c^3/(3n^2)+\cdots)$。最后结果为 $e^n(c/n)^{n-1}\sqrt{2\pi n}$ 乘以$1+n/12+O(n^{-2})$。

(当然，通过在积分中令$w=cz$，我们可以更快地得到这个结果。但是在这里给出的答案适用于机械地求马鞍点的方法，而不必试图更灵巧些。)

41. 再一次，纯结果只不过是以c^{n-1}来乘(21)；但是在这个情况下矩形通路的左边而不是右边是重要的。(顺便指出，当$c=-1$时，我们不能导出当x是实数和正数
134 时，使用汉克尔周线的(22)的一个类似式，因为在这个通路上的积分发散。但是通过z^x的通常的定义，当$n=x>0$时，积分的一条适当通路确实产生公式$-(\cos\ \pi x)/\Gamma(x)$。)

42. 当n是偶数时我们有$\oint e^{z^2}dz/z^n=0$。否则当n很大时，有角落$\pm\sqrt{n/2}\pm in$的矩形的左边和右边近似地贡献

$$\frac{e^{n/2}}{2\pi(n/2)^{n/2}}\int_{-\infty}^{\infty}\exp\left(-2t^2-\frac{(-it)^3}{3}\frac{2^{3/2}}{n^{1/2}}+\frac{(it)^4}{n}-\cdots\right)dt$$

我们可以限制$|t|\leqslant n^\epsilon$来证明，这个积分为$I_0+(I_4-\frac{4}{9}I_6)/n$，且其相对误差为$O(n^{9\epsilon-3/2})$。其中$I_k=\int_{-\infty}^{\infty}e^{-2t^2}t^k dt$。同以前一样，相对误差实际上是$O(n^{-2})$；我们推导出答案为：

$$\frac{1}{((n-1)/2)!}=\frac{e^{n/2}}{\sqrt{2\pi}(n/2)^{n/2}}\left(1+\frac{1}{12n}+O\left(\frac{1}{n^2}\right)\right)，n为奇数$$

(当$n=x>0$时，(22)的类似式是$\left(\left(\sin\frac{\pi x}{2}\right)^2\Big/\Gamma((x-1)/2)\right)$。

43. 令$f(z)=e^{e^z}/z^n$。当$z=-n+it$时，我们有$|f(z)|<en^{-n}$。当$z=t+2\pi in+i\pi/2$时，我们有$|f(z)|=|z|^{-n}<(2\pi n)^{-n}$。所以除非在一条通路上$z=\xi+it$，否则这个积分可予忽略；而在该通路上$|f|$随着$|t|$从0增为$\pi$而减少。当已经有$t=n^{\epsilon-1/2}$时，我们有$|f(z)|/f(\xi)=O(\exp(-n^{2\epsilon}/(\log\ n)^2))$。而且当$|t|>\pi$时，我们有$|f(z)|/f(\xi)<1/|1+i\pi/\xi|^n=\exp\left(-\frac{n}{2}\ln\left(1+\pi^2/\xi^2\right)\right)$

44. 在(25)中置$u=na_2t^2$来得到$\Re\int_0^\infty e^{-u}\exp(n^{-1/2}c_3(-u)^{3/2}+n^{-1}c_4(-u)^2+n^{-3/2}c_5(-u)^{5/2}$

$+\cdots)du/\sqrt{na_2u}$，其中 $c_k=(2/(\xi+1))^{k/2}(\xi^{k-1}+(-1)^k(k-1)!)/k!=a_k/a_2^{k/2}$。这个表达式导致

$$b_l=\sum_{\substack{k_1+2k_2+3k_3+\cdots=2l\\k_1+k_2+k_3+\cdots=m\\k_1,k_2,k_3,\cdots\geq 0}}\left(-\frac{1}{2}\right)^{\overline{l+m}}\frac{c_3^{k_1}}{k_1!}\frac{c_4^{k_2}}{k_2!}\frac{c_5^{k_3}}{k_3!}\cdots$$

即对于$2l$的分划的求和。例如 $b_1=\frac{3}{4}c_4-\frac{15}{16}c_3^2$。

45. 为了得到$\varpi_n/n!$我们在(26)的推导中以$e^z-(n+1)\ln z$代替$g(z)$。这个改变以$1/(1+it/\xi)$来乘以前答案中的被积函数，它是$1/(1-n^{-1/2}a(-u)^{1/2})$。其中 $a=-\sqrt{2/(\xi+1)}$。因此我们得到

$$b_l'=\sum_{\substack{k+k_1+2k_2+3k_3+\cdots=2l\\k_1+k_2+k_3+\cdots=m\\k,k_1,k_2,k_3,\cdots\geq 0}}\left(-\frac{1}{2}\right)^{\overline{l+m}}a^k\frac{c_3^{k_1}}{k_1!}\frac{c_4^{k_2}}{k_2!}\frac{c_5^{k_3}}{k_3!}\cdots$$

即$p(2l)+p(2l-1)+\cdots+p(0)$个项的求和；$b_1'=\frac{3}{4}c_4-\frac{15}{16}c_3^2+\frac{3}{4}ac_3-\frac{1}{2}a^2$。[利·莫泽和马·维曼以不同方式得到系数$b_l'$(*Trans. Royal Soc. Canada* (3) 49，第3节 (1955)，49-54)。他们是推导ϖ_n的一个渐近序列的最先者。当n变为$n+1$时，他们的近似值较(26)的结果要稍微不大精确些，因为它不能精确地通过马鞍点。公式(26)是由伊·约·古德(I. J. Good)给出的，*Iranian J. Science and Tech.* 4 (1975)，77-83。]

46. 当$n\to\infty$时，对于固定的k，等式(13)和(31)表明$\varpi_{nk}=(1-\xi/n)^k\varpi_n(1+O(n^{-1}))$。而当$k=n$时，这个近似式也成立，但有相对误差$O((\log n)^2/n)$。 135

47. 步骤(H1，⋯，H6)分别被执行(1，ϖ_n，$\varpi_n-\varpi_{n-1}$，ϖ_{n-1}，ϖ_{n-1}，$\varpi_{n-1}-1$)次。在H4中的循环置$j\leftarrow j-1$总共$\varpi_{n-2}+\varpi_{n-3}+\cdots+\varpi_1$次，在H6中的循环置$b_j\leftarrow m$总共$(\varpi_{n-2}-1)+\cdots+(\varpi_1-1)$次。比率$\varpi_{n-1}/\varpi_n$近似地为$(\ln n)/n$，而且$(\varpi_{n-2}+\cdots+\varpi_1)/\varpi_n\approx(\ln n)^2/n^2$。

48. 在下式中，我们能很容易地检验求和和积分的交换。

$$\frac{e\varpi_x}{\Gamma(x+1)}=\frac{1}{2\pi i}\oint\frac{e^{e^z}}{z^{x+1}}dz=\frac{1}{2\pi i}\oint\sum_{k=0}^{\infty}\frac{e^{kz}}{k!z^{x+1}}dz$$
$$=\sum_{k=0}^{\infty}\frac{1}{k!}\frac{1}{2\pi i}\oint\frac{e^{kz}}{z^{x+1}}dz=\sum_{k=0}^{\infty}\frac{1}{k!}\frac{k^x}{\Gamma(x+1)}$$

49. 如果$\xi=\ln n-\ln\ln n+x$，我们有$\beta=1-e^{-x}-\alpha x$。因此由拉格朗日的反演公式(习题4.7－8)，

$$x=\sum_{k=1}^{\infty}\frac{\beta^k}{k}[t^{k-1}]\left(\frac{f(t)}{1-\alpha f(t)}\right)^k=\sum_{k=1}^{\infty}\sum_{j=0}^{\infty}\frac{\beta^k}{k}\alpha^j\binom{k+j-1}{j}[t^{k-1}]f(t)^{j+k}$$

其中$f(t)=t/(1-e^{-t})$。所以从方便的恒等式

$$\left(\frac{z}{1-e^{-z}}\right)^m=\sum_{n=0}^{\infty}\left[\begin{matrix}m\\m-n\end{matrix}\right]\frac{z^n}{(m-1)(m-2)\cdots(m-n)}$$

可得出结果。

(当$n\geqslant m$时，这个恒等式须小心地解释；如同在*CMath*的等式(7.59)中说明的那样，z^n的系数是m的n次多项式。)

在本题中的这一公式是路·康姆特给出的(*Comptes Rendus Acad. Sci.* (A) **270**(Paris，1970)，1085-1088)，他识别出由尼·戈·德·布鲁因以前计算过的系数(*Asymptotic Methods in Analysis* (1958)，25-28)。对于$n>e$的收敛性，是由杰弗里、科里斯(Corless)、哈里(Hare)和克努特证明的(*Comptes Rendus Acad. Sci.* (I) **320** (1995)，1449-1452)。他们还推导出稍微更快地收敛的一个公式。

(方程$\xi e^{\xi}=n$也有复根。在本习题的公式中，我们通过使用$\ln n+2\pi im$代替$\ln n$可得到全部的根；当$m\neq 0$时，这个和快速地收敛。参见科里斯、戈恩尼特(Gonnet)、哈里、杰弗里和克努特，*Advances in Computational Math.* **5** (1996)，347-350。)

50. 令$\xi=\xi(n)$，然后$\xi'(n)=\xi/((\xi+1)n)$。因而对于$|k|<n+1/e$，可证明泰勒级数

$$\xi(n+k)=\xi+k\xi'(n)+\frac{k^2}{2}\xi''(n)+\cdots$$

是收敛的。

其实，还有更多得多是真的，因为通过解析连续到负实轴的树函数$T(z)$，可得到函数$\xi(n)=-T(-n)$。(在$z=e^{-1}$处树函数有一个二次的奇异性，在绕开这个奇异性之后，在$z=0$处，我们遇到一个对数的奇异性，作为其中二次奇异性仅出现在级0处的有趣多级黎曼表面的一部分。)树函数的微分满足$z^kT^{(k)}(z)=R(z)^k p_k(R(z))$，其中$R(z)=T(z)/(1-T(z))$且$p_k(x)$是由$p_{k+1}(x)=(1+x)^2\ p_k'(x)+k(2+x)p_k(x)$定义的$k-1$次多项式。例如:

136 $$p_1(x)=1,\ p_2(x)=2+x,\ p_3(x)=9+10x+3x^2,\ p_4(x)=64+113x+70x^2+15x^3$$

(顺便指出，$p_k(x)$的系数枚举称为格里格(greg)树的生物进化树：$[x^j]\ p_k(x)$是带有j个未加标号的节点和k个已标号节点的有向树的个数，其中的叶必须是已标号了的，而未标号节点必须至少有两个子女。参见约·费尔森斯坦(J. Felsenstein)，*Systematic Zoology* **27** (1978)，27-33；莱·理·福尔德兹(L. R. Foulds)和罗·威·罗宾逊(R. W. Robinson)，*Lecture Notes in Math.* **829** (1980)，110-126；科·弗赖特(C.Flight)，*Manuscripta* **34** (1990)，122-128。)如果$q_k(x)=p_k(-x)$，通过归纳法我们可以证明，对于$0<x<1$，$(-1)^m q_k^{(m)}(x)\geqslant 0$。因此对于所有的$k, m\geqslant 1$，当$x$由0变成1时，$q_k(x)$从$k^{k-1}$单调地减少成为$(k-1)!$。由此得出

$$\xi(n+k)=\xi+\frac{kx}{n}-\left(\frac{kx}{n}\right)^2\frac{q_2(x)}{2!}+\left(\frac{kx}{n}\right)^3\frac{q_3(x)}{3!}-\cdots,\qquad x=\frac{\xi}{\xi+1}$$

其中的部分和如果$k>0$则交替地超出或低于正确值。

51. 有两个马鞍点 $\sigma=\sqrt{n+5/4}-1/2$ 和 $\sigma'=-1-\sigma$。对在$\sigma\pm im$和 $\sigma'\pm im$处有两个角落的一个矩形通路进行积分表明，当$n\to\infty$时仅仅σ是有关系的(尽管 σ'贡献大约 $e^{-\sqrt{n}}$ 的一个相对误差，当n很小时，它可能是重要的)。几乎像在(25)中那样论证，但通过$g(z)=z+z^2/2-(n+1)\ln z$，我们发现，t_n 可通过

$$\frac{n!}{2\pi}\int_{-n^{\epsilon}}^{n^{\epsilon}} e^{g(\sigma)-a_2t^2+a_3it^3+\cdots+a_l(-it)^l+O(n^{(l+1)\epsilon-(l-1)/2})}\mathrm{d}t,\qquad a_k=\frac{\sigma+1}{k\sigma^{k-1}}+\frac{[k=2]}{2}$$

很好地近似。如同在习题44中那样，这个积分推广成

$$\frac{n!e^{(n+\sigma)/2}}{2\sigma^{n+1}\sqrt{\pi a_2}}(1+b_1+b_2+\cdots+b_m+O(n^{-m-1}))$$

对于$k\geqslant 3$，这次$c_k=(\sigma+1)\sigma^{1-k}(1+1/(2\sigma))^{-k/2}/k$，因此$(2\sigma+1)^{3k}\sigma^k b_k$ 是σ的$2k$次多项式；例如

$$b_1=\frac{3}{4}c_4-\frac{15}{16}c_3^2=\frac{8\sigma^2+7\sigma-1}{12\sigma(2\sigma+1)^3}$$

特别是，斯特林近似和b_1项(在我们插入对于σ的公式后)产生

$$t_n=\frac{1}{\sqrt{2}}n^{n/2}e^{-n/2+\sqrt{n}-1/4}\left(1+\frac{7}{24}n^{-1/2}-\frac{119}{1152}n^{-1}-\frac{7933}{414720}n^{-3/2}+O(n^{-2})\right)$$

这是比等式5.1.4－(53)要更精确得多的一个结果，而且是以更简单的方式得到的。

52. 令 $G(z)=\sum_k \Pr(X=k)z^k$，使得第j个累加数κ_j 是$j![t^j]\ln G(e^t)$。在情况(a)中我们有 $G(z)=e^{e^{\xi z}}-e^{\xi}$；因此

$$\ln G(e^t)=e^{\xi e^t}-e^{\xi}=e^{\xi}(e^{\xi(e^t-1)}-1)=e^{\xi}\sum_{k=1}^{\infty}(e^t-1)^k\frac{\xi^k}{k!},\qquad \kappa_j=e^{\xi}\sum_k\left\{{k\atop j}\right\}\xi^k[j\neq 0]$$

情况(b)是对偶情况类：这里$\kappa=j=\varpi_j$ $[j\neq 0]$，因为

$$G(z)=e^{e^{-1}-1}\sum_{j,k}\left\{{k\atop j}\right\}e^{-j}\frac{z^k}{k!}=e^{e^{-1}-1}\sum_j\frac{(e^{z-1}-e^{-1})^j}{j!}=e^{e^{z-1}-1}$$

[如果在情况(a)中$\xi e^{\xi}=1$，我们有$\kappa_j=e\varpi$ $[j\neq 0]$。但若在该情况下$\xi e^{\xi}=n$，则均值是$\kappa_1=n$，且方差σ^2是$(\xi+1)n$。因此，习题45的公式指出，均值n以 $1/\sqrt{2\pi\sigma}$ 的近似概率出现，而相对误差为$O(1/n)$。这个发现导致以另一个方式证明该公式。] 137

53. 如同在等式1.2.10－(23)中那样，我们可以写$\ln G(e^t)=\mu t+\sigma^2t^2/2+\kappa_3t^3/3!+\cdots$，而且有一个正常数$\delta$ 使得当$|t|<\delta$ 时，$\sum_{j=3}^{\infty}|\kappa_j||t|^j/j!<\sigma^2t^2/6$。因此如果$0<\epsilon<1/2$，当$n\to\infty$时，对于某个常数$c>0$我们能证明

$$[z^{\mu n+r}]G(z)^n = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{G(e^{it})^n\,dt}{e^{it(\mu n+r)}}$$
$$= \frac{1}{2\pi}\int_{-n^{\varepsilon-1/2}}^{n^{\varepsilon-1/2}}\exp\left(-irt-\frac{\sigma^2t^2n}{2}+O(n^{3\varepsilon-1/2})\right)dt+O(e^{-cn^{2\varepsilon}})$$

对于$n^{\varepsilon-1/2} < |t| < \delta$，被积函数以$\exp(-\sigma^2 n^{2\varepsilon}/3)$的绝对值为界；而且当$\delta < |t| < \pi$时它的数量至多为$\alpha^n$，其中$\alpha=\max|G(e^{it})|$小于1。因为由我们的假设，个别的项$p_k e^{kit}$不总处于一条直线上。因此，

$$[z^{\mu n+r}]G(z)^n = \frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\left(-irt-\frac{\sigma^2t^2n}{2}+O(n^{3\varepsilon-1/2})\right)dt+O(e^{-cn^{2\varepsilon}})$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\left(-\frac{\sigma^2n}{2}\left(t+\frac{ir}{\sigma^2n}\right)^2-\frac{r^2}{2\sigma^2n}+O(n^{3\varepsilon-1/2})\right)dt+O(e^{-cn^{2\varepsilon}})$$
$$= \frac{e^{-r^2/(2\sigma^2n)}}{\sigma\sqrt{2\pi n}}+O(n^{3\varepsilon-1})$$

以类似的公式考虑$\kappa_3,\kappa_4,\cdots$，对任意大的m，我们可以把这个估计改进成$O(n^{-m})$。因此对于$\varepsilon=0$，这个结果也为真。[事实上，这样的改进导致“埃杰沃什扩展”。按照它$[z^{\mu n+r}]G(z)^n$渐近为

$$\frac{e^{-r^2/(2\sigma^2n)}}{\sigma\sqrt{2\pi n}}\sum_{\substack{k_1+2k_2+3k_3+\cdots=m\\ k_1+k_2+k_3+\cdots=l\\ k_1,k_2,k_3,\cdots\geq 0\\ 0\leq s\leq l+m/2}}\frac{(-1)^s(2l+m)^{\underline{2s}}}{\sigma^{4l+2m-2s}2^s s!}\frac{r^{2l+m-2s}}{n^{l+m-s}}\frac{1}{k_1!k_2!\cdots}\left(\frac{\kappa_3}{3!}\right)^{k_1}\left(\frac{\kappa_4}{4!}\right)^{k_2}\cdots$$

绝对误差为$O(n^{-p/2})$，其中隐藏在O中的常数仅依赖于p和G，但不依赖于r或n——如果我们把求和限定为对于$m<p-1$的情况的话。例如，当$p=3$时我们得到

$$[z^{\mu n+r}]G(z)^n = \frac{e^{-r^2/(2\sigma^2n)}}{\sigma\sqrt{2\pi n}}\left(1-\frac{\kappa_3}{2\sigma^4}\left(\frac{r}{n}\right)+\frac{\kappa_3}{6\sigma^6}\left(\frac{r^3}{n^2}\right)\right)+O\left(\frac{1}{n^{3/2}}\right)$$

而且当$p=4$时，有7个以上的项。参见帕·罗·契比雪夫(P. L. Chebyshev)，*Zapiski Imp. Akad. Nauk* 55(1887)，No.6，1-16；*Acta Math.* 14 (1890)，305－315；弗·威·埃杰沃什(F. Y. Edgeworth)，*Trans. Cambridge Phil.* Soc. 20 (1905)，36-65，113-141；哈·克拉梅尔(H. Cramér)，*Skandinavisk Aktuarietidsskrift* 11 (1928)，13-74，141-180。]

54. 公式(40)等价于$\alpha=s\coth s+s$，$\beta=s\coth s-s$。

55. 令$c=\alpha e^{-\alpha}$。牛顿迭代$\beta_0=c$，$\beta_{k+1}=(1-\beta_k)ce^{\beta_k}/(1-ce^{-\beta_k})$迅速地提高到正确值，除非$\alpha$极端地靠近1。例如，当$\alpha=\ln 4$时，$\beta_7$和$\ln 2$的差别小于$10^{-75}$。

56. (a) 对n用归纳法，$g^{(n+1)}(z)=(-1)^n\left(\frac{\sum_{k=0}^{n}\left\langle{n \atop k}\right\rangle e^{(n-k)z}}{\alpha(e^z-1)^{n+1}}-\frac{n!}{z^{n+1}}\right)$。

(b) $\sum_{k=0}^{n}\left\langle {n \atop k} \right\rangle e^{k\sigma}/n! = \int_0^1 \cdots \int_0^1 \exp\left(\lfloor u_1+\cdots+u_n \rfloor \sigma\right) du_1 \cdots du_n$

$$< \int_0^1 \cdots \int_0^1 \exp\left((u_1+\cdots+u_n)\sigma\right) du_1 \cdots du_n = (e^\sigma - 1)^n/\sigma^n$$

下界是类似的，因为 $\lfloor u_1+\cdots+u_n \rfloor > u_1+\cdots+u_n-1$ 。

(c) 于是$n!(1-\beta/\alpha)<(-\sigma)^n g^{(n+1)}(\sigma)<0$，而且我们只需验证$1-\beta/\alpha<2(1-\beta)$，即$2\alpha\beta<\alpha+\beta$。但由习题54，$\alpha\beta<1$和$\alpha+\beta>2$。

57. (a) 如同在答案56(c)中那样，$n+1-m=(n+1)(1-1/\alpha)<(n+1)(1-\beta/\alpha)=(n+1)\sigma/\alpha \leqslant 2N$。(b) 当$\alpha$增加时，量$\alpha+\alpha\beta$增加，因为相对于$\alpha$的导数是$1+\beta+\beta(1-\alpha)/(1-\beta)=(1-\alpha\beta)/(1-\beta)+\beta>0$。因此$1-\beta<2(1-1/\alpha)$。

58. (a) $|e^{\sigma+it}-1|^2/|\sigma+it|^2=(e^{\sigma+it}-1)(e^{\sigma-it}-1)/(\sigma^2+t^2)$ 相对于t的导数是 $(\sigma^2+t^2)\sin t - t\left(2\sin\frac{t}{2}\right)^2 - \left(2\sinh\frac{\sigma}{2}\right)^2 t$ 乘以一个正函数。对于$0<t\leqslant 2\pi$，这个导数总是负的，因为它小于 $t^2\sin t - t\left(2\sin\frac{t}{2}\right)^2 = 8u\sin u\cos u(u-\tan u)$，其中$t=2u$。

令 $s=2\sinh\frac{\sigma}{2}$，当$\sigma \geqslant \pi$和$2\pi \leqslant t \leqslant 4\pi$时，此导数仍然是负的，因为我们有$t \leqslant 4\pi \leqslant s^2-\sigma^2/(2\pi) \leqslant s^2-\sigma^2/t$。类似地，当$\sigma \geqslant 2\pi$时，对于$4\pi \leqslant t \leqslant 168\pi$，这个导数保持为负；这个证明越来越容易。

(b) 令 $t=u\sigma/\sqrt{N}$。于是(41)和(42)证明

$$\int_{-\tau}^{\tau} e^{(n+1)g(\sigma+it)}\mathrm{d}t =$$

$$\frac{(e^\sigma-1)^m}{\sigma^n\sqrt{N}}\int_{-N^\epsilon}^{N^\epsilon}\exp\left(-\frac{u^2}{2}+\frac{(-iu)^3 a_3}{N^{1/2}}+\cdots+\frac{(-iu)^l a_l}{N^{l/2-1}}+O(N^{(l+1)\epsilon-(l-1)/2})\right)\mathrm{d}u$$

其中$(1-\beta)a_k$是α和β的$k-1$次多项式，且$0 \leqslant a_k \leqslant 2/k$。(例如，$6a_3=(2-\beta(\alpha+\beta))/(1-\beta)$且$24a_4=(6-\beta(\alpha^2+4\alpha\beta+\beta^2))/(1-\beta)$。)被积函数的单调性表明，在余下的范围上积分是可忽略的。现在加入尾部，把积分扩展到$-\infty<u<\infty$，并且以$c_k=2^{k/2}a_k$使用答案44的公式来定义$b_1, b_2, \cdots$。

(c) 我们将证明$|e^z-1|^m\sigma^{n+1}/((e^\sigma-1)^m|z|^{n+1})$在那三条通路上是指数地小的。如果$\sigma \leqslant 1$，则这个量小于$1/(2\pi)^{n+1}$ (因为，例如$e^\sigma-1>\sigma$)。如果$\sigma>1$，我们有$\sigma<2|z|$以及$|e^z-1| \leqslant e^\sigma-1$。

59. 在这种极端情况下，$\alpha=1+n^{-1}$和 $\beta=1-n^{-1}+\frac{2}{3}n^{-2}+O(n^{-3})$；因此 $N=1+\frac{1}{3}n^{-1}+O(n^{-2})$。前导项 $\beta^{-n}/\sqrt{2\pi N}$ 是 $e/\sqrt{2\pi}$ 乘以 $1-\frac{1}{3}n^{-1}+O(n^{-2})$。(注意 $e/\sqrt{2\pi}\approx 1.0844$。)答案58(b)中的量$a_k$结果为$1/k+O(n^{-1})$。因此到一次为止的正确的项为

$$\frac{b_j}{N^j}=[z^j]\exp\left(-\sum_{k=1}^{\infty}\frac{B_{2k}z^{2k-1}}{2k(2k-1)}\right)+O\left(\frac{1}{n}\right)$$

即在对应于斯特林近似式

$$\frac{1}{1!}\sim\frac{e}{\sqrt{2\pi}}\left(1-\frac{1}{12}+\frac{1}{288}+\frac{139}{51840}-\frac{571}{2488320}-\cdots\right)$$

的(发散)级数中的诸项。

60. (a) 长度n的m进制串且其中m个数字均出现，这样的串的个数为$m!\left\{ {n \atop m} \right\}$。
139 而且容斥原理把这个量表达成为$\binom{m}{0}m^n-\binom{m}{1}(m-1)^n+\cdots$ 。现在参见习题7.2.1.4－37。

(b) 我们有$(m-1)^n/(m-1)!=(m^n/m!)\,m\exp(n\ln(1-1/m))$，而且$\ln(1-1/m)$小于$-n^{\varepsilon-1}$。

(c) 在此情况下，$\alpha>n^{\varepsilon}$和$\beta=\alpha e^{-\alpha}e^{\beta}<\alpha e^{1-\alpha}$。因此$1<(1-\beta/\alpha)^{m-n}<\exp(nO(e^{-\alpha}))$；而且$1>e^{-\beta m}=e^{-(n+1)\beta/\alpha}>\exp(-n\,O(e^{-\alpha}))$。所以(45)变成为$(m^n/m!)(1+O(n^{-1})+O(ne^{-n^{\varepsilon}}))$。

61. 现在 $\alpha=1+\frac{r}{n}+O(n^{2\varepsilon-2})$ 和 $\beta=1-\frac{r}{n}+O(n^{2\varepsilon-2})$ ，因此$N=r+O(n^{2\varepsilon-1})$。而且等式(43)的情况$l=0$归结为

$$n^r\left(\frac{n}{2}\right)^r\frac{e^r}{r^r\sqrt{2\pi r}}\left(1+O(n^{2\varepsilon-1})+O\left(\frac{1}{r}\right)\right)$$

(这个近似式同诸如$\left\{ {n \atop n-1} \right\}=\binom{n}{2}$和$\left\{ {n \atop n-2} \right\}=2\binom{n}{4}+\binom{n+1}{4}$这样的恒等式很好地协调在一起；其实，当$r$是常数时，根据CMath的公式(6.42)和(6.43)，当$n\to\infty$时，我们有

$$\left\{ {n \atop n-r} \right\}=\frac{n^{2r}}{2^r r!}\left(1+O\left(\frac{1}{n}\right)\right)$$

62. 对于$1<n<10000$(而且在那些情况的5648个中 $m=\lfloor e^{\xi}-1\rfloor$)，这个断言为真。伊·罗·坎菲尔德(E.R.Canfield)和卡·破米兰茨(C.Pomerance)在一篇概述以前的工作及相关问题的优秀论文中，已经证明对于所有充分大的n，该命题成立。而且在两种情况下极大值出现仅当$e^{\xi}\bmod 1$极端接近于1/2时。[*Integers* 2 (2002)，A1，1-13。]

63. (a) 当$p_1=\cdots=p_n=p$时，结果成立，因为$a_{k-1}/a_k=(k/(n+1-k))\times((n-\mu)/\mu)<(n-\mu)/(n+1-\mu)<1$。由归纳法，当$p_n=0$或1时它也为真。对于一般的情况，考虑对于满足$p_1+\cdots+p_n=\mu$的$(p_1,\cdots,p_n)$的所有选择上$a_k-a_{k-1}$的极小值：如果$0<p_1<p_2<1$，令 $p_1'=p_1-\delta$ 和 $p_2'=p_2+\delta$ ，而且注意，对于仅仅依赖于$p_3,\cdots,p_n$的某个α，$a_k'-a_{k-1}'=a_k-a_{k-1}+\delta(p_1-p_2-\delta)\alpha$ 。在一个极小的点处，我们必定有$\alpha=0$；因此我们

可以选择δ使得或者 $p_1'=0$或者 $p_2'=1$。因此，当所有p_j都有三个值$\{0, 1, p\}$之一时，可实现极小值。但在这样的情况下，我们已经证明$a_k - a_{k-1}>0$。

(b) 把每一个p_j改成$1-p_j$，使μ变成$n-\mu$且使a_k变成a_{n-k}。

(c) $f(x)$没有为正的根。因此$f(z)/f(1)$有在(a)和(b)中的形式。

(d) 令$C(f)$是在f的系数序列中符号改变的个数，我们要来证明$C((1-x)^2 f)=2$。事实上，对于所有$m\geqslant 0$，$C((1-x)^m f)=m$。因为$C((1-x)^m)=m$，且当a和b为正时$C((a+bx)f)\leqslant C(f)$；因此$C((1-x)^m f)\leqslant m$。因此如果$f(x)$是无论什么样的任意非零多项式，$C((1-x)f)>C(f)$；因此$C((1-x)^m f)\geqslant m$。

(e) 由于 $\sum_k \left[{n \atop k}\right] x^k = x(x+1)\cdots(x+n-1)$，部分(c)对于$\mu=H_n$可直接应用。而且对于多项式 $f_n(x)=\sum_k \left\{{n \atop k}\right\} x^k$，我们可以以$\mu=\varpi_{n+1}/\varpi_n - 1$使用部分(c)，如果$f_n(x)$有$n$个实根的话。通过使用归纳法，可得到后边的命题，因为$f_{n+1}(x)=x(f_n(x)+f_n'(x))$：如果$a>0$，而且如果$f(x)$有$n$个实根，则函数$g(x)=e^{ax}f(x)$也是。而且当$x\to-\infty$时，$g(x)\to 0$。因此 $g'(x)=e^{ax}(af(x)+f'(x))$ 也有n个实根(即有一个根在很远的左边，而其他$n-1$是在$g(x)$的根之间)。

[参见埃·拉奎尔热(E. Laguerre)，*J. de Math.* (3) **9** (1883)，99－146；瓦·霍夫丁(W. Hoeffding)，*Annals Math. Stat.* **27** (1956)，713－721；约·牛·达洛茨(J. N. Darroch)，*Annals Math. Stat.* **35** (1964)，1317－1321；詹·皮特曼，*J. Combinatorial Theory* **A77** (1997)，297-303。]

64. 我们只需要用计算机代数从$\ln \varpi_n$中减去$\ln \varpi_{n-k}$即可。 140

65. 它是ϖ_n^{-1}乘以k块出现的次数加上在所有集合分划表中k块有序对出现的次数，即 $\left(\binom{n}{k}\varpi_{n-k}+\binom{n}{k}\binom{n-k}{k}\varpi_{n-2k}\right)/\varpi_n$，减去(49)的平方。渐近地，$(\xi^k/k!)(1+O(n^{4\epsilon-1}))$。

66. 当$n=100$时，对于分划$7^1\,6^2\,5^4\,4^6\,3^7\,2^6\,1^4$和$7^1\,6^2\,5^4\,4^6\,3^8\,2^5\,1^3$，实现(48)的极大值。

67. M^k的预期值是ϖ_{n+k}/ϖ_n。由(50)，均值因此是$\varpi_{n+1}/\varpi_n=n/\xi+\xi/(2(\xi+1)^2)+O(n^{-1})$，而方差为

$$\frac{\varpi_{n+2}}{\varpi_n}-\frac{\varpi_{n+1}^2}{\varpi_n^2}=\left(\frac{n}{\xi}\right)^2\left(1+\frac{\xi(2\xi+1)}{(\xi+1)^2 n}-1-\frac{\xi^2}{(\xi+1)^2 n}+O\left(\frac{1}{n^2}\right)\right)=\frac{n}{\xi(\xi+1)}+O(1)$$

68. 在一个分划的所有部分中，非零分量的极大个数是$n=n_1+\cdots+n_m$；它出现当且仅当所有组成部分为0或1。极大的级也等于n。

69. 在步骤M3的开始，如果$k>b$且$l=r-1$，则转到M5。在步骤M5中，如果$j=a$和$(v_j-1)(r-l)<u_j$，则转到M6而不是减少v_j。

70. (a) $\left|{n-1 \atop r-1}\right|+\left|{n-2 \atop r-1}\right|+\cdots+\left|{r-1 \atop r-1}\right|$，因为$\left|{n-k \atop r-1}\right|$包含有$k$个0的块$\{0,\cdots,0,1\}$。总数，也称

为$p(n-1,1)$，是$p(n-1)+\cdots+p(1)+p(0)$。

(b) 如果我们交换$n-1\leftrightarrow n$，则$\{1,\cdots,n-1,n\}$的r块分划中恰有 $N=\left\{{n-1 \atop r}\right\}+\left\{{n-2 \atop r-2}\right\}$ 个是相同的。所以答案为$N+\frac{1}{2}\left(\left\{{n \atop r}\right\}-N\right)=\frac{1}{2}\left(\left\{{n \atop r}\right\}+N\right)$。它也是满足$\max(a_1,\cdots,a_n)=r-1$和$a_{n-1}<a_n$的限制增长串$a_1\cdots a_n$的个数。因此这总数是$\frac{1}{2}(\varpi_n+\varpi_{n-1}+\varpi_{n-2})$。

71. $\left\lfloor\frac{1}{2}(n_1+1)\cdots(n_m+1)-\frac{1}{2}\right\rfloor$，因为有分成两部分的$(n_1+1)\cdots(n_m+1)-2$个合成，而且这些中的一半不是在词典顺序下，除非所有n_j都是偶数。(参见习题7.2.1.4-31。对于多达5部分的公式已由爱·麦·赖特(E.M.Wright)给出，*Proc. London Math. Soc.* (3) **11** (1961)，499-510。)

72. 是的。对于$0\le j,\ k\le n$，以下算法在$\Theta(n^4)$个步骤中计算$a_{jk}=p(j,k)$：对于所有j和k，由$a_{jk}\leftarrow 1$开始。然后对于$l=0,1,\cdots,n$和$m=0,1,\cdots,n$(以任何次序)，对于$j=l,\cdots,n$和$k=m,\cdots,n$(以递增的顺序)，如果$l+m>1$，则置$a_{jk}\leftarrow a_{jk}+a_{(j-l)(k-m)}$。

参见表A－1，一个类似的方法在$O(n_1\cdots n_m)^2$步内计算$p(n_1,\cdots,n_m)$。在已经引用的文章中，契玛和莫兹金已经推导了递归关系

$$n_1p(n_1,\cdots,n_m)=\sum_{l=1}^{\infty}\sum_{k_1,\cdots,k_m\ge 0}k_1p(n_1-k_1l,\cdots,n_m-k_ml)$$

但这个有趣的公式仅在某些情况下才对计算有帮助。

表A－1 多重分划数

n	0	1	2	3	4	5	6
$p(0,n)$	1	1	2	3	5	7	11
$p(1,n)$	1	2	4	7	12	19	30
$p(2,n)$	2	4	9	16	29	47	77
$p(3,n)$	3	7	16	31	57	97	162
$p(4,n)$	5	12	29	57	109	189	323
141 $p(5,n)$	7	19	47	97	189	339	589

n	0	1	2	3	4	5
$P(0,n)$	1	2	9	66	712	10457
$P(1,n)$	1	4	26	249	3274	56135
$P(2,n)$	2	11	92	1075	16601	325269
$P(3,n)$	5	36	371	5133	91226	2014321
$P(4,n)$	15	135	1663	26683	537813	13241402
$P(5,n)$	52	566	8155	149410	3376696	91914202

73. 是的，当有m个1和n个2时，令$P(m,n)=p(1,\cdots,1,2,\cdots,2)$，于是$P(m,0)=\varpi_m$，而且我们可以使用递归式

$$2P(m,n+1)=P(m+2,n)+P(m+1,n)+\sum_k\binom{n}{k}P(m,k)$$

当我们以两个不同元素x和x'来代替$P(m,n+1)$的多重集合中的一对x时，我们通过考虑发生什么情况来证明这个递归式。我们得到$2P(m,n+1)$分划，表示$P(m+2,n)$，除非在$P(m+1,n)$的情况下，其中x和x'属于同一个块中，或者在$\binom{n}{k}P(m,n-k)$的情况下，其中含有x和x'的诸块相等并且有k个附加的元素。

注：参见表A－1，另一个递归式，它对于计算不大有用，是

$$P(m+1,n)=\sum_{j,k}\binom{n}{k}\binom{n-k+m}{j}P(j,k)$$

序列$P(0, n)$首先是由爱·凯·罗伊德(E. K. Lloyd)(*Proc. Cambridge Philos. Soc.* 103 (1988), 277-284)和吉·拉贝利(G. Labelle)(*Discrete Math.* 217 (2000), 237-248)研究的，他们通过完全不同的方法来计算它。习题70(b)表明$P(m, 1)=(\varpi_m+\varpi_{m+1}+\varpi_{m+2})/2$；一般地说，$P(m, n)$可以阴影记号$\varpi^m q_n(\varpi)$写出，其中$q_n(x)$是由生成函数$\sum_{n=0}^{\infty} q_n(x)z^n/n!=\exp((e^z+(x+x^2)z-1)/2)$定义的$2n$次的一个多项式。因此，由习题31

$$\sum_{n=0}^{\infty}P(m,n)\frac{z^n}{n!}=e^{(e^z-1)/2}\sum_{k=0}^{\infty}\frac{\varpi_{(2k+m+1)(k+m+1)}}{2^k}\frac{z^k}{k!}$$

作为更一般得多的情况的一个特殊情况，拉贝利证明$\{1, 1, \cdots, n, n\}$分成恰好r个块的分划的个数为

$$n![x^r z^n]e^{-x+x^2(e^z-1)/2}\sum_{k=0}^{\infty}e^{zk(k+1)/2}\frac{x^k}{k!}$$

75. 马鞍点方法产生$Ce^{An^{2/3}+Bn^{1/3}}/n^{55/36}$，其中$A=3\zeta(3)^{1/3}$，$B=\pi^2\zeta(3)^{-1/3}/2$以及$C=\zeta(3)^{19/36}(2\pi)^{-5/6}3^{-1/2}\exp(1/3+B^2/4+\zeta'(2)/(2\pi^2)-\gamma/12)$。[法·灿·奥尔洛克(F. C. Auluck)，*Proc. Cambridge Philos. Soc.* 49 (1953)，72－83；爱·麦·赖特，*American J. Math.* 80 (1958)，643-658。]

76. 使用$p(n_1, n_2, n_3, \cdots)\geqslant p(n_1+n_2, n_3, \cdots)$的事实，因此$P(m+2, n)\geqslant P(m, n+1)$，通过归纳法可以证明$P(m, n+1)\geqslant(m+n+1)P(m, n)$。因此

$$2P(m, n)\leqslant P(m+2, n-1)+P(m+1, n-1)+eP(m, n-1)$$

对这不等式进行迭代证明$2^n P(0, n)=(\varpi^2+\varpi)^n+O(n(\varpi^2+\varpi)^{n-1})=(n\varpi_{2n-1}+\varpi_{2n})(1+O((\log n)^3/n))$。(从习题75的答案中的生成函数，可以得到一个更精确的渐近公式。)

78. 3 3 3 3 2 1 0 0 0
1 0 0 0 2 2 3 2 0
2 2 1 0 0 2 1 0 2
2 1 0 2 2 0 0 1 3

(因为编码的分划必须都是(000000000))

79. 有432个这样的循环。但它们仅产生304个集合分划的不同循环，因为不同的循环可能描述相同的分划序列。例如(000012022332321)和(000012022112123)是在分划上等价的。 142

80. [参见钟金芳蓉、珀·迪亚科尼斯和罗·格拉罕，*Discrete Mathematics* 110 (1992)，52-55。]构造具有ϖ_{n-1}个顶点和ϖ_n个边的一个有向图；每个限制增长串$a_1\cdots a_n$定义从顶点$a_1\cdots a_{n-1}$到顶点$\rho(a_2\cdots a_n)$的一条有向弧，其中ρ是习题4中的

函数。(例如，从0100121到0110203的有向弧01001213。)每一个万有循环定义在这个有向图中的欧拉尾部。反之，每个欧拉尾部可用来定义在元素$\{0, 1, \cdots, n-1\}$上的限制增长的一个或多个万有序列。

如果我们令从每一个非零顶点$a_1 \cdots a_{n-1}$开始的最后出口通过有向弧$a_1 \cdots a_{n-1} a_{n-1}$，则由2.3.4.2节的方法，存在一个欧拉尾部。然而，这个序列可能不是循环。例如，当$n<4$时，没有万有循环存在；而当$n=4$时，万有序列000012030110100222定义不对应于任何万有循环的集合分划的一个循环。

对于$n \geqslant 6$，如果我们对于某些不同元素$\{u, v, x, y\}$，以$0^n xyx^{n-3}u(uv)^{\lfloor (n-2)/2 \rfloor}u^{[n \text{ odd}]}$开始的一个欧拉尾部作为出发点，则可以证明一个循环的存在性。对于$2 \leqslant k \leqslant n-4$，如果我们把$0^k\,121^{n-3-k}$的最后出口从$0^{k-1}\,121^{n-2-k}$改成为$0^{k-1}\,121^{n-3-k}2$，并且令$0121^{n-4}$和$01^{n-3}2$的最后出口分别为$010^{n-4}1$和$0^{n-3}10$，则这个模式是可能的。现在如果我们选择向后的循环个数，由此确定u和v，我们可以令x和y是不同于$\{0, u, v\}$的最小元素。

事实上我们可以作出结论，有这个极端特殊类型的万有循环的个数是巨大的——至少，当$n \geqslant 6$时

$$\left(\prod_{k=2}^{n-1} (k!(n-k))^{\left\{ {n-1 \atop k} \right\}}\right) / ((n-1)!(n-2)^3 3^{2n-5} 2^{2})$$

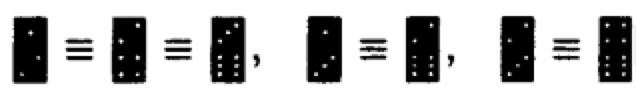
而且，至今还不知道它们之中是否有可容易地编码的。参见以下对于$n=5$的情况。

81. 注意$\varpi_5=52$，我们使用$\{1, 2, 3, 4, 5\}$的一个万有循环，其中的元素为13个梅花、13个方片、13个红桃、12个黑桃以及一个J。如同在上题答案中那样，通过使用欧拉尾部反复尝试，所发现的一个这样的循环是

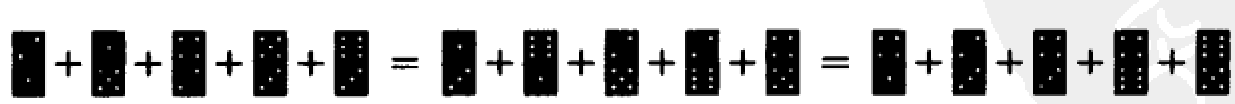
(♠♠♠♠♠♣♢♡J♣♡♢♡♠♣♣♢♡♣♢♣♡♢♢♣♡♣♡♣♣♣♣♢♣♢♢♣♢♠♠♢♡♡♡♠♡♡♡♠♠♢♢).

(事实上，如果我们转移到$a_k=a_{k-1}$作为一个最后的求助手段，而且如果我们尽可能快地引入J，则实际上有114 056个这样的循环。)如果我们把J叫做一个黑桃，则这个技巧仍以$\dfrac{47}{52}$的概率有效。

82. 有13 644个解，尽管如果我们认为



这个数减少成1981，最小的普通和是5/2，而最大是25/2。引人注目的解



是仅有获得普通和118/15的两个实质上不同的方式之一。[这个问题是由博·安·科迪姆斯基(B. A. Kordemsky)在*Matematicheskaîa Smekalka* (1954)中提出的，在英
143 语翻译中它是数78，*The Moscow Puzzles* (1972)。]

索引和词汇表

索引中的页码为英文原书页码，与本书中文部分边栏的页码一致。当一个索引条款所指页码包含相关习题时，也请参见该习题的答案以获取更多信息。这里对答案页未加索引，除非其中有习题中未包含的话题。

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